

MAPS BETWEEN CIRCLE BUNDLES: FIBER-PRESERVING, FINITENESS AND REALIZATION OF MAPPING DEGREE SETS

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ABSTRACT. Let E_i be an oriented circle bundle over a closed oriented aspherical n -manifold M_i , $i = 1, 2$. We prove the following geometric realization result, which extends the classical result for maps between circle bundles over surfaces to all dimensions ≥ 2 . Let $f: E_1 \rightarrow E_2$ be a map. If the induced homomorphism f_* on π_1 sends the fiber subgroup of $\pi_1(E_1)$ to the fiber subgroup of $\pi_1(E_2)$ injectively, then there is a fiber-preserving map in the homotopy class of f .

Using the above result and a computation of the mapping degree set of fiber-preserving maps between E_1 and E_2 , we obtain the following results on the finiteness and the realization problems of mapping degree sets.

(\mathcal{F}) The mapping degree set $D(E_1, E_2)$ is finite if M_2 is hyperbolic and the Euler class of E_2 is not torsion.

(\mathcal{R}) For any finite set A of integers containing 0 and any $n > 2$, A is the mapping degree set $D(M, N)$ for some closed oriented n -manifolds M and N .

Item (\mathcal{F}) extends the previously known 3-dimensional case to all dimensions ≥ 3 . Item (\mathcal{R}) gives a complete answer to the realization problem on finite sets of integers (containing 0) in all dimensions, including in particular the previously unknown cases in dimensions 4 and 5.

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1. INTRODUCTION

Let M, N be two closed, connected, oriented manifolds of the same dimension. The degree of a map $f: M \rightarrow N$, denoted by $\deg(f)$, is probably one of the oldest and most fundamental concepts in topology. The *set of mapping degrees* from M to N is defined by

$$D(M, N) := \{d \in \mathbb{Z} \mid \exists f: M \rightarrow N, \deg(f) = d\}.$$

When $M = N$, the *set of degrees of self-maps* $D(M, M)$ is denoted by $D(M)$.

The study of various properties of mapping degree sets for various classes of M and N , including the finiteness of $D(M, N)$, whether $\pm 1 \in D(M, N)$, as well as the exact computation of $D(M, N)$, has a long history. This topic was revolutionised by Thurston [Th] and Gromov [Gr] more than 40 years ago, and the most prominent notion that arose from these profound works was the *simplicial volume*. Thereafter, research on these questions became very active, rich in open problems and applications, and involves various methods from geometric topology, algebraic topology, differential geometry, representation theory, analysis, and others. For a (highly non-exhaustive) list of references featuring the variety of these techniques, we refer the reader to [BG], [KL], [LS], [BRW], [DLSW], [CMV], [Ne2], [BGM], and the bibliographies therein.

In this paper, we mainly study maps between circle bundles of the same dimension and their mapping degree sets. In the following we will describe our results and their applications.

1.1. Fiber-preserving maps. S^1 -bundles are primary examples of fiber bundles and constitute a significant class of manifolds. In particular, among the eight classes of geometric 3-manifolds in the sense of Thurston, six of them are finitely covered by S^1 -bundles over surfaces, which are called Seifert manifolds. The following result is classical in 3-manifold topology, which is important in the study of mapping class groups and non-zero degree maps of 3-manifolds.

Theorem A. *If M and N are circle bundles over aspherical surfaces and $f_*: \pi_1(M) \rightarrow \pi_1(N)$ sends the fiber subgroup to the fiber subgroup, then there is a fiber-preserving map in homotopy class of f .*

The proof of Theorem A appeared in Waldhausen's 1967 paper [Wal] for self-maps, and in Rong's 1993 paper [Ro] for the general case. Both proofs use the hierarchy theory of 2- and 3-manifolds.

When the surfaces in Theorem A are replaced by n -manifolds for $n > 2$, such a theorem was not known and our first goal is to establish such a result.

Theorem 1.1. *Let E_i be an oriented S^1 -bundle over a closed oriented aspherical n -manifold M_i , $i = 1, 2$, and let $f: E_1 \rightarrow E_2$ be a map. If the induced map f_* on π_1 sends the fiber subgroup of $\pi_1(E_1)$ to the fiber subgroup of $\pi_1(E_2)$ injectively, then there is a fiber-preserving map in the homotopy class of f .*

Remark 1. The proofs of Theorem A and that of Theorem 1.1 are different. The proof of Theorem A uses the hierarchy theory of 2- and 3-manifolds, and such a hierarchy theory is not available in higher dimensions. The proof of Theorem 1.1 uses the theory of S^1 -bundles, in particular that oriented S^1 -bundles over M are principal bundles and whose isomorphism classes are in 1-1 correspondence with $H^2(M, \mathbb{Z})$, and the gauge class group of an S^1 -bundle over M is isomorphic to $[\pi_1(M), \mathbb{Z}] = H^1(M, \mathbb{Z})$.

Our next goal is to describe the mapping degree sets of fiber-preserving maps between S^1 -bundles. Given two closed oriented circle bundles E_1 and E_2 of the same dimension, we define the *fiber-preserving mapping degree set* as

$$D_{FP}(E_1, E_2) := \{d \in \mathbb{Z} \mid \exists \text{ fiber-preserving map } f: E_1 \rightarrow E_2, \text{ such that } \deg(f) = d\}.$$

Theorem 1.2. *Let M_i be a closed oriented n -manifold and let $E_i \rightarrow M_i$ be an oriented S^1 -bundle with Euler class $e_i \in H^2(M_i; \mathbb{Z})$, $i = 1, 2$. Then*

$$D_{FP}(E_1, E_2) = \{0\} \cup \{k \cdot \deg(f) \mid k \neq 0, f: M_1 \rightarrow M_2, \deg(f) \neq 0, \text{ and } f^\#(e_2) = ke_1\},$$

where $f^\#: H^2(M_2; \mathbb{Z}) \rightarrow H^2(M_1; \mathbb{Z})$ is the induced homomorphism on second cohomology.

1.2. Applications.

1.2.1. *Finiteness of mapping degree sets.* A primary question about a mapping degree set $D(M, N)$ is whether it is finite. We say a numerical invariant v of n -manifolds is a domination invariant if for any map $f: M \rightarrow N$, we have $v(M) \geq |\deg(f)| \cdot v(N)$.

Clearly $D(M, N)$ is finite for any M if $v(N)$ is positive and finite for some domination invariant v . The simplicial volume is the most important domination invariant. It is positive for hyperbolic manifolds, but it vanishes for circle bundles. On the other hand, for an S^1 -bundle E over a hyperbolic surface with non-zero Euler class, $w(E) > 0$ for the Seifert volume w , which is a domination invariant introduced by Brooks and Goldman [BG], thus $D(M, E)$ is finite for any 3-manifold M .

So far there is no known non-zero and finite domination invariant for S^1 -bundles over n -manifolds, with $n \geq 3$ [Ne2]. The first significant consequence of Theorems 1.1 and 1.2 is on the finiteness of mapping degree sets between S^1 -bundles. For brevity, we will often use \tilde{M}_a to denote the oriented circle bundle over a closed oriented n -manifold M with Euler class $e(\tilde{M}_a) = a \in H^2(M; \mathbb{Z})$. Our finiteness result is the following one.

Theorem 1.3 (Finiteness Theorem). *Suppose N is a closed oriented hyperbolic n -manifold and $b \in H^2(N; \mathbb{Z})$ is not torsion. Then $D(\tilde{M}_a, \tilde{N}_b)$ is a finite set for any closed oriented aspherical n -manifold M and any $a \in H^2(M; \mathbb{Z})$.*

Remark 2. Theorem 1.3 implies that the self-mapping degree set $D(\tilde{N}_b) \subset \{0, \pm 1\}$. This fact about $D(\tilde{N}_b)$ was proved in [Ne2, Theorem 1.3]. A direct argument shows that a non-zero degree map $f: \tilde{M}_a \rightarrow \tilde{N}_b$ sends the fiber subgroup to the fiber subgroup (see the proof of Corollary 4.1, or [JS], [Sc], [Ro], [Ne2]). Starting from this point, the proof of Theorem

1.3 and that of [Ne2, Theorem 1.3] are different. The proof in [Ne2] uses a group theoretic idea developed in [Ne1] and some dynamical argument, since self-maps can be iterated. The maps in Theorem 1.3 cannot be iterated, and the proof of Theorem 1.3 relies on Theorems 1.1 and 1.2.

