Optimal mesh design for the finite element approximation of reaction–diffusion problems

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SUMMARY

We consider the numerical approximation of singularly perturbed problems, and in particular reaction–diffusion problems, by the $h$ version of the finite element method. We present guidelines on how to design non-uniform meshes both in one and two dimensions that are asymptotically optimal as the mesh-width tends to zero. We also present the results of numerical computations showing that robust, optimal rates can be achieved even in the pre-asymptotic range. Copyright © 2001 John Wiley & Sons, Ltd.

KEY WORDS: singularly perturbed reaction–diffusion problem; finite element method; exponentially graded mesh

1. INTRODUCTION

The numerical solution of singularly perturbed boundary value problems has received much attention recently and there are numerous papers and even books written on this subject (see References [1–3] and the references therein). Problems of this type arise in many areas, such as fluid mechanics and heat transfer as well as problems in structural mechanics posed over thin domains. The solution of singularly perturbed problems will, in general, contain boundary layers along the boundary of the domain. These layers complicate the numerical approximation, and the method must be carefully tailored to account for their presence. In particular, the method must be robust in the sense that the error in the approximation should not deteriorate as the singular perturbation parameter tends to zero.

Currently, there are two known ways to alleviate this problem in the context of the finite element method (FEM). The first is through the use of the $h$ version on piecewise uniform meshes; these called Shishkin meshes and their use is becoming quite popular. It has been shown [4, 5] that robust quasi-optimal (algebraic) error bounds can be obtained using this

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technique. The second alternative is through the use of the $p/hp$ version of the FEM with appropriately designed meshes [6–10]). The advantage of this second approach is the exponential rates of convergence that can be established when the domain is smooth [11].

In this paper we will consider the problem of designing non-uniform meshes to be used with the $h$ version of the FEM, both in one and two dimensions. These meshes will be asymptotically optimal (i.e. as $h \to 0$), in the sense that the approximation error is minimized, and robust, optimal (algebraic) rates can be established. In Section 2, this will be done for the one-dimensional case (as was originally suggested in References [6–8]). Using a tensor product argument, the design of the mesh will be extended to two dimensions, in Section 3. Finally, in Section 4 we will present the results of numerical computations that illustrate the robust performance of this method even in the pre-asymptotic range.

Throughout the paper, $H^k(\Omega)$ will denote the Sobolev space of order $k$ on a domain $\Omega \subset \mathbb{R}^2$, with $H^0(\Omega) = L^2(\Omega)$, and $\| \cdot \|_{k,\Omega}$, $| \cdot |_{k,\Omega}$ denoting the norm and seminorm as usual. (The subscript indicating the domain will be omitted when no confusion occurs.) Also, $H^0_0(\Omega) = \{ u \in H^1(\Omega) : u = 0$ on $\partial \Omega \}$. For $I = [a,b] \subset \mathbb{R}$, we will use $H^k(I)$ and $H^1_0(I) = \{ u \in H^1(I) : u(a) = u(b) = 0 \}$. The letters $c$ and $C$, or without dependencies on variables, will be used to denote generic positive constants, possibly not the same in each occurrence.

2. MESH DESIGN IN ONE-DIMENSION

2.1. The model problem

We consider the one-dimensional singularly perturbed problem

$$-\varepsilon^2 u''(x) + u(x) = f(x) \text{ in } I = (-1,1)$$

$$u(\pm 1) = 0$$  \hspace{1cm} (1)

where $\varepsilon \in (0,1]$ is a parameter that can approach 0. The case of constant coefficients is considered for simplicity and we note that our results apply to the more general case of non-constant coefficients, with minor modifications. The variational formulation of (1) reads: find $u_\varepsilon \in H^1_0(I)$ such that

$$B(u_\varepsilon, v) = F(v) \quad \forall v \in H^1_0(I)$$  \hspace{1cm} (2)

where

$$B(u, v) = \int_I \{ \varepsilon^2 u'v' + uv \} \, dx, \quad F(v) = \int_I vf \, dx$$  \hspace{1cm} (3)

