Solution of the planar Newtonian stick–slip problem with the singular function boundary integral method

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SUMMARY

A singular function boundary integral method (SFBIM) is proposed for solving biharmonic problems with boundary singularities. The method is applied to the Newtonian stick–slip flow problem. The streamfunction is approximated by the leading terms of the local asymptotic solution expansion which are also used to weight the governing biharmonic equation in the Galerkin sense. By means of the divergence theorem the discretized equations are reduced to boundary integrals. The Dirichlet boundary conditions are weakly enforced by means of Lagrange multipliers, the values of which are calculated together with the singular coefficients. The method converges very fast with the number of singular functions and the number of Lagrange multipliers, and accurate estimates of the leading singular coefficients are obtained. Comparisons with the analytical solution and results obtained with other numerical methods are also made. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: biharmonic equation; boundary singularity; Stokes flow; stick–slip problem; Lagrange multipliers; singular coefficients

1. INTRODUCTION

In the past few decades, many different numerical methods have been proposed for the treatment of boundary singularities in plane elliptic boundary value problems, in order to improve the solution accuracy and resolve the convergence difficulties occurring in the neighbourhood of such singular points. These methods range from special mesh-refinement schemes to sophisticated techniques that incorporate, directly or indirectly, the form of the local asymptotic expansion, which is known in many occasions. An exhaustive survey of treatment of singularities in elliptic boundary value problems is provided in the recent articles by Li and Lu [1], Dosiyev [2] and Shi et al. [3]. Knowledge of the coefficients appearing in the local solution expansion is often desired in many engineering applications. These coefficients, referred to as...
singular coefficients or generalized stress intensity factors [4], are calculated either directly (see Reference [5] and references therein) or by post-processing the numerical solution [6, 7].

In the past few years, we have developed the singular function boundary integral method (SFBIM) for Laplacian problems with boundary singularities [5, 8, 9], in which the unknown singular coefficients are calculated directly. The solution is approximated by the leading terms of the local asymptotic solution expansion which are also used to weight the governing equation in the Galerkin sense. With a double application of Green’s theorem, the discretized equations are reduced to boundary integrals over those parts of the boundary that do not involve the singular point. The Dirichlet boundary conditions are weakly enforced by means of Lagrange multipliers, which are calculated simultaneously with the singular coefficients. The method has been tested on standard Laplacian problems, yielding extremely accurate estimates of the leading singular coefficients, and exhibiting exponential convergence with respect to the number of singular functions [5, 8, 9].

The objective of the present paper is to extend the SFBIM to biharmonic problems with boundary singularities. For that purpose we have chosen to solve the Newtonian planar stick–slip problem, which is a benchmark Stokes flow problem used to test various numerical methods proposed in the literature for the solution of viscous and non-Newtonian flows, such as the extrude–swell flow. This concerns the extrusion of a liquid from a slit or an axisymmetric die into the atmosphere. Due to the relaxation of stresses, the fluid swells as it exits the die. Swelling is particularly pronounced in the case of elastic fluids, but it is also observed in the Newtonian case, provided that the Reynolds number is sufficiently low. The stick–slip problem is a special case of the extrude–swell problem: in the limit of infinite surface tension, no swelling occurs, and the free surface becomes flat (in the case of slit die). A boundary inverse-square-root stress singularity appears at the exit of the die due to the sudden change of the boundary conditions from the wall to the flat free surface, which is the cause of numerical difficulties that become more severe in the case of non-Newtonian or viscoelastic fluids [10, 11].

The creeping planar stick–slip problem was solved analytically by Richardson who used a Wiener–Hopf technique [12] and by Sturges who used the method of matched eigenfunction expansions [13]. Both methods have been used by Trogdon and Joseph [14] to obtain analytical solutions for the round stick–slip problem.

Direct estimates of the leading singular coefficients in the case of the planar stick–slip problem have been reported by various researchers who employed a variety of numerical methods and techniques to incorporate the leading terms of the local asymptotic solution (which is equivalent to subtracting the leading terms of the singularity). Kelmanson employed a direct modified boundary integral equation method (BIEM) incorporating a subtraction of the singular terms technique that accelerated the rate of convergence, and reported estimates for the leading four coefficients [15]. Estimates for these coefficients have also been reported by Georgiou et al. [16] who solved the problem using the integrated singular basis function method (ISBFM). In this method, the singular functions are directly subtracted from the original problem formulation which leads to a modified problem with the regular part of the solution and the singular coefficients as unknowns. The smooth problem is then solved using finite elements. The integrals involving singular contributions are reduced to boundary ones by means of a double integration by parts and the original essential boundary conditions are enforced by means of Lagrange multipliers. These two features are encountered also with the SFBIM that we propose in the present work. However, the two methods are quite different.
for the following reasons:

(a) With the ISBFM the problem is formulated in terms of the two velocity components and the pressure, while with the SFBIM it is formulated in terms of the streamfunction.

(b) In the ISBFM the unknown fields are the smooth parts of the primary variables which are approximated by means of standard polynomial basis functions. In the SFBIM, however, there is no subtraction of the singularity and the unknown field, i.e. the stream function, is approximated as a linear combination of the leading terms of the local asymptotic expansion. It is clear that such an approximation is valid only if the domain of the problem falls within the domain of convergence of the local solution.

(c) In the ISBFM, the discretized equations are double integrals, whereas in the SFBIM, these are boundary ones. Hence, the dimension of the problem is reduced by one, and the computational cost of the SFBIM is considerably lower than that of the ISBFM.

Karageorghis [17] obtained direct estimates of the first four singular coefficients using a modified method of fundamental solutions (MFS) that was based on the direct subtraction of the leading terms of the singular local solution. A similar method was also used by Poullikkas et al. [18] who subtracted only the leading term of the singular local solution assuming that its form is unknown and part of the problem.

Georgiou et al. [19] developed a singular finite element method (SFEM), in which special elements incorporating the radial form of the local singularity expansion are employed in a small region around the singular point, in order to resolve the convergence difficulties and improve the accuracy of the global solution. They obtained more accurate results than those achieved with ordinary elements and calculated the leading singular coefficients by post-processing the finite element solution. A similar post-processing technique has been employed by Salamon et al. [20] who obtained accurate results near the singularity using high resolution finite elements with quasi-orthogonal mesh generation and local, adaptive mesh refinement with irregular imbedded elements. Ngamaramvaranggul and Webster [21] developed a semi-implicit Taylor–Galerkin/pressure-correction finite element method (STGFEM) for free surface flows and applied it to various Newtonian flows including the plane and axisymmetric stick–slip and extrudate–swell problems. More recently, Normandin et al. [22] solved the Newtonian stick–slip problem using a finite element Galerkin technique associated with stream-tube analysis and presented comparisons of the computed streamlines with previous results.