1.2.2. *The realization problem.* The problem of realizing an arbitrary set of integers (containing zero) as a mapping degree set appeared in the literature rather recently, although it has been present over the last decades, at least implicitly. It was shown in [NWW, Theorem 1.3] that there exist uncountably many infinite sets $A \subseteq \mathbb{Z}$ with $0 \in A$ that are not equal to $D(M, N)$, for any closed oriented n -manifolds M, N . So the following refined version of the realization problem arose.

Problem 1. [NWW, Problem 1.4] Suppose A is a finite set of integers containing 0. Are there closed oriented n -manifolds M and N such that $A = D(N, N)$?

Our interest in S^1 -bundles also stems from the fact that S^1 -bundles over surfaces are instrumental for answering Problem 1. At first, given a non-zero integer k , the set $\{0, k\}$, the smallest non-zero subset of \mathbb{Z} containing 0, is realized by $D(\tilde{\Sigma}_1, \tilde{\Sigma}_k)$, see [NWW, Lemma 3.5]. Here $\tilde{\Sigma}_k$ is the S^1 -bundle over a closed oriented hyperbolic surface Σ with Euler number k . Then by using connected sums of non-trivial S^1 -bundles over hyperbolic surfaces, and their products, many finite subsets of \mathbb{Z} are realized as mapping degree sets [NWW, Theorems 1.7 and 1.9]. Soon after these results, C. Costoya, V. Muñoz, and A. Viruel, gave a complete positive answer to Problem 1 in a stronger form, by only using connected sums of S^1 -bundles over surfaces, and some arithmetic combinatorics [CMV, Prop. 2.2].

Theorem B. [CMV, Theorem A]. *If A is a finite set of integers containing 0, then $A = D(M, N)$ for some closed oriented connected 3-manifolds M, N .*

Combining Theorem B and the fact that every mapping degree set of n -manifolds is the mapping degree set of some $(n + k)$ -manifolds for any $k \geq 3$ [NSTWW, Theorem 2.2], we proved the following result.

Theorem C. [NSTWW, Theorem 2.3] *For each positive integer $n \neq 1, 2, 4, 5$, every finite set of integers containing 0 is the mapping degree set of a pair of n -manifolds.*

Theorem C obviously fails for $n = 1, 2$. One motivation of the present work is to show that Theorem C also holds for $n = 4$ and 5, therefore giving a complete solution to Problem 1 in all dimensions. Indeed, as we shall explain below, our study of maps between S^1 -bundles yields this consequence. We first prove the following result.

Theorem 1.4. *For each non-zero integer k and $n = 4, 5$, there exist closed oriented aspherical n -manifolds of the form $\tilde{N}_b, \tilde{N}_{kb}$ such that*

$$D(\tilde{N}_b, \tilde{N}_{kb}) = \{0, k\}.$$

By applying the arithmetic combinatorial construction of [CMV, Prop 2.2], Theorems 1.4 and C, imply the following result.

Theorem 1.5 (Realization Theorem). *For every finite set A of integers containing 0 and each integer $n > 2$, there exist closed, oriented n -manifolds M and N such that $D(M, N) = A$.*

2. FIBER-PRESERVING: FROM GROUP LEVEL TO TOPOLOGY LEVEL

In this section, we will prove Theorem 1.1. We begin by recalling some facts on S^1 -bundles, see [MS], [Ha2], [Mo].

Let M_i be a closed oriented n -manifold, and let $S^1 \xrightarrow{j_i} E_i \xrightarrow{p_i} M_i$ be an oriented S^1 -bundle, $i = 1, 2$. We say that a map $\tilde{f}: E_1 \rightarrow E_2$ is *fiber-preserving* if it maps each S^1 -fiber of E_1 to an S^1 -fiber of E_2 , and in this case, it induces a map $f: M_1 \rightarrow M_2$. A fiber-preserving map $\tilde{f}: E_1 \rightarrow E_2$ is a *bundle map*, if the restriction of \tilde{f} on each S^1 fiber is a homeomorphism. Suppose now $M_1 = M_2 = M$. We say that a fiber-preserving map $\tilde{f}: E_1 \rightarrow E_2$ is *vertical* if it induces the identity map on M . Finally, two S^1 -bundles $E_1 \xrightarrow{p_1} M$ and $E_2 \xrightarrow{p_2} M$ are isomorphic if there is a bundle map $\tilde{f}: E_1 \rightarrow E_2$ such that $p_1 = p_2 \circ \tilde{f}$.

For each map $f: M \rightarrow N$ and an S^1 -bundle \tilde{N} over N , we have the pull-back bundle $f^*(\tilde{N})$ over M and a bundle map $\tilde{f}: f^*(\tilde{N}) \rightarrow \tilde{N}$ such that the following diagram is commutative.

$$(1) \quad \begin{array}{ccc} f^*(\tilde{N}) & \xrightarrow{\tilde{f}} & \tilde{N} \\ \downarrow p_1 & & \downarrow p_2 \\ M & \xrightarrow{f} & N \end{array}$$

Some basic facts about S^1 -bundles are stated in the following.

Theorem 2.1.

- (i) *Any oriented S^1 -bundle E over M is isomorphic to a pull-back bundle of the universal S^1 -bundle $S^\infty \rightarrow \mathbb{C}P^\infty$, i.e., there is a map $f: M \rightarrow \mathbb{C}P^\infty$ such that the following diagram commutes.*

$$(2) \quad \begin{array}{ccc} E \cong f^*(S^\infty) & \xrightarrow{\tilde{f}} & S^\infty \\ \downarrow p_1 & & \downarrow p \\ M & \xrightarrow{f} & \mathbb{C}P^\infty \end{array}$$

Moreover, f is unique up to homotopy. Therefore, there is a one-to-one correspondence between isomorphism classes of oriented S^1 -bundles over M and elements of $[M, \mathbb{C}P^\infty]$, the set of homotopy classes of maps $M \rightarrow \mathbb{C}P^\infty$.

- (ii) *Since $\mathbb{C}P^\infty$ is a model of the Eilenberg-MacLane space $K(\mathbb{Z}, 2)$, we have*

$$[M, \mathbb{C}P^\infty] \cong [M, K(\mathbb{Z}, 2)] \cong H^2(M; \mathbb{Z}).$$

Therefore, there is a one-to-one correspondence between isomorphism classes of oriented circle bundles E over M and elements in $H^2(M; \mathbb{Z})$, which sends E to its Euler class $e(E) \in H^2(M; \mathbb{Z})$.

- (iii) Since the embedding of $SO(2) \subset \text{Diff}_0(S^1)$, respectively $SO(2) \subset \text{Homeo}_0(S^1)$, is a homotopy equivalence, there is a one-to-one correspondence between isomorphism classes of oriented S^1 -bundles over M and isomorphism classes of principal S^1 -bundles over M in the smooth category, respectively in the topological category. (As usual, by $\text{Diff}_0(M)$, respectively $\text{Homeo}_0(M)$, we denote the component of the diffeomorphism group, respectively the homeomorphism group, of M containing the identity.)

In the rest of the paper, we omit the term ‘‘isomorphism classes’’.

Now we rewrite Theorem 1.1 in the following way.

Theorem 2.2 (Theorem 1.1). *Let $S^1 \xrightarrow{j_i} E_i \xrightarrow{p_i} M_i$ be an oriented S^1 -bundle over a closed oriented aspherical n -manifold M_i , $i = 1, 2$ and let $\tilde{f}: E_1 \rightarrow E_2$ be a map. If the induced map \tilde{f}_* on π_1 sends the fiber subgroup to the fiber subgroup injectively, then there is a fiber-preserving map in the homotopy class of \tilde{f} .*

Before proving Theorem 2.2, we first show some preliminary results.