The solution of (2) contains, in general, boundary layers as $\varepsilon \to 0$. In Reference [6] the following asymptotic expansion for $u_\varepsilon$, which shows the explicit nature of the boundary layers, was derived (under the assumption that the right-hand side function $f$ in (1) is smooth enough, e.g. $f \in H^{4M}(I)$ for some $M \in \mathbb{N}$):

$$u_\varepsilon(x) = u_0(x) + A_\varepsilon e^{-(1+x)/\varepsilon} + B_\varepsilon e^{-(1-x)/\varepsilon}$$  \hspace{1cm} (4)
In (4) $u_\varepsilon(x)$ is a smooth function and $A_\varepsilon,B_\varepsilon$ are smooth coefficients, that are uniformly bounded independently of $\varepsilon$. In fact, $|u_\varepsilon|_{\ell}$ is uniformly bounded independently of $\varepsilon$, where $\ell$ depends on the regularity of $f$.

The above decomposition shows that if $f$ is smooth, then approximating $u_\varepsilon(x)$ depends entirely on the ability to approximate the ‘typical’ one-dimensional boundary layer function $u_{BL}(x) := \exp(-x/\varepsilon)$.

2.2. The finite element method

We will be using spaces of piecewise polynomials on $I$ characterized by the mesh-degree combination $\Sigma = (\Delta, \tilde{p})$, defined as follows. Given $m + 1 \geq 2$ nodal points

$$-1 =: x_0 < x_1 < \ldots < x_{m-1} < x_m := 1$$

we define the mesh $\Delta$ by

$$\Delta = \{I_i\}_{i=1}^m, \quad I_i = (x_{i-1}, x_i), \quad h_i = x_i - x_{i-1}, \quad i = 1, \ldots, m$$

and set $\tilde{p} = (p_1, \ldots, p_m)$ to be the degree vector. Let $\Pi_p(I)$ be the set of polynomials on $I$ of degree less than or equal to $p$ and $C^k(I)$ to be the set of functions on $I$ with $k$ continuous derivatives. We then define the space of piecewise polynomials as

$$S(\Sigma) \equiv S(\Delta, \tilde{p}) = \{u \in C^0(I) : u|_{I_i} \in \Pi_p(I), \ I_i \in \Delta\}$$

Note that $S(\Sigma) \subset H^1(I)$ with $\dim(S(\Sigma)) = \sum_i^m p_i + 1$. Also define $S_0(\Sigma) = S(\Sigma) \cap H^1_0(I)$ and set $N = \dim(S_0(\Sigma)) = \dim(S(\Sigma)) - 2$. (We will omit the dependence on $\Sigma$ when no confusion occurs, and simply write $S$ or $S_0$.)

The discrete version of (2) then reads: find $u_\varepsilon^S \in S_0$ such that

$$B(u_\varepsilon^S,v) = F(v) \quad \forall v \in S_0$$

We have

$$B(u_\varepsilon - u_\varepsilon^S,v) = 0 \quad \forall v \in S_0$$

so that the error $(u_\varepsilon - u_\varepsilon^S)$ satisfies

$$\|u_\varepsilon - u_\varepsilon^S\|_\varepsilon \leq \inf_{\zeta \in S_0} \|u_\varepsilon - \zeta\|_\varepsilon$$

where the energy norm $\|\cdot\|_\varepsilon$ is defined by

$$\|v\|_\varepsilon^2 := \varepsilon^2 \|v\|^2_1 + \|v\|^2_0$$

As noted above, the approximation of $u_\varepsilon$ depends on the ability to approximate $u_{BL}(x) = \exp(-x/\varepsilon)$ over $I = (0,1)$. For notational convenience, we will consider the approximation of $u_{BL}$ for the remainder of this section and define

$$\Phi(\varepsilon,S) = \inf_{v \in S} \|u_{BL} - v\|_\varepsilon$$
Suppose $\Delta$ is chosen to be *uniform*, with $h := h_i = 1/m$, and the $h$-version is used, in which $\bar{\rho} = (p, p, \ldots, p)$ is kept fixed and $h \to 0$. Then, the usual finite element error analysis (see e.g. Reference [12]) shows that

$$\Phi(\varepsilon, S) \leq C(p)\varepsilon^{-p+1/2}N^{-p}$$

(11)