The stick–slip flow of non-Newtonian and viscoelastic fluids has also received considerable attention due to the convergence difficulties associated with the presence of the singularity and the relevance to the extrudate–swell problem. We discuss briefly the literature with emphasis on works reporting results for the Newtonian case. Tanner and Huang [10] applied the J-integral method for solving the planar stick–slip flow of power-law fluids, corrected the numerical estimate of singularity strength in Richardson’s Newtonian analysis [12], and calculated the first singular coefficient for various power-law exponents. Owens and Phillips [23] presented solutions of the planar stick–slip problem obtained with a spectral domain decomposition method (DDM) for an Oldroyd-B fluid. In a subsequent article, they applied an algebraic mapping to treat the flow domain without truncation and computed the singular coefficients using a post-processing technique [24]. Baaijens investigated the numerical stability of a discontinuous Galerkin method using a Phan–Thien–Tanner model incorporating monotonicity enforcement [25] and applied low-order discontinuous Galerkin methods to solve the

The rest of the paper is organized as follows: in Section 2, the Newtonian planar stick–slip problem is introduced and the governing equations and the local asymptotic solution expansion are presented. The SFBIM is developed in Section 3, where four different formulations of the method corresponding to different techniques of imposing the Dirichlet boundary conditions are presented. The numerical results are given in Section 4, where the fast convergence of the method with respect to the number of singular functions is demonstrated and comparisons are made with the analytical solution [12] and the results obtained with the boundary integral equation method of Kelmanson [15], the spectral domain decomposition method of Owens and Phillips [24], the high-resolution finite element method of Salamon et al. [20], the STGFEM of Ngamaramvaranggul and Webster [21], and other methods. The conclusions are summarized in Section 5.

2. GOVERNING EQUATIONS AND ASYMPTOTIC SOLUTION

The planar stick–slip problem is the idealization of the extrusion of a Newtonian fluid between parallel plates at infinite surface tension. The geometry of the flow is depicted in Figure 1. Due to symmetry, only the upper half of the flow domain is considered, i.e. boundary part $S_D$ denotes the plane of symmetry. Boundary parts $S_A$ and $S_B$ represent the wall and the flat free surface, respectively. The latter is flat in the limit of infinite surface tension. Finally, $S_C$ and $S_E$ are, respectively, the artificial inlet and outlet boundaries.

In the creeping case, the flow is governed by the biharmonic equation [27]:

$$\nabla^4 \psi = 0 \quad \text{in } \Omega$$

(1)

where $\psi$ is the streamfunction defined by

$$u_x \equiv \frac{\partial \psi}{\partial y} \quad \text{and} \quad u_y \equiv -\frac{\partial \psi}{\partial x}$$

(2)

$u_x$ and $u_y$ being the velocity components in the $x$ and $y$ directions, respectively.

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The boundary conditions of the flow are also depicted in Figure 1. Along the wall $S_A$, there is no slip and no penetration (i.e. the two velocity components are zero). Along the free surface, both $u_y$ and the $xy$-stress component are zero which leads to $\nabla^2 u = 0$. The inflow and outflow planes are taken at a distance $L$ before and after the die exit. This distance is assumed to be sufficiently large so that the flow corresponds to the fully developed Poiseuille flow at the inflow plane and to a plug (i.e. uniform) flow at the outflow plane. Finally, along the symmetry plane, the vertical velocity component and the shear stress are zero, i.e. the centreline is a slip surface. The stick–slip flow is characterized by the presence of a stress singularity at the exit $O$ caused by the sudden change in the boundary conditions, from no slip (stick) along the wall $S_A$ to full slip along the flat free surface $S_B$.

After using the transformation $\psi = u + 1$, the problem of Figure 1 is transformed as follows [23]:

$$\nabla^4 u = 0 \quad \text{in } \Omega$$

with

$$u = 0, \quad \frac{\partial u}{\partial y} = 0 \quad \text{on } S_A$$
$$u = 0, \quad \nabla^2 u = 0 \quad \text{on } S_B$$
$$\left. \frac{\partial \nabla^2 u}{\partial x} \right|_{S_C} = 0, \quad \frac{\partial u}{\partial x} = 0 \quad \text{on } S_C$$
$$u = -1, \quad \nabla^2 u = 0 \quad \text{on } S_D$$
$$u = \frac{1}{2} y(3 - y^2) - 1, \quad \frac{\partial u}{\partial x} = 0 \quad \text{on } S_E$$

The transformed problem is also shown in Figure 2. Note that the weak condition

$$\left. \frac{\partial (\nabla^2 u)}{\partial x} \right|_{S_C} = 0$$

along $S_C$ can be replaced by the stronger Dirichlet condition

$$u = y - 1$$

which leads to a different formulation, since with the SFBIM, imposing Dirichlet conditions requires the introduction of (unknown) Lagrange multipliers.
The asymptotic solution in the neighbourhood of the singularity can be expressed in terms of an eigenfunction expansion of the form [12,15,16]:

\[ u(r, \theta) = \sum_{j=1}^{\infty} a_j r^{\mu_j+1} f(\theta, \mu_j), \quad (r, \theta) \in \Omega \]  

where \((r, \theta)\) are the polar co-ordinates centred at the singular point, \(\mu_j\), with \(j = 1, 2, \ldots\), are the singularity powers arranged in ascending order, the functions \(f(\theta, \mu_j)\) represent the \(\theta\)-dependence of the eigensolution, and \(x_j\) are the unknown singular coefficients determined by the global flow.

The functions \(W_j \equiv r^{\mu_j+1} f(\theta, \mu_j)\) are referred to as singular functions. The local solution (7) consists of even and odd solutions, the corresponding singular functions of which will be denoted by \(W^e_j\) and \(W^o_j\), respectively. In the case of even solutions [12],

\[ W^e_j \equiv r^{\mu_j+1} f_1(\theta, \mu_j) \]  

with

\[ f_1(\theta, \mu_j) = \cos(\mu_j + 1) \theta - \cos(\mu_j - 1) \theta, \quad \mu_j = j - \frac{1}{2}, \quad j = 1, 2, \ldots \]  

whereas in the case of odd solutions,

\[ W^o_j \equiv r^{\mu_j+1} f_2(\theta, \mu_j) \]  

with

\[ f_2(\theta, \mu_j) = (\mu_j - 1) \sin(\mu_j + 1) \theta - (\mu_j + 1) \sin(\mu_j - 1) \theta, \quad \mu_j = j + 1, \quad j = 1, 2, \ldots \]  

Thus the first singular function is

\[ W^e_1 = r^{3/2} \left( \cos \frac{3\theta}{2} - \cos \frac{\theta}{2} \right) \]

which indicates that the velocity gradients and the stresses vary as the inverse square root of the radial distance from the singular point.