Lemma 2.3. *Let $1 \rightarrow Z \rightarrow G \xrightarrow{p} \Gamma \rightarrow 1$ and $1 \rightarrow Z' \rightarrow G' \xrightarrow{p'} \Gamma' \rightarrow 1$ be two short exact sequences of groups given by central extensions. Let $i: Z \rightarrow Z'$ be an isomorphism and let $\psi: \Gamma \rightarrow \Gamma'$, $\phi_j: G \rightarrow G'$ be homomorphisms such that the following diagram commutes for $j = 1, 2$.*

$$(3) \quad \begin{array}{ccccccccc} 1 & \longrightarrow & Z & \longrightarrow & G & \xrightarrow{p} & \Gamma & \longrightarrow & 1 \\ & & \downarrow & & \downarrow \phi_j & & \downarrow \psi & & \downarrow \\ 1 & \longrightarrow & Z' & \longrightarrow & G' & \xrightarrow{p'} & \Gamma' & \longrightarrow & 1. \end{array}$$

Then there is a self-homomorphism $\kappa: G \rightarrow G$ such that $\phi_1 \circ \kappa = \phi_2$. Moreover, there exists a homomorphism $\lambda: \Gamma \rightarrow Z$, such that $\kappa(g) = \lambda(p(g)) \cdot g$ for any $g \in G$.

Proof. Here we naturally consider Z as a subgroup of G , and Z' as a subgroup of G' . So we have $\phi_j|_Z = i$ for $j = 1, 2$. We define $\kappa: G \rightarrow G$ by

$$\kappa(g) = i^{-1}(\phi_2(g) \cdot \phi_1(g)^{-1}) \cdot g.$$

Note that $\phi_2(g) \cdot \phi_1(g)^{-1} \in Z'$ holds since

$$p'(\phi_2(g) \cdot \phi_1(g)^{-1}) = p'(\phi_2(g)) \cdot p'(\phi_1(g))^{-1} = \psi(p(g)) \cdot \psi(p(g))^{-1} = e.$$

We check that κ is indeed a group homomorphism:

$$\begin{aligned}
 \kappa(g_1) \cdot \kappa(g_2) &= i^{-1}(\phi_2(g_1) \cdot \phi_1(g_1)^{-1}) \cdot g_1 \cdot i^{-1}(\phi_2(g_2) \cdot \phi_1(g_2)^{-1}) \cdot g_2 \\
 &= i^{-1}(\phi_2(g_1) \cdot \phi_1(g_1)^{-1} \cdot \phi_2(g_2) \cdot \phi_1(g_2)^{-1}) \cdot g_1 \cdot g_2 \\
 &= i^{-1}(\phi_2(g_1) \cdot \phi_2(g_2) \cdot \phi_1(g_2)^{-1} \cdot \phi_1(g_1)^{-1}) \cdot (g_1 \cdot g_2) \\
 &= i^{-1}(\phi_2(g_1 \cdot g_2) \cdot \phi_1(g_1 \cdot g_2)^{-1}) \cdot (g_1 \cdot g_2) = \kappa(g_1 \cdot g_2).
 \end{aligned}$$

Here the second and third equalities hold since Z and Z' are in the center of G and G' , respectively.

We check that $\phi_1 \circ \kappa = \phi_2$:

$$\begin{aligned}
 \phi_1 \circ \kappa(g) &= \phi_1(i^{-1}(\phi_2(g) \cdot \phi_1(g)^{-1}) \cdot g) = \phi_1(i^{-1}(\phi_2(g) \cdot \phi_1(g)^{-1})) \cdot \phi_1(g) \\
 &= (\phi_2(g) \cdot \phi_1(g)^{-1}) \cdot \phi_1(g) = \phi_2(g).
 \end{aligned}$$

For the moreover part, we first define $\tilde{\lambda} : G \rightarrow Z$ by

$$\tilde{\lambda}(g) = i^{-1}(\phi_2(g) \cdot \phi_1(g)^{-1}).$$

We can check that $\tilde{\lambda}$ is a homomorphism, by the computation that proves κ is a homomorphism. For any $z \in Z$, we have $\tilde{\lambda}(z) = i^{-1}(\phi_2(z) \cdot \phi_1(z)^{-1}) = i^{-1}(i(z) \cdot i(z^{-1})) = e$. So $Z < \ker \tilde{\lambda}$, and $\tilde{\lambda}$ induces a homomorphism $\lambda : \Gamma \rightarrow Z$ such that $\tilde{\lambda} = \lambda \circ p$. By definitions of κ and $\tilde{\lambda}$, we have $\kappa(g) = \tilde{\lambda}(g) \cdot g = \lambda(p(g)) \cdot g$. \square

In fact, $\kappa : G \rightarrow G$ is a group isomorphism, but we will not need this fact in our proof of Theorem 2.2.

Let us put the above lemma in the context of circle bundles. Suppose $S^1 \xrightarrow{j} E \xrightarrow{p} M$ is an oriented S^1 -bundle over a closed oriented manifold M so that $\pi_1(S^1)$ lies in the center of $\pi_1(E)$ (e.g. when M is aspherical). Given a homomorphism $\phi : \pi_1(E) \rightarrow \pi_1(E)$ that sends the fiber subgroup $\pi_1(S^1) < \pi_1(E)$ to itself, the restriction of ϕ gives a homomorphism $\phi| : \pi_1(S^1) \rightarrow \pi_1(S^1)$, and ϕ induces a homomorphism $\bar{\phi} : \pi_1(M) \rightarrow \pi_1(M)$.

Proposition 2.4. *Suppose $\phi : \pi_1(E) \rightarrow \pi_1(E)$ is an isomorphism that sends the fiber subgroup to itself, such that $\phi| = id_{\pi_1(S^1)}$ and $\bar{\phi} = id_{\pi_1(M)}$. Then there exists a bundle isomorphism $f_\phi : E \rightarrow E$ that induces ϕ on π_1 .*

Proof. We apply Lemma 2.3 to the following commutative diagram, with $\phi_1 = id$ and $\phi_2 = \phi$.

$$\begin{array}{ccccccc}
 1 & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(E) & \xrightarrow{p_*} & \pi_1(M) & \longrightarrow & 1 \\
 & & \downarrow id & & \downarrow \phi_j & & \downarrow id & & \\
 1 & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(E) & \xrightarrow{p_*} & \pi_1(M) & \longrightarrow & 1.
 \end{array}$$

Then there is a homomorphism $\lambda : \pi_1(M) \rightarrow \pi_1(S^1)$, such that $\phi(g) = \lambda(p_*(g)) \cdot g$.

By Theorem 2.1 (iii), we can assume that E is a principal S^1 -bundle, so we have an S^1 -action on E . We do not distinguish between left and right actions since S^1 is abelian.

Since S^1 is a $K(\pi, 1)$ -space, there exists $f: M \rightarrow S^1$ realizing $\lambda: \pi_1(M) \rightarrow \pi_1(S^1)$. Without loss of generality, we assume the basepoint $m_0 \in M$ is mapped to $1 \in S^1$, and we identify S^1 with the fiber $p^{-1}(m_0) \subset E$ over m_0 . Define $f_\phi: E \rightarrow E$ by

$$f_\phi(x) = f(p(x)) \bullet x,$$

where \bullet is the S^1 -action. Then $f_\phi: E \rightarrow E$ is clearly a bundle isomorphism.