where $N = O(1/h)$ (often called the number of degrees of freedom). As $\varepsilon \to 0$, estimate (11) shows that the rate of convergence deteriorates, hence the method is not robust. Our goal is to design *non-uniform graded* meshes for which the estimate

$$\Phi(\varepsilon, S) \leq C(p)\varepsilon^{1/2}N^{-p}$$

(12)

holds for $N$ sufficiently large, with $C$ independent of $\varepsilon$. We will focus our attention on meshes $\Delta$ that can be defined in terms of $m$ and a *grading* function $\Gamma(x) : (0, 1) \to (0, 1)$, satisfying the following properties:

$$\Gamma \in C^1([0, 1])$$

(13)

$$\Gamma'(x) > 0 \text{ in } (0, 1)$$

(14)

$$\Gamma(0) = 0, \quad \Gamma(1) = 1$$

(15)

The uniform mesh mentioned earlier falls into this category with $\Gamma_{\text{unif}}(x) = x$.

Suppose $\Gamma(x)$ and $m \geq 1$ are given. Then the mesh $\Delta = \{x_i\}_{i=0}^m$ is defined in terms of $\Gamma(x)$ by

$$\Gamma(x_i) = \frac{i}{m}, \quad 0 \leq i \leq m$$

(16)

Meshes $\Delta$ defined by (16) via a grading function $\Gamma$ satisfying (13)–(15) will be denoted by $\Delta_\Gamma$. For any $\Gamma(x)$ satisfying (13)–(15), we will also denote

$$A_{p, \varepsilon}(\Gamma) = \left[ \int_0^1 (\Gamma'(x))^{-2p} e^{-2\varepsilon x} dx \right]^{1/2}$$

(17)

It was established in Reference [8] that as $N = mp \to \infty$,

$$\Phi(\varepsilon, S) \approx A_{p, \varepsilon}(\Gamma) \varepsilon^{-p}N^{-p}$$

(18)

where $\Gamma(x)$ satisfies (13)–(15) and $\Sigma = (\Delta_\Gamma, \bar{\rho})$ with $\bar{\rho}$ uniform. Comparing (18) and (12) we see that the asymptotically optimal rate (12) holds for the subspace $S(\Sigma)$, if and only if there exists a constant $C = C(p)$, independent of $\varepsilon$ such that

$$A_{p, \varepsilon}^2(\Gamma) = \int_0^1 (\Gamma'(x))^{-2p} e^{-2\varepsilon x} dx \leq C(p)\varepsilon^{2p+1} \quad \forall \varepsilon \in (0, 1]$$

(19)

Moreover, if $\Gamma(x)$ is chosen to be independent of $\varepsilon$, i.e. $S(\Sigma)$ is chosen to be the same space for all $\varepsilon$, then the criterion (19) fails to hold and the non-optimal rate of convergence given by (11) holds. This is the case for the uniform mesh for which $\Gamma_{\text{unif}}(x) = x$.

We will now show that there are meshes for which (19) is satisfied, so that the asymptotically optimal rate (12) will hold. In fact, we will derive an *asymptotically optimal*
mesh $\Delta_{\Gamma^*}$, corresponding to the grading function $\Gamma^*(x)$. By optimality we mean that among all $\Gamma(x)$ satisfying (13)--(15), $\Gamma^*(x)$ will be the grading function for which the error (18) is a minimum. Obviously, this is equivalent to minimizing $A^2_{p,\varepsilon}(\Gamma)$.

To this end, let $\Gamma(x) = \Gamma^*(x) + \delta\eta(x)$, where $\Gamma$ and $\Gamma^*$ satisfy (13)--(15). Then, by (14), $\eta$ vanishes at $x = 0$ and $x = 1$, which implies

$$\int_0^1 \eta'(x) \, dx = 0$$

(20)

For the minimization, we get the necessary condition $\frac{d}{d\delta}[A^2_{p,\varepsilon}(\Gamma^* + \delta\eta)]_{\delta=0} = 0$ for any $\eta$ satisfying (20). This yields

$$-2p \int_0^1 \left[ \frac{d}{dx}(\Gamma^*(x)) \right]^{-2p-1} e^{-2x/\varepsilon} \eta'(x) \, dx = 0$$

which gives

$$\frac{d}{dx}(\Gamma^*(x)) = K \exp \left[ \frac{-2x}{(2p+1)\varepsilon} \right], \quad K \in \mathbb{R}$$