In what follows we will be using the symbols \(x_j\) and \(\beta_j\) for the singular coefficients corresponding to the even and odd singular functions, respectively. Thus, the local solution is written as follows:

\[ u = \sum_{j=1}^{\infty} x_j W^e_j + \sum_{j=1}^{\infty} \beta_j W^o_j \]  

3. THE SFBIM

In the SFBIM [5] the solution of problem (3)–(4) is approximated by the leading terms of the local solution expansion (12). By employing the first \(N_x\) terms in both sums of (12) the approximate solution \(\tilde{u}\) is

\[ \tilde{u} = \sum_{j=1}^{N_x} \tilde{x}_j W^e_j + \sum_{j=1}^{N_x} \tilde{\beta}_j W^o_j \]
where \( \tilde{z}_j \) and \( \tilde{\beta}_j \) are the approximations of the singular coefficients. Obviously, the total number of singular functions involved in approximation (13) is \( 2N_{\alpha} \).

By applying Galerkin’s principle, the governing equation is weighted by the singular functions used in the approximation of the solution. Hence, the following set of discretized equations is obtained:

\[
\int_{\Omega} \nabla^4 \tilde{u} W_k^i \, dV = 0, \quad i = 1, 2, \ldots, N_{\alpha}, \quad k = 1, 2 \quad (14)
\]

By applying Green’s theorem twice and taking into account that the singular functions \( W_k^i \) are biharmonic, the above volume integrals are reduced to boundary ones:

\[
\int_{\partial \Omega} \left( \frac{\partial \tilde{u}}{\partial n} \nabla^2 W_k^i - \tilde{u} \frac{\partial (\nabla^2 W_k^i)}{\partial n} \right) \, dS + \int_{\partial \Omega} \left( \frac{\partial (\nabla^2 \tilde{u})}{\partial n} W_k^i - \nabla^2 \tilde{u} \frac{\partial W_k^i}{\partial n} \right) \, dS = 0, \quad i = 1, 2, \ldots, N_{\alpha}, \quad k = 1, 2 \quad (15)
\]

where \( \partial \Omega = S_A \cup S_B \cup S_C \cup S_D \cup S_E \). The dimension of the problem is, thus, reduced by one, which leads to a considerable reduction of the computational cost. Since \( W_k^i \) satisfy exactly the boundary conditions along \( S_A \) and \( S_B \), the above integral along these boundary segments, is identically zero. Therefore,

\[
\int_{S_A \cup S_B \cup S_E} \left( \frac{\partial \tilde{u}}{\partial n} \nabla^2 W_k^i - \tilde{u} \frac{\partial (\nabla^2 W_k^i)}{\partial n} \right) \, dS + \int_{S_C \cup S_D \cup S_E} \left( \frac{\partial (\nabla^2 \tilde{u})}{\partial n} W_k^i - \nabla^2 \tilde{u} \frac{\partial W_k^i}{\partial n} \right) \, dS = 0,
\]

\( i = 1, 2, \ldots, N_{\alpha}, \quad k = 1, 2 \quad (16) \)

In the SFBIM the Dirichlet boundary conditions are imposed by means of Lagrange multipliers which replace the normal derivative of the solution \( u \). In the problem under study Dirichlet boundary conditions appear only along boundary parts \( S_D \) and \( S_E \). Since along \( S_E \) the normal derivative \( \partial u/\partial x \) vanishes, Lagrange multipliers are chosen to replace \( \partial (\nabla^2 u)/\partial x \) in the boundary integrals of Equation (16). Boundary parts \( S_D \) and \( S_E \) are partitioned into three-node elements and the corresponding Lagrange multipliers, denoted respectively by \( \lambda_D \) and \( \lambda_E \), are expanded in terms of quadratic basis functions \( M^j \):

\[
\lambda_D = \frac{\partial \tilde{u}}{\partial n} = \sum_{j=1}^{N_{\alpha}} \lambda_D^j M^j \quad \text{on} \quad S_D \quad (17)
\]

and

\[
\lambda_E = \frac{\partial (\nabla^2 \tilde{u})}{\partial n} = \sum_{j=1}^{N_{\alpha}} \lambda_E^j M^j \quad \text{on} \quad S_E \quad (18)
\]

where \( N_{\alpha} \) and \( N_{\alpha} \) are the numbers of the discrete Lagrange multipliers \( \lambda_D \) and \( \lambda_E \) along the corresponding boundaries. The nodal values of \( \lambda_D \) and \( \lambda_E \) are additional unknowns of the problem. The required \( N_{\alpha} + N_{\alpha} \) additional equations are obtained by weighting the Dirichlet boundary conditions along \( S_D \) and \( S_E \) by the quadratic basis functions \( M^j \) in the Galerkin...
sense. The following linear system of $2N_a + N\lambda_D + N\lambda_E$ discretized equations is thus obtained:

$$
\int_{S_c} \left( -\bar{u} \frac{\partial (\nabla^2 W_k^i)}{\partial x} - \nabla^2 \bar{u} \frac{\partial W_k^i}{\partial x} \right) \, dy \\
+ \int_{S_D} \left( -\lambda_D \nabla^2 W_k^i + \bar{u} \frac{\partial (\nabla^2 W_k^i)}{\partial y} - \frac{\partial (\nabla^2 \bar{u})}{\partial y} \right) \, dx \\
+ \int_{S_E} \left( -\lambda_E W_k^i + \bar{u} \frac{\partial (\nabla^2 W_k^i)}{\partial x} + \nabla^2 \bar{u} \frac{\partial W_k^i}{\partial x} \right) \, dy = 0, \quad i = 1, \ldots, N_a, \quad k = 1, 2
$$

(19)

$$
\int_{S_D} \bar{u} M^i \, dx = -\int_{S_D} M^i \, dx, \quad i = 1, 2, \ldots, N\lambda_D
$$

(20)

$$
\int_{S_E} \bar{u} M^i \, dy = \int_{S_E} \left[ \frac{1}{2} y(3 - y^2) - 1 \right] M^i \, dy, \quad i = 1, 2, \ldots, N\lambda_E
$$

(21)

The above linear system is not symmetric. This can be written in block form as follows:

$$
\begin{bmatrix}
K & K_D' & K_E \\
K_D'' & O & O \\
K_E^T & O & O
\end{bmatrix}
\begin{bmatrix}
X_{\bar{z}, \bar{\beta}} \\
\Lambda_D \\
\Lambda_E
\end{bmatrix} =
\begin{bmatrix}
O \\
B \\
C
\end{bmatrix}
$$

(22)

where $X_{\bar{z}, \bar{\beta}}$, $\Lambda_D$ and $\Lambda_E$ are the vectors of the unknowns:

$$
X_{\bar{z}, \bar{\beta}} = [\bar{z}_1, \ldots, \bar{z}_{N_z}, \bar{\beta}_1, \ldots, \bar{\beta}_{N_{\beta}}]^T
$$

$$
\Lambda_D = [\lambda_D^1, \lambda_D^2, \ldots, \lambda_D^{N\lambda_D}]^T
$$

$$
\Lambda_E = [\lambda_E^1, \lambda_E^2, \ldots, \lambda_E^{N\lambda_E}]^T
$$

It should be noted that the integrands in the above equations are non-singular and all integrations are carried out far from the boundaries causing the singularity. Note that the stiffness matrix is not symmetric and that it becomes singular if $N\lambda > 2N_a$ where $N\lambda = N\lambda_D + N\lambda_E$. The above formulation will be referred to as Formulation A. We have also considered three alternative formulations which are briefly discussed below.