It suffices to show that f_ϕ induces ϕ on π_1 . For $f: [0, 1] \rightarrow S^1$ and $g: [0, 1] \rightarrow E$, we define $f \bullet g: [0, 1] \rightarrow E$ by $(f \bullet g)(t) = f(t) \bullet g(t)$. Take the basepoint $e_0 \in p^{-1}(m_0) \subset E$ that corresponds to $1 \in S^1$ under the identification between S^1 and $p^{-1}(m_0)$. For any based loop $\gamma: [0, 1] \rightarrow E$ such that $\gamma(0) = \gamma(1) = e_0$, we need to check that $(f_\phi)_*([\gamma]) = \phi([\gamma]) \in \pi_1(E, e_0)$. Here, $(f_\phi)_*([\gamma])$ is represented by $f_\phi \circ \gamma: [0, 1] \rightarrow E$, with

$$f_\phi \circ \gamma(t) = f(p(\gamma(t))) \bullet \gamma(t) = ((f \circ p \circ \gamma) \bullet \gamma)(t).$$

Also, we have

$$\phi([\gamma]) = \lambda(p_*([\gamma])) \cdot [\gamma] = f_*(p_*([\gamma])) \cdot [\gamma] = [f \circ p \circ \gamma] \cdot [\gamma] = [(f \circ p \circ \gamma) \cdot \gamma].$$

Here the \cdot in $(f \circ p \circ \gamma) \cdot \gamma$ denotes the concatenation of paths. So $\phi([\gamma])$ is represented by

$$(f \circ p \circ \gamma) \cdot \gamma = ((f \circ p \circ \gamma) \cdot c_1) \bullet (c_{e_0} \cdot \gamma) \simeq (f \circ p \circ \gamma) \bullet \gamma.$$

Here c_1 and c_{e_0} denote the constant maps to $1 \in S^1$ and $e_0 \in E$ respectively, and \simeq denotes homotopy relative to $\{0, 1\}$.

So we verified that $(f_\phi)_*([\gamma]) = \phi([\gamma])$ for any $[\gamma] \in \pi_1(E, e_0)$, thus $(f_\phi)_* = \phi$ holds. \square

Lemma 2.5. *Suppose $S^1 \xrightarrow{j_i} E_i \xrightarrow{p_i} M$ are oriented S^1 -bundles, $i = 1, 2$, where M is aspherical. Suppose $\phi: \pi_1(E_1) \rightarrow \pi_1(E_2)$ is an isomorphism such that the following diagram commutes.*

$$\begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(E_1) & \xrightarrow{(p_1)_*} & \pi_1(M) & \longrightarrow & 1 \\ & & \downarrow id & & \downarrow \phi & & \downarrow id & & \\ 1 & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(E_2) & \xrightarrow{(p_2)_*} & \pi_1(M) & \longrightarrow & 1. \end{array}$$

Then E_1 and E_2 are isomorphic as oriented S^1 -bundles.

Proof. This is a known fact. First the Euler classes of these two \mathbb{Z} -central extensions are equal to each other in $H^2(\pi_1(M); \mathbb{Z})$, see for example [FS, pages 234-235]. Since M is aspherical, we have a natural isomorphism $H^2(\pi_1(M); \mathbb{Z}) \rightarrow H^2(M; \mathbb{Z})$, and the image of the Euler class of each \mathbb{Z} -central extension is the Euler class of the S^1 -bundle. Then Theorem 2.1 (ii) implies that E_1 is isomorphic to E_2 . \square

Now we are ready to prove Theorem 2.2. By some bundle reductions, we reach the place to apply Lemma 2.5, Theorem 2.1, and Proposition 2.4, and to prove that there is a fiber-preserving map in the homotopy class of \tilde{f} .

Proof of Theorem 2.2. Below we use ϕ to denote the homomorphism $\tilde{f}_*: \pi_1(E_1) \rightarrow \pi_1(E_2)$. So we have the following commutative diagram:

$$(4) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(E_1) & \xrightarrow{(p_1)_*} & \pi_1(M_1) \longrightarrow 1 \\ & & \downarrow \phi| & & \downarrow \phi & & \downarrow \bar{\phi} \\ 1 & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(E_2) & \xrightarrow{(p_2)_*} & \pi_1(M_2) \longrightarrow 1. \end{array}$$

Step I. *Reduction to the case $M_1 = M_2$.* Since M_2 is aspherical, there exists a map $g: M_1 \rightarrow M_2$, such that $g_*: \pi_1(M_1) \rightarrow \pi_1(M_2)$ equals $\bar{\phi}$. Let $p'_1: E'_1 \rightarrow M_1$ be the pull-back bundle of $p_2: E_2 \rightarrow M_2$ via $g: M_1 \rightarrow M_2$. Then we have a fiber-preserving bundle map $\tilde{f}': E'_1 \rightarrow E_2$, and the following diagram commutes.

$$(5) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(E'_1) & \xrightarrow{(p'_1)_*} & \pi_1(M_1) \longrightarrow 1 \\ & & \downarrow id & & \downarrow \tilde{f}'_* & & \downarrow g_* = \bar{\phi} \\ 1 & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(E_2) & \xrightarrow{(p_2)_*} & \pi_1(M_2) \longrightarrow 1, \end{array}$$

Since the left vertical arrow in diagram (5) is the identity, the right square of diagram (5) gives the fiber product of $\bar{\phi}$ and $(p_2)_*$. Comparing the right squares of diagrams (4) and (5) and by the universal property of the fiber product, there exists a homomorphism $\psi: \pi_1(E_1) \rightarrow \pi_1(E'_1)$, such that

$$\tilde{f}_* = \phi = \tilde{f}'_* \circ \psi: \pi_1(E_1) \rightarrow \pi_1(E_2), \quad \text{and} \quad (p_1)_* = (p'_1)_* \circ \psi: \pi_1(E_1) \rightarrow \pi_1(M_1).$$

Since M_1 is aspherical, E'_1 is also aspherical by the homotopy exact sequence of fiber bundles. So there is a map $\tilde{h}: E_1 \rightarrow E'_1$ such that $\tilde{h}_* = \psi: \pi_1(E_1) \rightarrow \pi_1(E'_1)$, and we have

$$\tilde{f}_* = \tilde{f}'_* \circ \psi = \tilde{f}'_* \circ \tilde{h}_* = (\tilde{f}' \circ \tilde{h})_*.$$

Since E_2 is aspherical, we have that \tilde{f} and $\tilde{f}' \circ \tilde{h}$ are homotopic to each other. Since \tilde{f}' is fiber-preserving, it suffices to prove that $\tilde{h}: E_1 \rightarrow E'_1$ is homotopic to a fiber-preserving map.

Note that $\tilde{h}: E_1 \rightarrow E'_1$ has non-zero degree and we have the following commutative diagram:

$$(6) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(E_1) & \xrightarrow{(p_1)_*} & \pi_1(M_1) \longrightarrow 1 \\ & & \downarrow \psi| & & \downarrow \tilde{h}_* = \psi & & \downarrow id \\ 1 & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(E'_1) & \xrightarrow{(p'_1)_*} & \pi_1(M_1) \longrightarrow 1. \end{array}$$

Here $\psi|: \pi_1(S^1) \rightarrow \pi_1(S^1)$ is a homomorphism that sends to 1 to $k \neq 0 \in \mathbb{Z} \cong \pi_1(S^1)$. Up to changing orientation, we can assume that $k > 0$.

Step II. *Reduction to the case $k = 1$.* By the commutative diagram (6), we can check that $\tilde{h}_* = \psi: \pi_1(E_1) \rightarrow \pi_1(E'_1)$ is injective, $\pi_1(S^1) \cap \psi(\pi_1(E_1)) = k\mathbb{Z} < \pi_1(S^1) \cong \mathbb{Z}$, and $[\pi_1(E'_1): \psi(\pi_1(E_1))] = k$. Let $q: E''_1 \rightarrow E'_1$ be the k -sheet covering of E'_1 corresponding to

$\psi(\pi_1(E_1)) < \pi_1(E'_1)$. Then E''_1 has an induced S^1 -bundle structure over M_1 , and q is a fiber-preserving map.