(21)

Integrating (21) and using the condition (14) yields

$$\Gamma^*_{p,\varepsilon}(x) = C_{p,\varepsilon}^{-1} \left( 1 - \exp \left[ \frac{-2x}{(2p+1)\varepsilon} \right] \right)$$

(22)

where

$$C_{p,\varepsilon} = 1 - \exp \left[ \frac{-2}{(2p+1)\varepsilon} \right]$$

(23)

Let us now evaluate $A^2_{p,\varepsilon}(\Gamma^*)$. Using (21),

$$A^2_{p,\varepsilon}(\Gamma^*) = \int_0^1 K^{-2p} \exp \left[ \frac{-2x}{(2p+1)\varepsilon} \right] \, dx = K^{-2p} \frac{(2p+1)\varepsilon}{2} C_{p,\varepsilon}$$

where $K = C_{p,\varepsilon}^{-1}(2/(2p+1)\varepsilon)$. Hence, we see that

$$A^2_{p,\varepsilon}(\Gamma^*) = \left( C_{p,\varepsilon} \frac{2p+1}{2} \right)^{2p+1} \varepsilon^{2p+1} \leq \left( \frac{2p+1}{2} \right)^{2p+1} \varepsilon^{2p+1}$$

(24)

since $C_{p,\varepsilon} \leq 1$. This shows that (19) is satisfied and the asymptotically optimal rate (12) holds. The exact rate of convergence, which will be the best possible asymptotically, is given by (see Reference [8] for details)

$$\Phi(\varepsilon, S) \approx \left( C_{p,\varepsilon} \frac{2p+1}{2} \right)^{p+1/2} \varepsilon^{1/2} N^{-p}$$

(25)

Comparing (25) and (11), we see that the dependence on $\varepsilon$ is eliminated in (25). In fact, a positive power of $\varepsilon$ is present in the error estimate, which as will be illustrated in Section 4
ahead, amounts to the method performing better as $\varepsilon$ gets smaller. The meshes corresponding to (22) are very strongly graded near the origin. We refer to them as exponential meshes.

For the original problem (1) on $I = (-1, 1)$, the nodal points for the exponential mesh are obtained as follows. Construct $\{\tilde{x}_i, \ldots, \tilde{x}_{m/2}\}$ on $\tilde{I} = (0, 1)$ using

$$\Gamma^*(\tilde{x}_i) = \frac{i}{m} = -\frac{\varepsilon}{2} (2p + 1) \log \left(1 - 2 \frac{C_{p, \varepsilon} i}{m}\right), \quad i = 1, \ldots, m/2$$

with $C_{p, \varepsilon}$ as in (23). Then,

$$\Delta_{\Gamma^*} = \{-1, -1, +\tilde{x}_1, \ldots, -1 + \tilde{x}_{m/2}, 1 - \tilde{x}_1, \ldots, 1 - \tilde{x}_{m/2}, 1\}$$

Figure 1 shows an example of $\Delta_{\Gamma^*}$ with $m = 10$, for $\varepsilon = 0.025$ and $p = 1$.

The $h$ version convergence results presented here are only asymptotically optimal, i.e. for sufficiently large values of $N = mp$. Our numerical results in Section 4, however, will show that these rates are visible even for small values of $N$.

3. MESH DESIGN IN TWO DIMENSIONS

3.1. The model problem

We now consider the two-dimensional analog of (1)

$$-\varepsilon^2 \Delta u + u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$

$$u = 0 \quad \text{on } \partial \Omega$$

with $\partial \Omega$ smooth. Its variational formulation reads: find $u_\varepsilon \in H^1_0(\Omega)$ such that

$$B(u_\varepsilon, v) = F(v) \quad \forall v \in H^1_0(\Omega)$$

where

$$B(u, v) = \int_\Omega \{\varepsilon^2 \nabla u \cdot \nabla v + uv\} \, dx \, dy, \quad F(v) = \int_\Omega f v \, dx \, dy$$