3.1. Formulation B

The only difference between this formulation and Formulation A is that function $\lambda_D$ along boundary part $S_D$ replaces the normal derivative of the Laplacian of $u$:

$$
\lambda_D = \frac{\partial (\nabla^2 \bar{u})}{\partial y} = \sum_{j=1}^{N\lambda_D} \lambda_D^j M^i \quad \text{on } S_D
$$

(23)
instead of the normal derivative of \( u \). Therefore, the only change in the formulation is in the boundary integral along the centreline plane, \( S_D \), which becomes:

\[
\int_{S_D} \left( -\lambda_D W^i_k + \bar{u} \frac{\partial (\nabla^2 W^i_k)}{\partial y} - \frac{\partial \bar{u}}{\partial y} \nabla^2 W^i_k \right) \, dx
\]

In contrast to formulation A, the resulting linear system of equations is symmetric. As before the stiffness matrix is singular if \( N_\lambda > 2N_a \).

### 3.2. Formulation C

In this formulation, the weak boundary condition \( \partial \nabla^2 u / \partial x = 0 \) along \( S_C \) is replaced by

\[
u = y - 1 \quad \text{on} \quad S_C
\]  

(24)

The use of this essential boundary condition requires the introduction of an additional Lagrange multiplier function, \( \lambda_C \), which replaces the normal derivative of the Laplacian of \( u \) and is expressed in terms of quadratic basis functions \( M^j \):

\[
\lambda_C = \frac{\partial (\nabla^2 \bar{u})}{\partial x} = \sum_{j=1}^{N_{\lambda_C}} \lambda^j_C M^j \quad \text{on} \quad S_C
\]

(25)

As is the case with the other essential boundary conditions, condition (24) is weighted by means of the basis functions \( M^j \). A linear system of \( 2N_a + N_{\lambda_C} + N_{\lambda_D} + N_{\lambda_E} \) equations is thus obtained:

\[
\int_{S_C} \left( \lambda_C W^i_k - \bar{u} \frac{\partial (\nabla^2 W^i_k)}{\partial x} - \nabla^2 \bar{u} \frac{\partial W^i_k}{\partial x} \right) \, dy \\
+ \int_{S_D} \left( -\lambda_D \nabla^2 W^i_k + \bar{u} \frac{\partial (\nabla^2 W^i_k)}{\partial y} - \frac{\partial (\nabla^2 \bar{u})}{\partial y} W^i_k \right) \, dx \\
+ \int_{S_E} \left( -\lambda_E W^i_k + \bar{u} \frac{\partial (\nabla^2 W^i_k)}{\partial x} + \nabla^2 \bar{u} \frac{\partial W^i_k}{\partial x} \right) \, dy = 0, \quad i = 1, \ldots, N_a, \quad k = 1, 2
\]

(26)

\[
\int_{S_C} \bar{u} M^i \, dy = \int_{S_C} (y - 1) M^i \, dy, \quad i = 1, 2, \ldots, N_{\lambda_C}
\]

(27)

\[
\int_{S_D} \bar{u} M^i \, dx = - \int_{S_D} M^i \, dx, \quad i = 1, 2, \ldots, N_{\lambda_D}
\]

(28)

\[
\int_{S_E} \bar{u} M^i \, dy = \int_{S_E} \left[ \frac{1}{2} y(3 - y^2) - 1 \right] M^i \, dy, \quad i = 1, 2, \ldots, N_{\lambda_E}
\]

(29)

The above linear system is not symmetric. The stiffness matrix is singular if \( N_\lambda > 2N_a \) where here \( N_\lambda = N_{\lambda_C} + N_{\lambda_D} + N_{\lambda_E} \).
3.3. Formulation D

As in Formulation B, the Lagrange multiplier function $\lambda_D$, used to impose the Dirichlet boundary conditions along $S_D$, replaces the normal derivative of the Laplacian of $u$ (Equation (23)). The resulting linear system of discretized equations is the same as that of Formulation C except from the integral along $S_D$ in Equation (25), which becomes:

$$\int_{S_D} \left( -\lambda_D W_k^l + \bar{u} \frac{\partial (\nabla^2 W_k^l)}{\partial y} - \frac{\partial \bar{u}}{\partial y} \nabla^2 W_k^l \right) d\Gamma$$

As in formulation B, the system of the discretized equations is symmetric.

4. NUMERICAL RESULTS

Calculations have been carried out with all four formulations presented in Section 3. In order to implement the SFBIM, the boundary parts $S_C$, $S_D$ and $S_E$ (i.e. the boundary parts away from the singularity) are subdivided into quadratic elements. Specifically, we employ $N_E$ elements over each one of boundaries $S_E$, $S_C$ and $N_D$ elements over boundary $S_D$. Thus, the total number of Lagrange multipliers in formulations A and B is $N_{\lambda} = N_{\lambda E} + N_{\lambda D}$ and in formulations C and D is $N_{\lambda} = N_{\lambda C} + N_{\lambda D} + N_{\lambda E} = 2N_{\lambda E} + N_{\lambda D}$ (where $N_{\lambda E} = 2N_E + 1$ and $N_{\lambda D} = 2N_D + 1$). As in Reference [5], the integrals in all formulations are calculated numerically by subdividing each quadratic element into 10 subintervals and using a 15-point Gauss–Legendre quadrature over each subinterval. Unless otherwise indicated, the semi-length $L$ of the domain has been taken equal to 3.

As already mentioned, the number of the singular functions $2N_s$ should be much greater than the number of Lagrange multipliers $N_{\lambda}$, since otherwise the stiffness matrix is ill-conditioned or singular. On the other hand, large values of $2N_s$ should be avoided because the contributions of the high-order singular functions become either negligible (for $r<1$) or very large (for $r>1$) beyond the limits double precision can handle.