Let $\tilde{k}: E_1 \rightarrow E'_1$ be the lifting map such that $\tilde{h} = q \circ \tilde{k}: E_1 \rightarrow E'_1$. It suffices to prove that \tilde{k} is homotopic to a fiber-preserving map. On the group level, by diagram (6), we have the following commutative diagram where $\tilde{k}_*: \pi_1(E_1) \rightarrow \pi_1(E'_1)$ is an isomorphism.

$$(7) \quad \begin{array}{ccccccc} 1 & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(E_1) & \xrightarrow{(p_1)_*} & \pi_1(M_1) \longrightarrow 1 \\ & & \downarrow id & & \downarrow \tilde{k}_* & & \downarrow id \\ 1 & \longrightarrow & \pi_1(S^1) & \longrightarrow & \pi_1(E''_1) & \xrightarrow{(p''_1)_*} & \pi_1(M_1) \longrightarrow 1, \end{array}$$

Step III. Finishing the proof. Since M_1 is aspherical, we have a natural isomorphism $H^2(\pi_1(M_1); \mathbb{Z}) \cong H^2(M; \mathbb{Z})$. Let $e_1, e''_1 \in H^2(M; \mathbb{Z}) \cong H^2(\pi_1(M_1); \mathbb{Z})$ be the Euler classes of the oriented S^1 -bundles $p_1: E_1 \rightarrow M_1$ and $p''_1: E''_1 \rightarrow M_1$, respectively. The commutative diagram (7) implies $e_1 = e''_1$ by Lemma 2.5. Then Theorem 2.1 (ii) implies that $p_1: E_1 \rightarrow M_1$ and $p''_1: E''_1 \rightarrow M_1$ are isomorphic oriented S^1 -bundles over M_1 .

Finally Proposition 2.4 implies the existence of a (fiber-preserving) bundle isomorphism $\tilde{k}': E_1 \rightarrow E''_1$, such that $\tilde{k}'_* = \tilde{k}_*: \pi_1(E_1) \rightarrow \pi_1(E''_1)$. Since E''_1 is aspherical, \tilde{k} is homotopic to the fiber-preserving map \tilde{k}' . This finishes the proof. \square

3. MAPPING DEGREE SETS OF FIBER-PRESERVING MAPS BETWEEN S^1 -BUNDLES

In this section, we will prove Theorem 1.2.

First we state a fact about vertical maps that will be used below.

Lemma 3.1. *Let M_i be a closed oriented n -manifold, and let $S^1 \xrightarrow{j_i} E_i \xrightarrow{p_i} M_i$ be an oriented S^1 -bundle, $i = 1, 2$. Suppose $f: E_1 \rightarrow E_2$ is a fiber-preserving non-zero degree map that induces $\bar{f}: M_1 \rightarrow M_2$. Then we have a factorisation*

$$f: E_1 \xrightarrow{v} E'_1 \xrightarrow{f'} E_2,$$

where v is a vertical map, and f' is a bundle map that induces $\bar{f}: M_1 \rightarrow M_2$.

Proof. This can be proved by a standard pull-back argument. Let E'_1 be the pull-back bundle of $E_2 \xrightarrow{p_2} M_2$ via $\bar{f}: M_1 \rightarrow M_2$, where

$$E'_1 = \{(x_1, e_2) \mid x_1 \in M_1, e_2 \in E_2, \bar{f}(x_1) = p_2(e_2)\}.$$

Then we have an S^1 -bundle $q: E'_1 \rightarrow M_1$ over M_1 defined by $q(x_1, e_2) = x_1$, and a bundle map $f': E'_1 \rightarrow E_2$ defined by $f'(x_1, e_2) = e_2$ that induces $\bar{f}: M_1 \rightarrow M_2$.

We define $v: E_1 \rightarrow E'_1$ by $v(e_1) = (p_1(e_1), f(e_1))$. This is a fiber-preserving map that induces the identity map on M_1 and satisfies $f = f' \circ v$. \square

From now on, we will often use \tilde{M}_a to denote the oriented circle bundle over a closed oriented n -manifold M with Euler class $e(\tilde{M}_a) = a \in H^2(M; \mathbb{Z})$.

Proposition 3.2. *Suppose M is a closed oriented n -manifold, $a, b \in H^2(M; \mathbb{Z})$ and $k \neq 0$ is an integer. Then the following are equivalent.*

- (1) $ka = b$.
- (2) *There exists a vertical map $\tilde{M}_a \rightarrow \tilde{M}_b$ of degree k .*

Proof. We first recall the obstruction definition of the Euler class of an oriented S^1 -bundle $p_a: \tilde{M}_a \rightarrow M$.

First we assume that M admits a CW-complex structure X , denote its 1-skeleton by $X^{(1)}$, and take a section $s: X^{(1)} \rightarrow \tilde{M}_a$ such that $p_a \circ s$ equals the inclusion $X^{(1)} \rightarrow M$. The Euler class $a \in H^2(M; \mathbb{Z})$ is represented by a cellular 2-cocycle $\alpha \in C^2(X)$ defined as follows. For any 2-cell Δ^2 of X with an orientation, $\alpha(\Delta^2)$ equals the number of $s(\partial\Delta^2)$ wrapping around the S^1 -fiber. This makes sense since the restriction of the S^1 -bundle on Δ^2 is trivial.

Now we prove the proposition.

(2) \Rightarrow (1): If we have a vertical map $v: \tilde{M}_a \rightarrow \tilde{M}_b$ of degree k , then the restriction of v on each S^1 -fiber of \tilde{M}_a is a degree- k map to the image S^1 -fiber of \tilde{M}_b . We take a section $s: X^{(1)} \rightarrow \tilde{M}_a$ of $\tilde{M}_a \rightarrow M$ as in the obstruction definition of Euler class. Since $v: \tilde{M}_a \rightarrow \tilde{M}_b$ is vertical, $v \circ s: X^{(1)} \rightarrow \tilde{M}_b$ is a section of $\tilde{M}_b \rightarrow M$. For any 2-cell Δ^2 of X with an orientation, the number of $v \circ s(\partial\Delta^2)$ wrapping around the S^1 -fiber of \tilde{M}_b equals k times the number of $s(\partial\Delta^2)$ wrapping around the S^1 -fiber of \tilde{M}_a . So we have $\beta = k\alpha \in C^2(X)$, and $b = ka \in H^2(M; \mathbb{Z})$ holds.

(1) \Rightarrow (2): Suppose that $b = ka \in H^2(M; \mathbb{Z})$. By Theorem 2.1 (iii), we can consider all S^1 -bundles as principal S^1 -bundles. Then we have a group embedding

$$\mathbb{Z}_k \subset S^1 \subset \text{Homeo}_0(\tilde{M}_a),$$

and the \mathbb{Z}_k -action induces an S^1 -bundle

$$S^1/\mathbb{Z}_k \rightarrow \tilde{M}_a/\mathbb{Z}_k \cong \tilde{M}_c \rightarrow M$$

for some $c \in H^2(M; \mathbb{Z})$, and we have a vertical map $v_k: \tilde{M}_a \rightarrow \tilde{M}_c$ of degree k . By the previous paragraph, we have $c = ka$, thus $c = b$. So v_k is a vertical map from \tilde{M}_a to \tilde{M}_b of degree k , as desired.

For a general M , we can take a homotopy equivalence $h: X \rightarrow M$ from a CW-complex X to M (see for example Corollary A.12 of [Ha2]), and apply a similar argument as above. \square

We rewrite Theorem 1.2 as the statement below.

Theorem 3.3 (Theorem 1.2). *Suppose M and N are closed oriented n -manifolds, $a \in H^2(M; \mathbb{Z})$ and $b \in H^2(N; \mathbb{Z})$. Then the mapping degree set of fiber-preserving maps from \tilde{M}_a to \tilde{N}_b is given by*

$$D_{FP}(\tilde{M}_a, \tilde{N}_b) = \{0\} \cup \{k \cdot \deg(f) \mid k \neq 0, f: M \rightarrow N, \deg(f) \neq 0 \text{ such that } f^\#(b) = ka\},$$

where $f^\#: H^2(N; \mathbb{Z}) \rightarrow H^2(M; \mathbb{Z})$ is the induced homomorphism.

Proof. We first prove that the left-hand set is a subset of the right-hand set. Suppose $\tilde{f}: \tilde{M}_a \rightarrow \tilde{N}_b$ is a fiber-preserving map which induces $f: M \rightarrow N$. Then by Lemma 3.1, we have $\tilde{f} = \tilde{f}_2 \circ \tilde{f}_1$, where $\tilde{f}_1: \tilde{M}_a \rightarrow \tilde{M}_c$ is a vertical map, and $\tilde{f}_2: \tilde{M}_c \rightarrow \tilde{N}_b$ is a bundle map that induces $f: M \rightarrow N$. For the bundle map, we have $e(\tilde{M}_c) = f_2^\#(e(\tilde{N}_b))$, that is $c = f^\#(b)$. For the vertical map, we have $ka = c$ by Proposition 3.2, where k is the degree of \tilde{f}_1 . So we have

$$ka = f^\#(b).$$

By the product formula of mapping degrees of fiber bundles, we have

$$\deg(\tilde{f}) = k \cdot \deg(f)$$

and the proof of the first part is done.