The solution to (29) will, once again, contain boundary layers along the boundary of the domain. In particular, $u_\varepsilon$ can be decomposed into a smooth part, a boundary layer part and a smooth remainder. For this decomposition, we will define boundary fitted co-ordinates in a region near the boundary. With $\rho_0$ the minimum radius of curvature of $\partial \Omega$ and $\tilde{n}_z$ the outward unit normal at a point $z = z(\theta) = (X(\theta), Y(\theta)) \in \partial \Omega$, we define

$$\Omega_\rho = \{z - \rho \tilde{n}_z : z \in \partial \Omega, \ 0 < \rho < \rho_0\}$$
Then, the correspondence

\[(\rho, \theta) \rightarrow z - \rho \tilde{\eta}_z = (X(\theta) - \rho Y'(\theta), Y(\theta) + \rho X'(\theta))\]

defines the boundary fitted co-ordinates. In Reference [9] it was shown that under appropriate assumptions on the right-hand side function \(f\) (e.g. \(f \in H^{M+2}(\Omega)\) for some \(M \in \mathbb{N}_0\)),

\[u_\varepsilon = u_S + \chi u_{BL} + r_\varepsilon\]

where \(\chi \in C^\infty([0, \infty))\) is a cut-off function with \(|\chi^{(m)}(t)| \leq C(\rho_0, m)\), \(m = 0, 1, \ldots\),

\[u_S(x, y) = \sum_{i=0}^{M} \varepsilon^{2i} \Delta^{(i)} \mathcal{F}(x, y)\]

and in \(\Omega_0\)

\[u_{BL} = \sum_{i=0}^{M} S_i(\theta) \rho^i e^{-\eta/\varepsilon}\]

with \(S_i(\theta)\) smooth and independent of \(\varepsilon\). The remainder in (32) satisfies

\[\|r_\varepsilon\|_{k, \Omega} \leq C\varepsilon^{M+3/2-k}, \quad 0 \leq k \leq M + 3/2\]

where \(C \in \mathbb{R}\) is independent of \(\varepsilon\). The key observation above is that just like the one-dimensional case, if \(f\) is smooth then the difficulty in approximating \(u_\varepsilon\) lies entirely within the boundary layer term \(u_{BL}\). Moreover, (34) shows that the boundary layer effect is essentially one-dimensional, namely in a direction normal to the boundary. The design of the mesh, as described next, will take these observations into account.

### 3.2. The Finite Element Method

With \(S_N \subset H^1_0(\Omega)\) a finite-dimensional subspace of dimension \(N\), we seek \(u_N^\varepsilon \in S_N\) such that

\[B(u_N^\varepsilon, v) = F(v) \quad \forall v \in S_N\]

Once again, we have

\[\|u_\varepsilon - u_N^\varepsilon\|_{\varepsilon, \Omega} = \inf_{v \in S_N} \|u_\varepsilon - v\|_{\varepsilon, \Omega}\]

with the energy norm defined, as before, by

\[\|u\|_{\varepsilon, \Omega}^2 = \varepsilon^2 |u|_{1, \Omega}^2 + \|u\|_{0, \Omega}^2\]

The space \(S_N\) will be defined as usual: let \(\Delta = \{K_i\}_{i=1}^m\) be a regular subdivision \((\text{see Reference [12]})\) of the domain \(\Omega\) consisting of triangles and/or quadrilaterals. Denote by \(\Pi_p(K)\) the space of polynomials of \(K\) of degree \(\leq p\) in each variables if \(K\) is the reference square and of total degree \(\leq p\) if \(K\) is the reference triangle. With \(\vec{p} = (p_1, \ldots, p_m)\) the polynomial degree vector, we set

\[S_N = \{u \in H^1_0(\Omega) : u|_{K_i} \in \Pi_{\vec{p}}(K_i), \ K_i \in \Delta, \ i = 1, \ldots, m\}\]

Since boundary layer vanishes away from the boundary, the mesh over the entire domain \(\Omega\) will be constructed in two steps. First over \(\Omega_0\), given by (31), and then over \(\Omega_1 = \Omega \setminus \Omega_0\).
Figure 2. Examples of the two-dimensional mesh design near $\delta \Omega$.

