On the finite element approximation of 4\textsuperscript{th} order singularly perturbed eigenvalue problems

Christos Xenophontos
Department of Mathematics and Statistics
University of Cyprus

joint work with D. Savvidou (UCY) and H.-G. Roos (TU-Dresden)
The Model Problem

*Singularly perturbed 4th order eigenvalue problem:*

Find \(0 \neq u(x) \in C^4(I), \lambda \in \mathbb{C}\) such that

\[
\varepsilon^2 u^{(4)}(x) - \left(\alpha(x)u'(x)\right)' + \beta(x)u(x) = \lambda u(x) \quad \text{in} \quad I = (0, 1)
\]

\[
u(0) = u(1) = u'(0) = u'(1) = 0
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$$u(0) = u(1) = u'(0) = u'(1) = 0$$

where $\varepsilon \in (0, 1]$ is a given small parameter and $\alpha(x), \beta(x) \geq 0$, are given sufficiently smooth functions.
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where $\lambda_k(0)$ are the eigenvalues of the reduced/limiting problem. If $\lambda_k(0)$ is real, then so is $\lambda_k(\varepsilon)$.
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- $\lambda_k(\varepsilon)$ can be expanded as a power series in $\varepsilon$. 
Theorem: [Moser, 1955]

Each eigenfunction $u_k$ can be decomposed as

$$u_k = u_k^S + u_k^{BL,+} + u_k^{BL,-}$$

where $u_k^S$ denotes the smooth part, $u_k^{BL,+}$ denotes the left boundary layer and $u_k^{BL,-}$ denotes the right boundary layer. Moreover, for $n = 0, 1, 2, \ldots$
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\[
\left| \left( u_k^S \right)^{(n)} (x) \right| \leq C_k,
\]

\[
\left| \left( u_k^{BL,+} \right)^{(n)} (x) \right| \leq C_k \varepsilon^{1-n} e^{-\alpha x / \varepsilon}, \quad \left| \left( u_k^{BL,-} \right)^{(n)} (x) \right| \leq C_k \varepsilon^{1-n} e^{-\alpha (1-x) / \varepsilon}
\]
Variational Formulation

Find

\[ u_k \in H_0^2(I) = \left\{ u \in H^2(I) : u(0) = u'(0) = u(1) = u'(1) = 0 \right\} \]

and \( \lambda_k \in \mathbb{C} \) such that

\[ B(u_k, v) = \lambda_k \langle u_k, v \rangle \quad \forall \ v \in H_0^2(I) \]

where \( \langle \cdot, \cdot \rangle \) is the usual \( L^2(I) \) inner product and

\[ B(u, v) = \varepsilon^2 \langle u'', v'' \rangle + \langle \alpha u', v' \rangle + \langle \beta u, v \rangle \]
Discretization

We seek \( u_h^k \in V_h \subset H^2_0(I), \lambda_h^k \in \mathbb{C} \) s. t.

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We define the energy norm as

$$\| u \|_E^2 = \varepsilon^2 \| u'' \|_{L^2(I)}^2 + \| u' \|_{L^2(I)}^2 + \| u \|_{L^2(I)}^2 \quad \forall \, u \in H^2_0(I)$$
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and we have

\[
B(u, u) \geq \gamma \|u\|_E^2 \quad \forall \, u \in H_0^2(I)
\]
In order to define the finite element space $V_h$ let

$$
\Delta = \{0 = x_0 < x_1 < \ldots < x_N = 1\}
$$

be an arbitrary mesh on $I = (0, 1)$ and set

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I_j = \left( x_{j-1}, x_j \right), h_j = x_j - x_{j-1}, j = 1, \ldots, N
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$$V_h = \left\{ u \in H^2_0(I) : u|_{I_j} \in P_p(I_j), j = 1, \ldots, N \right\}$$
Definition:

Let $\{x_i\}_{i=0}^N$ be an arbitrary partition of the interval $(a, b)$ and suppose that for a sufficiently smooth function $f(x)$, $x \in (a, b)$, the values $f(x_i) = y_i \in \mathbb{R}$, $f'(x_i) = y'_i \in \mathbb{R}$ are given. Then, there exists a unique polynomial $f^I \in P_{2N+1}(a, b)$, called the Hermite interpolant of $f$, given by

$$f^I(x) = \sum_{i=0}^{N} (y_i H_{0,i}(x) + y'_i H_{1,i}(x))$$

where, with $L_i(x)$ the Lagrange polynomial of degree $N$ associated with node $x_i$,

$$H_{0,i}(x) = \left[1 - 2(x - x_i) \frac{dL_i}{dx}(x_i)\right] L_i^2(x), \quad H_{1,i}(x) = (x - x_i) L_i^2(x)$$
Theorem:

Let $u \in C^{2n+2}([a, b])$ and let $\Delta = \{x_i\}_{i=0}^N$ be a mesh on $[a, b]$, with maximum mesh size $h$ and with $N$ a multiple of $n$. If $u^I$ is the piecewise Hermite interpolant of $u$, having degree at most $2n+1$ on each subinterval $[x_{i-1}, x_i], \ i = 1, \ldots, N$ then
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$$
\left\| \left( u^I - u \right)^{(k)} \right\|_{L^\infty(I)} \leq C h^{2n+2-k} \left\| u^{(2n+2)} \right\|_{L^\infty(I)}, \quad k = 0, 1, \ldots, 2n+1
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**Theorem:**

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$$\left\| (u^I - u)^{(k)} \right\|_{L^\infty(I)} \leq Ch^{2n+2-k} \left\| u^{(2n+2)} \right\|_{L^\infty(I)}, \ k = 0, 1, \ldots, 2n+1$$

In our setting

$$\left\| (u^I - u)^{(k)} \right\|_{L^\infty(I)} \leq Ch^{p+1-k} \left\| u^{(p+1)} \right\|_{L^\infty(I)}, \ k = 0, 1, \ldots, p$$
**Definition: Exponentially graded mesh**

With $N > 4$ a multiple of 4 we define

$$\phi(t) = -\ln(1 - 4C_{p, \epsilon} t), \quad t \in [0, 1/4 - 1/N]$$

$$C_{p, \epsilon} = 1 - \exp\left(-\frac{\alpha}{(p+1)\epsilon}\right)$$
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$$
x_j = \begin{cases} 
\frac{\varepsilon}{\alpha} (p+1)\phi(j/N) & , \ j = 0, 1, \ldots, N/4 - 1 \\
\frac{x_{N/4} - x_{N/4-1}}{2 + N/2} (j - N/4 + 1) & , \ j = N/4, \ldots, 3N/4 \\
1 - \frac{\varepsilon}{\alpha} (p+1)\phi\left(\frac{N - j}{N}\right) & , \ j = 3N/4 + 1, \ldots, N 
\end{cases}
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\frac{x_{3N/4} - x_{N/4-1}}{2 + N/2} (j - N/4 + 1), & j = N/4, \ldots, 3N/4 \\
1 - \frac{\varepsilon}{\alpha} (p+1)\phi\left(\frac{N-j}{N}\right), & j = 3N/4 + 1, \ldots, N
\end{cases}$$
Lemma: [X., CMAM 2017]

Let \( u_{BL} \) denote either boundary layer and let \( u_{BL}^I \) be its Hermite interpolant based on the exponential mesh. Then

\[
\left\| \left( u_{BL} - u_{BL}^I \right)^{(k)} \right\|_{L^\infty(I)} \leq C \mathcal{E}^{1-k} N^{-(p+1-k)}, \quad k = 0, 1, ..., p
\]

and

\[
\left| u_{BL} - u_{BL}^I \right|_{H^2(I)} \leq C \mathcal{E}^{-1/2} N^{-p+1}
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Lemma: [X., CMAM 2017]

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$$\left\| (u_{BL} - u_{BL}^I)^{(k)} \right\|_{L^\infty(I)} \leq C \varepsilon^{1-k} N^{-(p+1-k)}, \ k = 0, 1, \ldots, p$$

and

$$\left| u_{BL} - u_{BL}^I \right|_{H^2(I)} \leq C \varepsilon^{-1/2} N^{-p+1}$$

Using the above lemma and assuming $N < \varepsilon^{-1}$, we establish
\[
\left\| \left( u - u^I \right)^{(k)} \right\|_{L^\infty(I)} \leq C \varepsilon^{1-k} N^{-(p+1-k)}, \quad k = 0, 1, \ldots, p
\]
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Sketch of proof: We use the decomposition of \( u \) into a smooth part and two boundary layers. The layers are handled by the previous lemma and the smooth part by the assumption \( N < \varepsilon^{-1} \).
Proposition:

For all $h \leq h_0$, with $h_0$ independent of $\varepsilon$, there holds

$$\lambda_k \leq \lambda_k^h \leq C_k \lambda_k \left(1 + h^{2p-2}\right)$$
Proposition:

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Sketch of proof: We use the classical techniques found in [Strang & Fix, 1973], utilizing the Ritz projection $Rw$ of $w \in H_0^2(I)$ onto $V_h$:
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$$B(w - Rw, v) = 0 \quad \forall \ v \in V_h$$
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$$B(w - Rw, \nu) = 0 \quad \forall \nu \in V_h$$

There holds $\|w - Rw\|_E \leq Ch^{p-1}$. 
Other tools used include the *minimax principle* and the fact that the Green’s function associated with our problem is uniformly bounded. Continuity of the bilinear form and Galerkin orthogonality are also utilized.
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For the approximation of the eigenfunctions, we have the following, under the assumption that all eigenvalues are distinct.
Proposition:

Assume that the eigenfunctions and their approximations are normalized and that all eigenvalues are distinct. Then

\[ \left\| u_k - u^h_k \right\|_E \leq C_k h^{p-1} \]
Proposition:

Assume that the eigenfunctions and their approximations are normalized and that all eigenvalues are distinct. Then

\[ \| u_k - u_k^h \|_E \leq C_k h^{p-1} \]

Sketch of proof: The main observation is the identity

\[ B(u_k - u_k^h, u_k - u_k^h) = \lambda_k \left\| u_k - u_k^h \right\|^2_{L^2(I)} + \lambda_k^h - \lambda_k \]
Then, for $h$ sufficiently small, there holds

$$\frac{\lambda_k}{|\lambda^h_k - \lambda_j|} \leq \rho \in \mathbb{R} \quad \forall \quad j \neq k$$
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hence

$$\left\| u_k - u_k^h \right\|_E^2 \leq \lambda_k \left\| u_k - u_k^h \right\|_{L^2(I)}^2 + \lambda_k^h - \lambda_k$$

$$\leq 2(1 + \rho) \left\| u_k - Ru_k \right\|_{L^2(I)}^2 \leq C_k h^{2(p-1)}$$
Numerical Results

We consider the problem

$$\varepsilon^2 u^{(4)}(x) - \left(e^x u'(x)\right)' + xu(x) = \lambda u(x) \text{ in } I = (0,1)$$

$$u(0) = u(1) = u'(0) = u'(1) = 0$$

No exact solution is available, so for the computations we use a reference solution obtained with twice as many degrees of freedom (DOF).
Approximation of the $1^{st}$ eigenvalue
Approximation of the 2\textsuperscript{nd} eigenvalue

\[ \epsilon = 10^{-j}, \ p = 3 \]

\[ \text{slope } \approx -4 \]
Approximation of the $1^{\text{st}}$ eigenfunction.
Approximation of the 2\textsuperscript{nd} eigenfunction

$\epsilon = 10^{-j}$, $p = 3$
Closing Remarks

We considered a 4\textsuperscript{th} order\textit{ singularly perturbed eigenvalue problem} and studied the performance of an $h$ FEM on the \textit{Exponentially Graded Mesh}.
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The derivative of the eigenfunctions features boundary layers. Once the layers are resolved, classical results give us the required convergence (including the ‘doubling effect’ for the eigenvalues).
Closing Remarks

We considered a 4\textsuperscript{th} order \textit{singularly perturbed eigenvalue problem} and studied the performance of an \textit{h FEM} on the \textit{Exponentially Graded Mesh}.

The derivative of the eigenfunctions features boundary layers. Once the layers are resolved, classical results give us the required convergence (including the ‘doubling effect’ for the eigenvalues).

Numerical results corroborate our theoretical findings.