Next, we prove the converse inclusion. Suppose $f: M \rightarrow N$ is a non-zero degree map and $f^\#(b) = ka$. Then we take the pull-back bundle of $\tilde{N}_b \rightarrow N$ as in diagram (1). Since the map \tilde{f} in (1) is a bundle map, we have

$$e(\tilde{f}^*(\tilde{N}_b)) = f^\#(e(\tilde{N}_b)) = f^\#(b),$$

where $f^\#: H^2(N; \mathbb{Z}) \rightarrow H^2(M; \mathbb{Z})$ is the induced homomorphism. It implies that

$$f^*(\tilde{N}_b) = \tilde{M}_{f^\#(b)} = \tilde{M}_{ka}.$$

So we have the right square of the following diagram (8).

$$(8) \quad \begin{array}{ccccc} \tilde{M}_a & \xrightarrow{\tilde{v}} & \tilde{M}_{ka} & \xrightarrow{\tilde{f}} & \tilde{N}_b \\ \downarrow p_0 & & \downarrow p_1 & & \downarrow p_2 \\ M & \xrightarrow{\text{id}} & M & \xrightarrow{f} & N \end{array}$$

By Proposition 3.2, we have a vertical map $\tilde{v}: \tilde{M}_a \rightarrow \tilde{M}_{ka}$ of degree k , so we have the left square of (8). Note also that $\deg(\tilde{f}) = \deg f$. Then

$$\deg(\tilde{f} \circ \tilde{v}) = \deg(\tilde{v}) \cdot \deg(\tilde{f}) = k \cdot \deg(f),$$

and the proof of the second inclusion is completed. \square

We end our discussion in this section with the next two observations, which will make the picture of fiber-preserving maps between S^1 -bundles more complete.

Lemma 3.4. *Suppose M and N are closed oriented n -manifolds and $a \in H^2(M; \mathbb{Z})$, $b \in H^2(N; \mathbb{Z})$. If there is a fiber-preserving map $\tilde{f}: \tilde{M}_a \rightarrow \tilde{N}_b$ of non-zero degree, then a is torsion if and only b is torsion.*

Proof. Let k be the degree of f on the S^1 -fiber, and let f be the induced map on the base manifolds $M \rightarrow N$. Then $ka = f^\#(b)$ by the proof of Theorem 1.2. Since \tilde{f} is a non-zero degree map, f is also a non-zero degree map and $k \neq 0$.

If b is a torsion class, then $f^\#(b)$ is torsion. So a is torsion since $k \neq 0$, and we proved one direction of the lemma. Conversely, if a is torsion, then $f^\#(b)$ is torsion. Since f is a non-zero degree map, the induced homomorphism on cohomology modulo torsion

$$f^\#: H^*(N; \mathbb{Z})/tors \rightarrow H^*(M; \mathbb{Z})/tors$$

is injective. So b is torsion, and we proved the other direction of the lemma. \square

We define the *mapping degree set of vertical maps* as

$$D_V(\tilde{M}_a, \tilde{M}_b) := \{d \in \mathbb{Z} \mid \text{there exists a vertical map } \tilde{f}: \tilde{M}_a \rightarrow \tilde{M}_b, \deg(\tilde{f}) = d\}.$$

Clearly $D_V(\tilde{M}_a, \tilde{M}_b) \subset D_{FP}(\tilde{M}_a, \tilde{M}_b)$.

Corollary 3.5. *Suppose $a, b \in H^2(M; \mathbb{Z})$. Then $D_V(\tilde{M}_a, \tilde{M}_b)$ is*

- (1) *the empty set, if $b \notin \langle a \rangle$;*
- (2) *an infinite set, if $b = ka$, $k \neq 0$ and a is torsion;*
- (3) *$\{k\}$, if $b = ka$, $k \neq 0$ and a is not torsion.*

Proof.

- (1) This clearly follows from Proposition 3.2.
- (2) Assume $l > 0$ is the minimal integer such that $la = 0$. Then $b = (k + ql)a$ for any integer $k + ql \neq 0$. By Proposition 3.2, there exists a vertical map $\tilde{M}_a \rightarrow \tilde{M}_b$ of degree $k + ql$.
- (3) By Proposition 3.2, there exists a vertical map $\tilde{M}_a \rightarrow \tilde{M}_b$ of degree k . Assume there exists a vertical map $\tilde{M}_a \rightarrow \tilde{M}_b$ of degree k' . Then by Proposition 3.2, $b = k'a$, which implies that $ka = k'a$. Since a is not torsion, we have $k = k'$.

\square

4. APPLICATIONS

The following consequence of Theorem 1.1 is essential in our further applications.

Corollary 4.1. *Let $S^1 \xrightarrow{j_i} E_i \xrightarrow{p_i} M_i$ be an oriented S^1 -bundle over a closed oriented aspherical n -manifold M_i , $i = 1, 2$ and let $\tilde{f}: E_1 \rightarrow E_2$ be a map of non-zero degree. If each finite index subgroup of $\pi_1(M_2)$ is center-free, then \tilde{f} is homotopic to a fiber-preserving map. As a consequence, we have*

$$D_{FP}(E_1, E_2) = D(E_1, E_2).$$

Proof. By Theorem 1.1, we need only to verify that the induced map \tilde{f}_* on π_1 sends the fiber group to the fiber group injectively.

Since $\tilde{f}: E_1 \rightarrow E_2$ is a non-zero degree map, $\tilde{f}_*(\pi_1(E_1)) < \pi_1(E_2)$ is a finite index subgroup. Let $q: E'_2 \rightarrow E_2$ be the finite cover of E_2 corresponding to $\tilde{f}_*(\pi_1(E_1))$. Then E'_2 has an induced oriented S^1 -bundle structure $p'_2: E'_2 \rightarrow M'_2$, where the base manifold M'_2 is a finite cover of M_2 , and q is a fiber-preserving map. Now we have a lifting map $\tilde{g}: E_1 \rightarrow E'_2$, such that $\tilde{f} = q \circ \tilde{g}$, and $\tilde{g}_*: \pi_1(E_1) \rightarrow \pi_1(E'_2)$ is surjective.

Since E_1 is an orientable S^1 -bundle and M_1 is aspherical, the fiber group $\pi_1(S^1)$ maps into $\pi_1(E_1)$ injectively and its image lies in the center of $\pi_1(E_1)$. We still denote this subgroup of $\pi_1(E_1)$ by $\pi_1(S^1)$. Since $\pi_1(M'_2)$, as a finite index subgroup of $\pi_1(M_2)$, is center-free, $(p'_2)_* \circ \tilde{g}_*(\pi_1(S^1)) < \pi_1(M'_2)$ must be the trivial subgroup, and $\tilde{g}_*(\pi_1(S^1)) < \pi_1(E'_2)$ must be contained in the fiber subgroup $\pi_1(S^1) < \pi_1(E'_2)$. So $\tilde{f}_*(\pi_1(S^1)) < \pi_1(E_2)$ must be contained in the fiber subgroup $\pi_1(S^1) < \pi_1(E_2)$. \square

The next result is well-known in the theory of hyperbolic manifolds and 3-manifold topology.

Lemma 4.2. *Suppose N is either a closed hyperbolic n -manifold, or a closed irreducible 3-manifold that is not a Seifert manifold. Then each finite index subgroup of $\pi_1(N)$ is center-free.*

4.1. Finiteness of mapping degree sets between S^1 -bundles. Now we prove Theorem 1.3.

Proof of Theorem 1.3. Let $\tilde{f}: \tilde{M}_a \rightarrow \tilde{N}_b$ be a map of non-zero degree. We will check that there are only finitely many possibilities for $\deg(\tilde{f})$. Since N is a closed hyperbolic manifold, it is aspherical and each finite index subgroup of $\pi_1(N)$ is center-free by Lemma 4.2. By Corollary 4.1, we may assume that \tilde{f} is a fiber-preserving map and induces $f: M \rightarrow N$. By Theorem 1.2, we have $\deg(\tilde{f}) = k \deg(f)$ and $ka = f^\#(b)$.