We begin by dividing $\partial \Omega$ into subintervals $(\theta_j, \theta_{j+1})$, $j = 1, \ldots, m - 1$, $\theta \in \partial \Omega$ and drawing the inward normal at $\theta_j$, $j = 1, \ldots, m$ of length $\rho_0$. We then connect each point $(\rho_j, \theta_j) = (\rho_0, \theta_j)$ using the curve $\rho = \rho_0$ (= constant) and hence obtain a subdivision of $\Omega_0$ (uniform in the $\theta$ direction), consisting of quadrilaterals. Using the one-dimensional exponential mesh in the $\rho$ direction we construct the two-dimensional mesh over $\Omega_0$ in a tensor product way. The mesh over $\Omega_1$ is then chosen to be compatible with the mesh for $\Omega_0$. Figure 2 shows two such examples, over a portion of $\Omega$. In the first example $\Omega_1$ includes quadrilateral elements and in the second a uniform triangulation.

Let us briefly comment on how this choice of $S_N$ will ensure the uniform and (asymptotically) optimal approximation of $u_\varepsilon$. With $u_\varepsilon = u_S + \varepsilon u_{BL} + r_\varepsilon$ and $v = v_1 + v_2 + v_3 \in S_N$, we have

$$\left\| u_\varepsilon - u^N_\varepsilon \right\|_{\varepsilon, \Omega} = \inf_{v \in S_N} \left\| u_\varepsilon - v \right\|_{\varepsilon, \Omega} \leq \left\| u_S - v_1 \right\|_{\varepsilon, \Omega} + \left\| \varepsilon u_{BL} - v_2 \right\|_{\varepsilon, \Omega} + \left\| r_\varepsilon - v_3 \right\|_{\varepsilon, \Omega}$$

Since $w^M_\varepsilon$ and $r^M_\varepsilon$ are smooth and independent of $\varepsilon$, the first and last terms above can be bounded using usual $h$ version estimates (see e.g. Reference [12]). For the boundary layer term, we have to approximate a sum of terms of the form

$$\mathcal{S}(\theta) \rho^e e^{-\rho/\varepsilon}$$

with $\mathcal{S}(\theta)$ smooth. By properly choosing $v_2 \in S_N$ we reduce the difficulty in the approximation to that inherited from the approximation of the one-dimensional function $e^{-\rho/\varepsilon}$ (see Reference [9] for the details of an analogous argument applied for the $p/hp$ version of the FEM). By combining the previous observations, we see that the solution $u_\varepsilon$ to (29) can be uniformly approximated, without any loss in the order of convergence. As the numerical computations of the Section 4 below will show, the following optimal rate is observed:

$$\left\| u_\varepsilon - u^N_\varepsilon \right\|_{\varepsilon, \Omega} \leq C \varepsilon^{1/2} N^{-p/2}$$

where $N = O(1/h^2)$ and $C \in \mathbb{R}$ is independent of $\varepsilon$. 

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4. NUMERICAL RESULTS

4.1. One-dimensional model

In this section we present the results of numerical computations for the problem (1) with \( f(x) = 1 \) and exact solution given by

\[
    u(x) = 1 - \frac{e^{(x+1)/\varepsilon} + e^{(1-x)/\varepsilon}}{1 + e^{3\varepsilon}} \quad (37)
\]

Using the exponential mesh from Section 2, we obtain log–log plots for the percentage relative error in the energy norm

\[
    E = 100 \times \frac{\| u - u^N \|_e}{\| u \|_e}
\]

versus \( N \), the number of degrees of freedom. Figures 3 and 4 show the results for \( p = 1 \) and 2, respectively, for various values of \( \varepsilon \). The robustness of the method is clearly visible from these plots, as is the positive power of \( \varepsilon \) in the error estimate (25), i.e. the method not only does not deteriorate as \( \varepsilon \to 0 \) but actually performs better. Another important observation is that the results of Section 2 seem to hold even in the pre-asymptotic range, even though estimate (25) holds for sufficiently large \( N \).