Systematic runs have been carried out in order to study the effects of both $N_{\lambda}$ and $N_{\lambda}$ on the numerical results. The effect of $2N_s$ on the leading singular coefficients can be observed in Tables I and II, which show results obtained using formulation A with $N_{\lambda} = 32$. Fast convergence is observed as $2N_s$ is increased and accurate estimates of the leading singular coefficients are obtained. However, at very high values of $2N_s$ (i.e. above $2N_s = 88$) slow divergence is observed due to the inaccuracies introduced by the high-order singular functions.

The convergence of the method with the number of Lagrange multipliers is shown in Tables III and IV which show the values of the leading singular coefficients calculated with $2N_s = 88$ and various values of $N_{\lambda} = N_{\lambda D} + N_{\lambda E}$. Again, fast convergence is observed initially but as $N_{\lambda}$ approaches the value of $2N_s$, the results start diverging slowly, which is attributed to the fact that the stiffness matrix becomes ill-conditioned. Our computations showed that the optimal values of $N_{\lambda}$ and $2N_s$ are $N_{\lambda} = 32$ and $2N_s = 88$. For higher values of $2N_s$ (e.g. $2N_s = 120$) satisfactory values of the singular coefficients are still obtained, but the quality of the global solution is not very good.

An indication of the quality of the solution is given by the smoothness of the calculated Lagrange multipliers. Thus, for the optimal combination $N_{\lambda} = 32$ and $2N_s = 88$ with formulation A, the calculated Lagrange multiplier function along boundary $S_D$ is smooth. As shown in
the same. Similar observations are made in Figures 3(b)–(d) with the results of formulations
plier function exhibits oscillations, while the values of the singular coe/SOcients are essentially
edge, there are no reports in the literature for the values of the higher-order coe/SOcients. The
2

Table I. Convergence of the leading singular coefficients \( z_i \) with \( 2N_z \);
\( N_z = 32 \), formulation A.

<table>
<thead>
<tr>
<th>( 2N_z )</th>
<th>( z_1 )</th>
<th>( z_2 )</th>
<th>( z_3 )</th>
<th>( z_4 )</th>
<th>( z_5 )</th>
<th>( z_{10} )</th>
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</thead>
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<td>0.2645007</td>
<td>0.030376</td>
<td>-0.021407</td>
<td>-0.002900</td>
<td>0.00022</td>
</tr>
<tr>
<td>86</td>
<td>0.6909882</td>
<td>0.2645004</td>
<td>0.030374</td>
<td>-0.021405</td>
<td>-0.002892</td>
<td>0.00021</td>
</tr>
<tr>
<td>88</td>
<td>0.6909882</td>
<td>0.2645004</td>
<td>0.030374</td>
<td>-0.021404</td>
<td>-0.002891</td>
<td>0.00021</td>
</tr>
<tr>
<td>90</td>
<td>0.6909882</td>
<td>0.2645002</td>
<td>0.030375</td>
<td>-0.021403</td>
<td>-0.002895</td>
<td>0.00021</td>
</tr>
<tr>
<td>92</td>
<td>0.6909885</td>
<td>0.2645045</td>
<td>0.030371</td>
<td>-0.021436</td>
<td>-0.002875</td>
<td>0.00034</td>
</tr>
<tr>
<td>100</td>
<td>0.6909884</td>
<td>0.2644998</td>
<td>0.030371</td>
<td>-0.021401</td>
<td>-0.002878</td>
<td>0.00020</td>
</tr>
<tr>
<td>110</td>
<td>0.6909883</td>
<td>0.2645004</td>
<td>0.030374</td>
<td>-0.021405</td>
<td>-0.002891</td>
<td>0.00021</td>
</tr>
<tr>
<td>120</td>
<td>0.6909883</td>
<td>0.2645003</td>
<td>0.030374</td>
<td>-0.021404</td>
<td>-0.002890</td>
<td>0.00021</td>
</tr>
<tr>
<td>130</td>
<td>0.6909882</td>
<td>0.2645003</td>
<td>0.030375</td>
<td>-0.021404</td>
<td>-0.002897</td>
<td>0.00020</td>
</tr>
<tr>
<td>140</td>
<td>0.6909873</td>
<td>0.2644989</td>
<td>0.030387</td>
<td>-0.021393</td>
<td>-0.002943</td>
<td>0.00019</td>
</tr>
<tr>
<td>150</td>
<td>0.6909883</td>
<td>0.2645008</td>
<td>0.030373</td>
<td>-0.021407</td>
<td>-0.002886</td>
<td>0.00021</td>
</tr>
</tbody>
</table>

Table II. Convergence of the leading singular coefficients \( \beta_i \), with \( 2N_z \);
\( N_z = 32 \), formulation A.

<table>
<thead>
<tr>
<th>( 2N_z )</th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
<th>( \beta_3 )</th>
<th>( \beta_4 )</th>
<th>( \beta_5 )</th>
<th>( \beta_{10} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>70</td>
<td>-0.0808635</td>
<td>-0.017115</td>
<td>0.001726</td>
<td>0.001231</td>
<td>-0.000282</td>
<td>-0.000001</td>
</tr>
<tr>
<td>80</td>
<td>-0.0808617</td>
<td>-0.017119</td>
<td>0.001720</td>
<td>0.001240</td>
<td>-0.000270</td>
<td>-0.000006</td>
</tr>
<tr>
<td>86</td>
<td>-0.0808619</td>
<td>-0.017119</td>
<td>0.001720</td>
<td>0.001238</td>
<td>-0.000271</td>
<td>-0.000005</td>
</tr>
<tr>
<td>88</td>
<td>-0.0808619</td>
<td>-0.017119</td>
<td>0.001720</td>
<td>0.001238</td>
<td>-0.000271</td>
<td>-0.000005</td>
</tr>
<tr>
<td>90</td>
<td>-0.0808617</td>
<td>-0.017119</td>
<td>0.001720</td>
<td>0.001238</td>
<td>-0.000270</td>
<td>-0.000005</td>
</tr>
<tr>
<td>92</td>
<td>-0.0808645</td>
<td>-0.017122</td>
<td>0.001729</td>
<td>0.001245</td>
<td>-0.000287</td>
<td>-0.000009</td>
</tr>
<tr>
<td>100</td>
<td>-0.0808621</td>
<td>-0.017117</td>
<td>0.001721</td>
<td>0.001234</td>
<td>-0.000273</td>
<td>-0.000003</td>
</tr>
<tr>
<td>110</td>
<td>-0.0808619</td>
<td>-0.017119</td>
<td>0.001720</td>
<td>0.001238</td>
<td>-0.000272</td>
<td>-0.000005</td>
</tr>
<tr>
<td>120</td>
<td>-0.0808619</td>
<td>-0.017118</td>
<td>0.001720</td>
<td>0.001237</td>
<td>-0.000271</td>
<td>-0.000005</td>
</tr>
<tr>
<td>130</td>
<td>-0.0808617</td>
<td>-0.017119</td>
<td>0.001720</td>
<td>0.001239</td>
<td>-0.000270</td>
<td>-0.000006</td>
</tr>
<tr>
<td>140</td>
<td>-0.0808590</td>
<td>-0.017121</td>
<td>0.001711</td>
<td>0.001231</td>
<td>-0.000258</td>
<td>-0.000008</td>
</tr>
<tr>
<td>150</td>
<td>-0.0808623</td>
<td>-0.017119</td>
<td>0.001721</td>
<td>0.001237</td>
<td>-0.000273</td>
<td>-0.000005</td>
</tr>
</tbody>
</table>