Since b is not torsion and $\tilde{f}: \tilde{M}_a \rightarrow \tilde{N}_b$ is a map of non-zero degree, a is not torsion by Lemma 3.4, so there is at most one integer k such that $ka = f^\#(b)$. To prove that there are only finitely many possibilities for $\deg(\tilde{f})$, we only need to prove that there are only finitely many possibilities for both $\deg(f)$ and $f^\#(b)$.

Since N is a closed oriented hyperbolic n -manifold, its simplicial volume satisfies $\|N\| > 0$ and $|\deg(f)| \cdot \|N\| \leq \|M\|$ for any map $f: M \rightarrow N$ [Th, 6.1.4, 6.1.2], thus

$$|\deg(f)| \leq \|M\|/\|N\|.$$

So $\deg(f)$ can only take finitely many values. Below we prove that there are only finitely many possibilities for $\tilde{\beta} = f^\#(b)$.

Since $H_2(M; \mathbb{Z})$ is finitely generated, we choose a finite generating set $\{\alpha_1, \dots, \alpha_m\}$ of $H_2(M; \mathbb{Z})$. By the Universal Coefficient Theorem, each cohomology class $\tilde{\beta} \in H^2(M; \mathbb{Z})$ is determined by $\{\tilde{\beta}(\alpha_1), \dots, \tilde{\beta}(\alpha_m)\} \in \mathbb{Z}^m$, up to a finite ambiguity with size $|\text{Tor}(H^2(M; \mathbb{Z}))|$.

Claim: For each $1 \leq i \leq m$, there are only finitely many possibilities for $f^\#(b)(\alpha_i)$.

Proof of Claim: We have

$$f^\#(b)(\alpha_i) = b(f_\#(\alpha_i)).$$

So, we only need to prove that there are only finitely many possibilities for $f_\#(\alpha_i)$. Let

$$L = \max\{\|\alpha_i\| \mid i = 1, \dots, m\}.$$

By the functorial property of the simplicial volume (cf. [Gr, p.8]), we have

$$L \geq \|\alpha_i\| \geq \|f_{\#}(\alpha_i)\|.$$

Since N is a closed orientable hyperbolic manifold, the simplicial volume is a genuine norm on the finite dimensional space $H_2(N, \mathbb{R})$, see [CW, Theorem 1.6] for example. Hence there are only finitely many integer homology classes whose image in $H_2(N, \mathbb{R})$ has simplicial volume less or equal than L . This proves the Claim, thus also proves the theorem. \square

4.2. Realizing finite sets of integers as mapping degree sets. In this section, we will prove the Realization Theorem 1.5. First, we prove the following:

Proposition 4.3. *Suppose N is a closed oriented aspherical n -manifold, every finite index subgroup of $\pi_1(N)$ is center-free, and $b \in H^2(N; \mathbb{Z})$ is not torsion.*

(1) *Assume that $D(N)$ is finite. Then for any non-zero integers k and m , we have*

$$D(\tilde{N}_{mb}, \tilde{N}_{kb}) \subset \begin{cases} \{0, \pm k/m\}, & \text{if } k \text{ is a multiple of } m, \\ \{0\}, & \text{otherwise.} \end{cases}$$

(2) *Assume that $D(N) = \{0, 1\}$ and $f^{\#}(b) = b$ holds for any degree one self-map $f: N \rightarrow N$. Then for any non-zero integers k and m , we have*

$$D(\tilde{N}_{mb}, \tilde{N}_{kb}) = \begin{cases} \{0, k/m\}, & \text{if } k \text{ is a multiple of } m, \\ \{0\}, & \text{otherwise.} \end{cases}$$

Moreover, the assumptions in (2) hold if N is a closed oriented hyperbolic n -manifold with $n \geq 3$ and the order of its isometry group $\text{Isom}(N)$ is odd.

Proof. Suppose $\tilde{f}: \tilde{N}_{mb} \rightarrow \tilde{N}_{kb}$ is a map of non-zero degree for some non-zero integer k . Since N is aspherical and every finite index subgroup of $\pi_1(N)$ is center-free, by Corollary 4.1, we may assume that \tilde{f} is a fiber-preserving map that induces $f: N \rightarrow N$. By Theorem 1.2, we have

$$(9) \quad \deg(\tilde{f}) = l \deg(f) \text{ and } l(mb) = f^{\#}(kb)$$

for some non-zero integer l . Since \tilde{f} is of non-zero degree, f is also of non-zero degree.

(1) Since $D(N)$ is finite, we have $\deg(f) = \pm 1$. By (9), we have $\deg(\tilde{f}) = \pm l$, and

$$(10) \quad lmb = f^{\#}(kb) = kf^{\#}(b),$$

so we obtain

$$(11) \quad f^{\#}(b) = \frac{ml}{k}b = \lambda b \in H^2(N; \mathbb{Q}).$$

We will show that the rational number λ is ± 1 . Since $f: N \rightarrow N$ is a degree ± 1 map, it induces surjections $f_{\#}: H_i(N; \mathbb{Z}) \rightarrow H_i(N; \mathbb{Z})$ for all $i \geq 0$. Since each self-surjection on each

finitely generated Abelian group is an isomorphism, f induces isomorphisms $f_{\#}: H_i(N; \mathbb{Z}) \rightarrow H_i(N; \mathbb{Z})$ for all $i \geq 0$. By algebraic duality, f induces isomorphisms $f^{\#}: H^i(N; \mathbb{Z}) \rightarrow H^i(N; \mathbb{Z})$ for all $i \geq 0$, and in particular $f^{\#}: H^2(N; \mathbb{Z}) \rightarrow H^2(N; \mathbb{Z})$ is an isomorphism. Note that (11) implies that b is an eigenvector of $f^{\#}: H^2(N; \mathbb{Z})/tors \rightarrow H^2(N; \mathbb{Z})/tors$ with rational eigenvalue λ . Since $f^{\#}$ is an isomorphism, the characteristic polynomial of $f^{\#}$ is an integer polynomial with leading coefficient 1 and constant ± 1 . So this rational eigenvalue λ has to be ± 1 , i.e. $f^{\#}(b) = \pm b$. By (10), we obtain $kb = \pm mlb$. Since b is not torsion, we have $k = \pm ml$, that is k is a multiple of m , and $\deg(\tilde{f}) \in \{\pm k/m\}$. Then we have

$$D(\tilde{N}_{mb}, \tilde{N}_{kb}) \subset \{0, \pm k/m\}.$$

In particular,

$$D(\tilde{N}_b) = D(\tilde{N}_b, \tilde{N}_b) \subset \{0, \pm 1\}.$$

(2) Now we have that $\deg(f) = 1$ and $f^{\#}(b) = b$. Assume that $D(\tilde{N}_{mb}, \tilde{N}_{kb})$ contains a non-zero integer l . By (9), applying the same argument to the present situation, we have

$$l \cdot mb = f^{\#}(kb) = kf^{\#}(b) = kb,$$

Since b is not torsion, we have $k = ml$, that is k is an integer multiple of m , and $\deg(\tilde{f}) = l = k/m$. Then Proposition 3.2 implies that $k/m \in D(\tilde{N}_{mb}, \tilde{N}_{kb})$ if k is a factor of m . So $D(\tilde{N}_{mb}, \tilde{N}_{kb})$ is $\{0, k/m\}$ if k is a multiple of m , and is $\{0\}$ otherwise.

Now we prove the ‘‘Moreover’’ part of this theorem. Since N is a closed oriented hyperbolic n -manifold, by Corollary 4.1, $\tilde{f}: \tilde{N}_{mb} \rightarrow \tilde{N}_{kb}$ induces a map $f: N \rightarrow N$ with $|\deg(f)| = 1$. By [Th, Theorem 6.4], every map $f: N \rightarrow N$ of $|\deg(f)| = 1$ is homotopic to an isometry when $n \geq 3$. Since the order of $\text{Isom}(M)$ is odd, f is homotopic to an isometry of odd order, and so $\deg(f) = 1$. We may assume that $f^r = id$ for some odd r .