We also compare the exponential mesh with the Shishkin mesh from Reference [4] for the same model problem. The Shishkin mesh on \((-1,1)\) is defined as follows: Let

\[
    \tau = \min \left( \frac{1}{4}, \varepsilon (p + 1) \ln(1 + m) \right)
\]

![Figure 3. Energy norm convergence for one-dimensional problem with exponential mesh.](image)

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and identify the transition points $-1 + \tau$ and $1 - \tau$. Then divide the intervals $(-1, -1 + \tau)$, $(1 - \tau, 1)$ using $m$ uniform subintervals and the interval $(-1 + \tau, 1 - \tau)$ using $2m$ uniform subintervals. This will produce a piecewise uniform mesh with a total of $4m$ elements. Figures 5 and 6 show the comparison between the Shishkin and exponential meshes for $p = 1$ and 2, respectively, and $\varepsilon = 10^{-3}, 10^{-6}$. (The behaviour of both methods for other values of $p$ and $\varepsilon$ is virtually the same as in the cases illustrated here.) Such a comparison clearly shows the difference between the quasi-optimal rate of the Shishkin mesh established in e.g. Reference [5], versus the (asymptotically) optimal one for the exponential mesh established in Reference [8].

4.2. Two-dimensional model

We now consider (28) with $f = 1$ and $\Omega$ being the unit circle. The exact solution is given (in polar coordinates) by

$$u(r, \theta) = u(r) = 1 - \frac{I_0(r/\varepsilon)}{I_0(1/\varepsilon)},$$

where $I_0(r)$ is the modified Bessel function of order 0. This is a ‘one-dimensional’ problem in $r$ and the exponential mesh (for a quarter of the domain) is given in Figure 7, in which $m = 9$, $\varepsilon = 0.1$ and $p = 1$. The same robustness and optimal convergence rates hold in this case as well, as can be seen from Figures 8 and 9. Since the problem is essentially one-dimensional, the observed rate of convergence is $O(N^{-p})$, which is better than the one given by (36). This is because $N = O(1/h)$ in this model problem, as opposed to $N = O(1/h^2)$ in the general...
Figure 5. Comparison of the exponential and Shishkin meshes for one-dimensional problem.

Figure 6. Comparison of the exponential and Shishkin meshes for one-dimensional problem.
Figure 7. Two-dimensional exponential mesh over a quarter circle.

Figure 8. Energy norm convergence for two-dimensional unit disc problem with exponential mesh.
Figure 9. Energy norm convergence for two-dimensional unit disc problem with exponential mesh.

Figure 10. Two-dimensional exponential mesh over a quarter of the unit square.
Figure 11. Energy norm convergence for two-dimensional unit square problem with exponential mesh.

Figure 12. Energy norm convergence for two-dimensional unit square problem with exponential mesh.
two-dimensional case. The presence of the positive power of $\varepsilon$ in (36) is also visible from these figures.

We finally consider (28) with $\Omega = (0,1) \times (0,1)$. In general, for $\Omega$ a polygonal domain, the solution will have corner singularities and the design of the mesh and/or method must be adjusted accordingly. To avoid this and concentrate only on the presence of boundary layers in the solution, we take $f$ to be such that the exact solution is given by

$$u(x,y) = \left(1 - \frac{e^{-x/\varepsilon} + e^{(x-1)/\varepsilon}}{1 + e^{-1/\varepsilon}}\right) \left(1 - \frac{e^{-y/\varepsilon} + e^{(y-1)/\varepsilon}}{1 + e^{-1/\varepsilon}}\right)$$

This example will illustrate how the exponential mesh in two dimensions can be very effective provided it is constructed for both the $x$ and $y$ directions in a tensor product way. Figure 10 shows the mesh for a quarter of the domain for the case of $\varepsilon = 0.1$, $p = 1$ and $m = 9$. As can be seen in Figures 11 and 12, the method is robust and the optimal convergence rate (36) is visible even in the pre-asymptotic range.

5. CONCLUSIONS

The design of a non-uniform, exponentially graded mesh was presented. It was shown that such a mesh yields the optimal convergence rates, uniformly in $\varepsilon$, when the $h$ version of the FEM is used to approximate the solution to singularly perturbed problems. In the case of two dimensions, the exponential mesh was constructed in a tensor product way and once again, uniform, optimal convergence rates were observed. When compared with other popular mesh design strategies, the exponential mesh performed in a very satisfactory way and even produced better results for the numerical examples considered here.

REFERENCES