Figure 3(a), for a slightly different value of \( N_z \) (i.e. \( N_z = 36 \)), the calculated Lagrange multiplier function exhibits oscillations, while the values of the singular coefficients are essentially the same. Similar observations are made in Figures 3(b)–(d) with the results of formulations B–D. Recall here that in formulations A and C, \( \lambda_D \) replaces \( \partial u / \partial y \) while in formulations B and D it replaces \( \partial (\nabla^2 u) / \partial y \). From the results of Figure 3, it is clear that formulations A and C are more stable. Since it does not require additional Lagrange multipliers along boundary \( S_C \), formulation A is to be preferred. The converged values of the leading singular coefficients with all formulations are depicted in Table V. Note the slight differences in the values of \( 2N_z \) and \( N_z \) required for convergence. In Table VI, the values of \( z_1 \), \( z_2 \), \( z_3 \) and \( \beta_1 \), calculated using formulation A, are compared with values reported in the literature. To our knowledge, there are no reports in the literature for the values of the higher-order coefficients. The value 0.690988 for \( z_1 \) agrees with the analytical solution up to the sixth significant digit and is much more accurate than previously reported values. The values of the other three coefficients.
compare well with numerical results reported in the literature, especially with those calculated by the spectral domain decomposition method of Owens and Phillips [24].

Once the streamfunction is known, the two velocity components are directly calculated by means of Equation (2). Plots of the axial velocity, \( u_x \), along the slip surface \( (y = 1) \) and the centreline \( (y = 0) \), computed using formulation A with \( 2N_s = 88 \) and \( N_s = 32 \), are shown in Figure 4. The calculated slip surface velocity is in good agreement with the predictions of Kelmanson [15] and Owens and Phillips [24], while the centreline velocity agrees very well with the high-resolution finite element predictions of Salamon et al. [20], which is also indicated in Table VII, where the values of \( u_x \) at three points of the domain are compared. A comparison against the analytical solution along the symmetry plane (as calculated by Ngamaramvaranggul and Webster [21]) is provided in Table VIII. The small differences observed are due to the fact that the domain used by Richardson [12] was shorter (\( L = 2 \)). However, the predictions of the present method are still much closer to the analytical solution than those of the STGFEM of Ngamaramvaranggul and Webster [21]. Finally, as illustrated in Table IX, the calculated value of \( u_x \) at (0.2,1) compares well with the analytical solution (based on the graphical information recorded in Reference [12]), and is better than the
Figure 3. Converged (solid) and oscillatory (dashed) Lagrange multiplier functions along boundary $S_D$ calculated with formulations A–D. 
(a) Formulation A: $N_j = 32$ (solid) and 36 (dashed) $2N_s = 88$; 
(b) Formulation B: $N_j = 32$ (solid) and 36 (dashed) $2N_s = 90$; 
(c) Formulation C: $N_j = 39$ (solid) and 43 (dashed) $2N_s = 90$; and 
(d) Formulation D: $N_j = 39$ (solid) and 43 (dashed) $2N_s = 90$. 
Table V. Converged values of singular coefficients with formulations A–D.

<table>
<thead>
<tr>
<th>Singular coefficient</th>
<th>Formulation A</th>
<th>Formulation B</th>
<th>Formulation C</th>
<th>Formulation D</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \beta_1 )</td>
<td>0.690988</td>
<td>0.690988</td>
<td>0.690989</td>
<td>0.690989</td>
</tr>
<tr>
<td>( \beta_2 )</td>
<td>0.264500</td>
<td>0.264500</td>
<td>0.264500</td>
<td>0.264500</td>
</tr>
<tr>
<td>( \beta_3 )</td>
<td>0.03037</td>
<td>0.03037</td>
<td>0.03037</td>
<td>0.03037</td>
</tr>
<tr>
<td>( \beta_4 )</td>
<td>-0.02140</td>
<td>-0.02140</td>
<td>-0.02140</td>
<td>-0.02140</td>
</tr>
<tr>
<td>( \beta_5 )</td>
<td>-0.00289</td>
<td>-0.00289</td>
<td>-0.00289</td>
<td>-0.00289</td>
</tr>
<tr>
<td>( \beta_6 )</td>
<td>0.00423</td>
<td>0.00423</td>
<td>0.00423</td>
<td>0.00423</td>
</tr>
<tr>
<td>( \beta_7 )</td>
<td>0.00042</td>
<td>0.00041</td>
<td>0.00041</td>
<td>0.00041</td>
</tr>
<tr>
<td>( \beta_8 )</td>
<td>-0.00093</td>
<td>-0.00093</td>
<td>-0.00093</td>
<td>-0.00093</td>
</tr>
<tr>
<td>( \beta_9 )</td>
<td>-0.00007</td>
<td>-0.00007</td>
<td>-0.00007</td>
<td>-0.00007</td>
</tr>
<tr>
<td>( \beta_{10} )</td>
<td>0.00002</td>
<td>0.0002</td>
<td>0.0002</td>
<td>0.0002</td>
</tr>
</tbody>
</table>

Table VI. Comparison of computed singular coefficients (formulation A with \( 2N_z = 88 \) and \( N_z = 32 \)) with the results of other numerical methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>( \beta_1 )</th>
<th>( \beta_2 )</th>
<th>( \beta_3 )</th>
<th>( \beta_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Modified BIEM(^*) [15]</td>
<td>0.69108</td>
<td>0.26435</td>
<td>0.04962</td>
<td>-0.07990</td>
</tr>
<tr>
<td>Singular FEM(^†) [19]</td>
<td>0.69173</td>
<td>0.27168</td>
<td>0.05013</td>
<td></td>
</tr>
<tr>
<td>ISBFM [16]</td>
<td>0.69104</td>
<td>0.26140</td>
<td>-0.01263</td>
<td></td>
</tr>
<tr>
<td>Modified MFS [17]</td>
<td>0.690984</td>
<td>0.274807</td>
<td>-0.022104</td>
<td>-0.043983</td>
</tr>
<tr>
<td>J-integral method [10]</td>
<td>0.6910</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Spectral DDM(^†) [24]</td>
<td>0.69035</td>
<td>0.26404</td>
<td>0.03069</td>
<td>-0.08051</td>
</tr>
<tr>
<td>High-resolution FEM(^†) [20]</td>
<td>0.69160</td>
<td>0.27183</td>
<td>0.05232</td>
<td></td>
</tr>
<tr>
<td>Modified MFS(^‡) [18]</td>
<td>0.69019</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SFBIM (present work)</td>
<td>0.690988</td>
<td>0.264500</td>
<td>0.03037</td>
<td>-0.080862</td>
</tr>
<tr>
<td>Analytical solution [12]</td>
<td>0.690988</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\(^*\) Extrapolated values.
\(^†\) Singular coefficients obtained by post-processing the numerical solution.
\(^‡\) The best reported estimate is listed.