Again by (11), ml/k is a real eigenvalue of $f^{\#}: H^2(N, \mathbb{Q}) \rightarrow H^2(N, \mathbb{Q})$. Since $(f^{\#})^r = id$, ml/k is a r -th root of unity. Since r is odd, we have $ml/k = 1$, that is $k = ml$. So $f^{\#}(b) = b$ holds by (10). \square

Next, we are going to prove Theorem 1.4. By Proposition 4.3 we only need to prove the following result.

Proposition 4.4. *For $n = 4, 5$, there exists a closed, orientable $(n - 1)$ -manifold N such that the following hold:*

- (a) N is aspherical and every finite index subgroup of $\pi_1(N)$ is center-free,
- (b) $D(N) = \{0, 1\}$,
- (c) there is a non-torsion class $b \in H^2(N; \mathbb{Z})$ such that $f^{\#}(b) = b$ for any degree one map $f: N \rightarrow N$.

Proof. We split the proof into the cases $n = 4$ and $n = 5$.

(1) *The case $n = 4$.* Let K_1 and K_2 be two hyperbolic knots in the 3-sphere S^3 , such that $\text{Isom}(E(K_1)) = \{id\}$ and $E(K_1)$ and $E(K_2)$ are not homeomorphic to each other. Here

$E(K_i)$ denotes the knot complement $S^3 \setminus N(K_i)$. Let S_1 and S_2 be Seifert surfaces of K_1 and K_2 , respectively. One can choose $K_1 = 8_{20}$ in the Appendix of [Ad], which is hyperbolic and has no symmetry [BZ, Table 1]. Let

$$M = E(K_1) \cup_{\phi} E(K_2)$$

be the closed oriented 3-manifold obtained by taking an orientation reversing homeomorphism $\phi: \partial E(K_1) \rightarrow \partial E(K_2)$ such that $\phi(\partial S_1) = \partial S_2$. Clearly, M is irreducible and is not a Seifert manifold, so M meets condition (a) by Lemma 4.2. By classical results in 3-manifold topology, the following statements hold.

- (i) $H_2(M; \mathbb{Z}) = \mathbb{Z}$, and indeed it is generated by $S = S_1 \cup S_2$.
- (ii) $M = E(K_1) \cup_{\phi} E(K_2)$ gives the JSJ decomposition of M , where the JSJ torus T is the image of $\partial E(K_i)$. Note that T is separating in M , and M is Haken. In particular, M is aspherical and every finite index subgroup of $\pi_1(M)$ is center-free by Lemma 4.2.
- (iii) The simplicial volume of M satisfies $\|M\| = \|E(K_1)\| + \|E(K_2)\| > 0$.

Suppose $f: M \rightarrow M$ is a map of non-zero degree. Since $\|M\| > 0$, we have $\deg(f) = \pm 1$. Since M is Haken, f is homotopic to a homeomorphism. By the JSJ theory, we may assume that the JSJ decomposition $M = E(K_1) \cup_{\phi} E(K_2)$ is invariant under f . Since $E(K_1)$ and $E(K_2)$ are not homeomorphic to each other, each $E(K_i)$ is invariant under f for $i = 1, 2$. Since $E(K_1)$ is a hyperbolic knot, we may assume that $f|_{E(K_1)}$ is a self-isometry, and we have that $f|_{E(K_1)}$ is the identity since $\text{Isom}(E(K_1)) = \{id\}$.

So we have $\deg(f) = 1$, and f maps the oriented meridian of $E(K_1)$ to itself. Since $H_1(M; \mathbb{Z}) \cong \mathbb{Z}$ is generated by the oriented meridian of $E(K_1)$, $f_{\#}: H_1(M; \mathbb{Z}) \rightarrow H_1(M; \mathbb{Z})$ is the identity. Then the Poincaré duality and the fact that $\deg(f) = 1$ imply that $f^{\#}: H^2(M; \mathbb{Z}) \rightarrow H^2(M; \mathbb{Z})$ is the identity. Therefore $f^{\#}(b) = b$ for any $b \in H^2(M; \mathbb{Z})$.

(2) *The case $n = 5$.* By a result of Belolipesky and Lubotszky [BL, Theorem 1.1], for each $k \geq 2$, and any finite group Γ , there exists a closed hyperbolic k -manifold M such that $\text{Isom}(M) = \Gamma$. Indeed M is an orientable manifold, which is observed by Weinberger (see [Mü, Section 3]).

Let M_i be a closed oriented hyperbolic k -manifold such that $\text{Isom}(M_i) \cong \mathbb{Z}_{2i+1}$, the cyclic group of order $2i+1$. Then the family $\{M_i\}$ contains infinitely many hyperbolic k -manifolds. If $k > 3$, it follows from H. C. Wang's theorem [Wa] that $\{\text{Vol}(M_i)\}_{i=1}^{\infty}$ is unbounded. Here we take $k = n - 1 = 4$. By the Gauss-Bonnet Theorem, the set of Euler characteristics $\{\chi(M_i)\}_{i=1}^{\infty}$, is unbounded from above. Hence the set of second Betti numbers $\{\beta_2(M_i)\}_{i=1}^{\infty}$ is unbounded. So there exists a hyperbolic 4-manifold M such that the order of $\text{Isom}(M)$ is odd and $\beta_2(M) > 0$. The conclusion $f^{\#}(b) = b$ for any non-torsion class $b \in H^2(N; \mathbb{Z})$ follows from the condition that M is a closed orientable hyperbolic 4-manifold with odd order $\text{Isom}(M_i)$ and the ‘‘Moreover’’ part of Proposition 4.3. So M meets the conditions (a), (b), and (c).

This finishes the proof of Proposition 4.4, which concludes the proof of Theorem 1.4. \square

Finally we are ready to prove the Realization Theorem.

Proof of Theorem 1.5. For $n = 3$, the theorem follows from Theorem B ([CMV]). For $n \geq 6$, the theorem follows from Theorem C ([NSTWW]). For $n = 4, 5$, by implementing Theorem 1.4, the proof follows by the same strategy as in [CMV]. So we only give an outline of the proof here.

For any finite set $A \subset \mathbb{Z}$ that contains 0, by Proposition 2.2 of [CMV], there exist a finite sequence $\{B(i)\}_{i=1}^k$, where each $B(i)$ is a finite sequence of non-zero integers such that

$$A = \bigcap_{i=1}^k S_{B(i)}.$$

Here $S_{B(i)}$ denotes the set of sums of subsequences of $B(i)$, which includes 0, as the sum of the empty subsequence.

For $n = 4, 5$, take a closed, oriented, aspherical $(n - 1)$ -manifold N as in Proposition 4.4, and take a non-torsion element $b \in H^2(N; \mathbb{Z})$. We take distinct prime numbers p_1, \dots, p_k that are greater than absolute values of all numbers in all the sequences $B(i)$, with $i = 1, \dots, k$.

For each $i = 1, \dots, k$, let $\alpha_i = p_i \cdot \prod_{\beta \in B(i)} \beta$. Then we construct the following closed oriented n -manifolds:

$$N_i = \tilde{N}_{\alpha_i b}, \quad M_i = \#_{\beta \in B(i)} \tilde{N}_{\frac{\alpha_i}{\beta} b},$$

where N_i is aspherical. By Lemma 3.5 of [NWW] and Theorem 1.4, we have

$$D(M_i, N_i) = S_{B(i)}, \text{ and } D(M_i, N_j) = \{0\} \text{ if } i \neq j.$$

Then Proposition 3.7 of [CMV] implies the existence of an integer $l \geq 0$, such that

$$D\left(\left(\#_{i=1}^k M_i\right)\#(\#^l S^{n-1} \times S^1), \#_{i=1}^k N_i\right) = \bigcap_{i=1}^k D(M_i, N_i) = \bigcap_{i=1}^k S_{B(i)} = A.$$

\square

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