The singular boundary element (SBEM) solution of Ingham and Kelmanson [28], the SFEM and ISBFM results of Georgiou et al. [16, 19] (as calculated by Ngamaramvaranggul and Webster [21]), and the STGFEM result [21]. It should be noted that Ngamaramvaranggul and Webster [21] used an incorrect value for the analytical solution (0.618040 instead of 0.572).

The pressure corresponding to the local solution (12) is given by [27]

\[
p(x, y) = p_1(x, y) - p_0
\]
Figure 4. Calculated axial velocity: (a) along the slip surface \( y = 1 \); and (b) along the centreline \( y = 0 \); formulation A with \( 2N_x = 88 \) and \( N_z = 32 \).

Table VII. Comparison of computed axial velocities (formulation A with \( 2N_x = 88 \) and \( N_z = 32 \)) with the high-resolution finite element predictions of Salamon et al. [20].

<table>
<thead>
<tr>
<th></th>
<th>( u_x(0,0.975) )</th>
<th>( u_x(0,0) )</th>
<th>( u_x(1,1) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Salamon et al. [20]</td>
<td>0.234855</td>
<td>1.34150</td>
<td>0.944542</td>
</tr>
<tr>
<td>Present work</td>
<td>0.234840</td>
<td>1.34151</td>
<td>0.944528</td>
</tr>
</tbody>
</table>

where

\[
p_1 = -4 \left[ \sum_{j=1}^{\infty} \left( j - \frac{1}{2} \right) x_j r^{j-(3/2)} \sin \left( j - \frac{3}{2} \right) \theta - \sum_{j=1}^{\infty} (j + 1)(j + 2) \beta_j r^j \cos(j \theta) \right] (31)
\]
Table VIII. Computed velocities along the symmetry plane (formulation A, $2N_x = 88$, $N_j = 32$) compared with other results in the literature.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>−1.0</td>
<td>1.4964</td>
<td>1.4959</td>
<td>1.4965</td>
</tr>
<tr>
<td>−0.8</td>
<td>1.4899</td>
<td>1.4892</td>
<td>1.4903</td>
</tr>
<tr>
<td>−0.6</td>
<td>1.4758</td>
<td>1.4749</td>
<td>1.4766</td>
</tr>
<tr>
<td>−0.4</td>
<td>1.4484</td>
<td>1.4479</td>
<td>1.4499</td>
</tr>
<tr>
<td>−0.2</td>
<td>1.4027</td>
<td>1.4035</td>
<td>1.4051</td>
</tr>
<tr>
<td>0.2</td>
<td>1.2798</td>
<td>1.2701</td>
<td>1.2665</td>
</tr>
<tr>
<td>0.4</td>
<td>1.1967</td>
<td>1.2006</td>
<td>1.1929</td>
</tr>
<tr>
<td>0.6</td>
<td>1.1308</td>
<td>1.1403</td>
<td>1.1310</td>
</tr>
<tr>
<td>0.8</td>
<td>1.0834</td>
<td>1.0996</td>
<td>1.0848</td>
</tr>
<tr>
<td>1.0</td>
<td>1.0516</td>
<td>1.0702</td>
<td>1.0530</td>
</tr>
</tbody>
</table>

Table IX. Velocity results at $x = 0.2$ on the slip surface.

<table>
<thead>
<tr>
<th>Method</th>
<th>Velocity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Analytical [12]</td>
<td>0.572</td>
</tr>
<tr>
<td>SBEM [28]</td>
<td>0.572608</td>
</tr>
<tr>
<td>SFEM [19]</td>
<td>0.571896</td>
</tr>
<tr>
<td>ISBFM [16]</td>
<td>0.571259</td>
</tr>
<tr>
<td>STGFEM [21]</td>
<td>0.619786</td>
</tr>
<tr>
<td>SFBIM</td>
<td>0.571958</td>
</tr>
</tbody>
</table>

Table X. Variation of the value of $p(0^−, 1)$ with the semi-length $L$ of the domain; formulation A with $2N_x = 88$ and $N_j = 32$.

<table>
<thead>
<tr>
<th>$L$</th>
<th>$p(0^−, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>1.3452</td>
</tr>
<tr>
<td>2.2</td>
<td>1.3454</td>
</tr>
<tr>
<td>2.4</td>
<td>1.3454</td>
</tr>
<tr>
<td>2.6</td>
<td>1.3454</td>
</tr>
<tr>
<td>2.8</td>
<td>1.3451</td>
</tr>
<tr>
<td>3.0</td>
<td>1.3454</td>
</tr>
<tr>
<td>3.2</td>
<td>1.3444</td>
</tr>
<tr>
<td>3.4</td>
<td>1.3454</td>
</tr>
</tbody>
</table>

and $p_0 = p(L, 1)$ so that the pressure at $(L, 1)$ is zero. It is clear that the pressure $p$ is at most as accurate as $p_0$, the accuracy of which deteriorates as the semi-length $L$ of the domain increases, given that the contributions of the singular functions become larger. This effect is illustrated in Table X, where we show the variation of $p(0^−, 1)$ with $L$. Note that $p_1(0^−, 1) = 0$ and thus

$$p(0^−, 1) = - p_0 = - p(L, 1)$$
Figure 5. (a) Calculated pressure along $y = 1$; and (b) calculated centreline pressure (continuous curve) compared with the analytical results by Richardson (points) [12]; formulation A with $2N_x = 88$ and $N_x = 32$.

Table XI. Comparison of the computed centreline pressure at $x = 0$ (formulation A with $2N_x = 88$ and $N_x = 32$) with the results of other numerical methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>$p(0,0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Finite differences [29]</td>
<td>1.100</td>
</tr>
<tr>
<td>STGFEM [21]</td>
<td>0.9980</td>
</tr>
<tr>
<td>SFBIM (present work)</td>
<td>1.0349</td>
</tr>
<tr>
<td>Analytical solution [12]</td>
<td>1.0348</td>
</tr>
</tbody>
</table>

In Figure 5(a), the pressure along the wall and the slip surface ($y = 1$) is plotted. This is in good agreement with the results of Salamon et al. [20]. Due to the singularity, the pressure goes to minus infinity as the die exit is approached from the right, while it remains finite.
Figure 6. Computed streamlines and contours of the two velocity components and the pressure; formulation A with $2N_s = 88$ and $N_A = 32$. (a) Stream-function, $\psi$; (b) horizontal velocity component, $u_x = \partial \psi / \partial y$; (c) vertical velocity component, $u_y = -\partial \psi / \partial x$; and (d) pressure, $p$.

Table XII. Converged values of the leading singular coefficients with formulation A, $N_A = 32$ and different values of the semi-length $L$ of the domain; the value of $2N_s$ ranges from 78 to 88, depending on the value of $L$.

<table>
<thead>
<tr>
<th>$L$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$\beta_1$</th>
<th>$\beta_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.0</td>
<td>0.690973</td>
<td>0.264524</td>
<td>-0.080870</td>
<td>-0.017112</td>
</tr>
<tr>
<td>2.1</td>
<td>0.690980</td>
<td>0.264514</td>
<td>-0.080866</td>
<td>-0.017115</td>
</tr>
<tr>
<td>2.2</td>
<td>0.690984</td>
<td>0.264508</td>
<td>-0.080864</td>
<td>-0.017117</td>
</tr>
<tr>
<td>2.3</td>
<td>0.690986</td>
<td>0.264504</td>
<td>-0.080863</td>
<td>-0.017117</td>
</tr>
<tr>
<td>2.4</td>
<td>0.690987</td>
<td>0.264503</td>
<td>-0.080862</td>
<td>-0.017118</td>
</tr>
<tr>
<td>2.5</td>
<td>0.690988</td>
<td>0.264502</td>
<td>-0.080862</td>
<td>-0.017118</td>
</tr>
<tr>
<td>2.6</td>
<td>0.690988</td>
<td>0.264501</td>
<td>-0.080862</td>
<td>-0.017118</td>
</tr>
<tr>
<td>2.7</td>
<td>0.690988</td>
<td>0.264501</td>
<td>-0.080862</td>
<td>-0.017118</td>
</tr>
<tr>
<td>2.8</td>
<td>0.690988</td>
<td>0.264500</td>
<td>-0.080862</td>
<td>-0.017118</td>
</tr>
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<td>2.9</td>
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Figure 7. Convergence of $\alpha_1$ and $\beta_1$ with the semi-length $L$ of the domain: formulation A, $N_\delta = 32$.

(and positive) for negative $x$. The negative value of the pressure implies that the fluid in the extrudate region is subjected to tension. Note that the discontinuity cannot be captured by standard numerical methods in which a continuous approximation is used for the pressure. In Figure 5(b), the computed centreline pressure is compared with the analytical results of Richardson [12] (as given by Ngamaramvaranggul and Webster [21]). The agreement is excellent, which is also seen in Table XI, where the computed value of $p(0,0)$ is compared with the analytical value and those of other numerical methods.

The streamlines as well as the contours of the two velocity components ($u_x$ and $u_y$) and the pressure, computed using formulation A with $2N_\delta = 88$ and $N_\delta = 32$, are shown in Figure 6. These show the re-adjustment of the flow from a parabolic to a uniform velocity profile and agree well with previous results in the literature [16,20,22]. The maximum value of $u_y$ is 0.19364 which compares well with the value of 0.1936 provided by Salamon et al. [20]. (Ngamaramvaranggul and Webster [21] computed the peak value to be 0.17.) According to our calculations, the maximum occurs approximately at the point $(0.109,0.712)$.

The effect of the length of the domain on the computations has also been studied. Table XII depicts the converged values of the first four singular coefficients for $N_\delta = 32$ and different values of the semi-length, $L$, of the domain. As expected, the values of the singular coefficients change dramatically with $L$ for small values of the latter, since the assumptions for fully developed and uniform flow along the inlet and outlet planes, respectively, are not valid when the two planes are taken close to the die exit. This effect is illustrated in Figure 7, where
the calculated values of \( \alpha_1 \) and \( \beta_1 \) are plotted versus \( L \). We observe that the value \( L = 3 \) is sufficiently high to assure the validity of the imposed inlet and outlet boundary conditions. At higher values of \( L \), the accuracy of the computed solutions starts deteriorating, due to the fact that the number \( N_{\alpha_0} \) of the corresponding Lagrange multipliers is kept fixed while the length of the boundary \( S_D \) increases. As already noted, increasing \( N_{\alpha_0} \) will not improve the accuracy, since it leads to ill-conditioning of the stiffness matrix.

5. CONCLUSIONS

The singular function boundary integral method (SFBIM) has been developed for solving a biharmonic problem with a boundary singularity, i.e. the Newtonian planar stick–slip problem in terms of the streamfunction. The solution is approximated by means of the leading singular functions defined by the local asymptotic solution expansion around the singularity. Hence, the method is restricted to Stokes problems with a boundary singularity for which the local solution is available. The proposed approximation is valid only if the domain of the problem is a subset of the domain of convergence of the local solution. If this is not the case, the domain can be partitioned into subdomains over which separate approximations, that obey appropriate compatibility conditions along the interfaces, may be used.

The main features of the SFBIM are as follows:

(a) The singular coefficients are calculated directly.
(b) The governing biharmonic equation is weighted by the singular functions in the Galerkin sense.
(c) The discretized equations are reduced to boundary integrals by means of a double application of the divergence theorem, which leads to a considerable reduction of the computational cost.
(d) The Dirichlet boundary conditions are weakly enforced by means of Lagrange multipliers which may replace either \( \partial u / \partial n \) or \( \partial ( \nabla^2 u ) / \partial n \) in the integrands of the discretized equations. The Lagrange multipliers are calculated together with the singular coefficients.

Four different formulations of the SFBIM, corresponding to different techniques of imposing the Dirichlet boundary conditions, have been investigated. Even though all formulations give about the same results, using a weaker instead of a Dirichlet condition along the outflow plane is a much better choice, since the number of Lagrange multipliers, \( N_l \) must be much lower than the number of singular functions, \( 2N_\alpha \) in order to avoid ill-conditioning of the stiffness matrix. Moreover, the best choice for the Lagrange multipliers along the symmetry plane is to replace the normal derivative of the solution and not the normal derivative of its Laplacian.

The SFBIM converges very fast with the number of singular functions and the number of Lagrange multipliers, and accurate estimates of the leading singular coefficients are obtained. In particular, the value 0.690988 for the leading singular coefficient agrees well with the analytical solution up to the sixth significant digit. The effect of the length of the domain on the values of the leading singular coefficients has also been investigated. Finally, the numerical results for the velocity components and the pressure compare very well with the analytical solution of Richardson [12] and the high-resolution finite element results of Salamon et al. [20].

REFERENCES