The \( h_p \) Finite Element Method for Singularly Perturbed Problems in Smooth Domains

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We consider the numerical approximation of singularly perturbed elliptic problems in smooth domains. The solution to such problems can be decomposed into a smooth part and a boundary layer part. We present guidelines for the effective resolution of boundary layers in the context of the \( h_p \) finite element method and we construct tensor product spaces that approximate these layers uniformly at a near-exponential rate.

Keywords: Boundary layer, singularly perturbed problem, \( h_p \) version.

1. Introduction

We study the finite element approximation of elliptic boundary value problems whose solutions contain boundary layers. Boundary layers are rapidly varying solution components that have support in a narrow neighborhood of the boundary of the domain. They arise in the study of plates and shells in structural mechanics and in heat transfer problems with small thermal coefficients. In 11, it was shown that boundary layers are prominent in the study of shells, where their effective resolution was of outmost importance. Also, in 1, boundary layers arising in plate theory (and in particular in the Reissner-Mindlin plate model) was discussed. The results presented here provide the groundwork for future directions in the approximation of boundary layers arising in plate and shell theory.

In one-dimension, boundary layers have the general form

\[
u_\varepsilon(x) = \exp(-x/\varepsilon), \quad x \in I \subset \mathbb{R}
\]  

(1.1)

where \( \varepsilon \in (0, 1] \) is a small parameter that can approach zero. Solutions to singularly perturbed elliptic boundary value problems contain functions such as (1.1). The approximation of these functions was studied in 13, where an \( h_p \) finite element method scheme with robust convergence properties was presented. In particular, it was shown that adding an extra element of size \( O(\varepsilon) \) near the boundary gives an \( h_p \) scheme that approximates boundary layers uniformly in \( \varepsilon \) at an exponential
rate of convergence. (Here $\varepsilon$ is the thickness of the layer and $p$ is the degree of the approximating polynomial.) The importance of the results in $^3$ lies in the fact that solutions to two-dimensional problems exhibit similar effects that tend to be of one-dimensional nature. That is, the solution contains components of the type (1.1), where $x$ denotes the distance normal to the boundary, c.f. $^1$, $^9$.

In this paper, we extend the one-dimensional ideas to two-dimensional problems over smooth domains, whose solutions contain parts of the form

$$ u_{HL}(\theta, \rho) = C(\theta) e^{-\rho/\varepsilon}, \quad (1.2) $$

where $C(\theta)$ is smooth. In (1.2), $\theta, \rho$ denote, respectively, the arc length and the normal distance to the boundary, of a point $x$ in a neighborhood of $\partial \Omega$. Using the results of $^3$ we will construct two-dimensional tensor product finite element spaces in the $(\rho, \theta)$ coordinates, for approximating functions of the type (1.2). We will consider a singularly perturbed elliptic boundary value problem and provide a decomposition for its solution into a smooth part, a boundary layer part (like (1.2)) and a smooth remainder. Using tensor product spaces, we will approximate each component of the solution, obtaining an arbitrary high algebraic convergence rate, uniform in the energy norm. Finally, we will illustrate the validity of our results through numerical computations in the case when $\Omega$ is the unit disk. (The results presented here are taken from $^{16}$.)

Throughout this paper, $H^k(\Omega)$ will denote the Sobolev space of order $k \in \mathbb{N}_0$ on a domain $\Omega \subset \mathbb{R}^2$, with $H^0(\Omega) = L^2(\Omega)$, and $\| \cdot \|_{k, \Omega}$, $| \cdot |_{k, \Omega}$ denoting the norm and seminorm as usual. Whenever there is no confusion we omit the subscript $\Omega$. The set $C^n(\Omega)$ will denote continuous functions with $n$ continuous derivatives and $\Pi_p(\Omega)$ will denote the set of polynomials of degree $\leq p$ in each variable, over $\Omega$. The letters $K, C$ (without any dependencies on variables) will be used to denote generic positive constants, possibly not the same in each occurrence.

2. The Model Problem and its Regularity

We will consider the singularly perturbed model boundary value problem

$$ L_\varepsilon u^\varepsilon \equiv -\varepsilon^2 \Delta u^\varepsilon + u^\varepsilon = f \text{ in } \Omega, $$

$$ u^\varepsilon = 0 \text{ on } \partial \Omega, $$

where $\Omega \subset \mathbb{R}^2$ is a smooth domain with $\partial \Omega$ analytic, and $\varepsilon \in (0,1]$. (This model problem is the two-dimensional analog of the boundary value problem considered in $^3$.) We cast (2.1) into an equivalent variational form that reads: Find $u^\varepsilon \in H^1_0(\Omega)$ such that

$$ B_\varepsilon(u^\varepsilon, v) = F(v) \quad \forall \ v \in H^1_0(\Omega), $$

where

$$ B_\varepsilon(u, v) = \int_\Omega \{ \varepsilon^2 \nabla v \nabla u + uv \} \, dx dy, $$

$$ F(v) = \int_\Omega f v \, dx dy. $$

and

\[ F(v) = \int_{\Omega} f \, dx \, dy. \quad (2.4) \]

For every \( f \in L^2(\Omega) \) there exists a unique solution \( u^\varepsilon \in H^1_0(\Omega) \), to (2.2), for every \( \varepsilon \in (0,1] \). We will show that \( u^\varepsilon \) can be decomposed into a smooth part \( u^\varepsilon_M \), a boundary layer part \( u^{\text{BL}}_M \) and a smooth remainder. For this purpose, we define boundary fitted coordinates in a region

\[ \Omega_0 = \{ z - \rho \hat{m}_z^n \mid z \in \partial \Omega, 0 < \rho < \rho_0 < \text{min. radius of curvature of } \partial \Omega \}, \quad (2.5) \]

near the boundary, by the correspondence

\[ (\rho, \theta) \rightarrow z - \rho \hat{m}_z^n = \left( X(\theta) - \rho X'(\theta), Y(\theta) + \rho Y'(\theta) \right). \]

In (2.5), \( z = z(\theta) = (X(\theta), Y(\theta)) \) and \( \hat{m}_z^n \) denotes the outward unit normal at \( z \in \partial \Omega \) (see Figure 1). We have the following decomposition theorem whose proof appears in the appendix (see also \(^8\) for other decompositions).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{boundary_fitted_coordinates.png}
\caption{Boundary fitted coordinates.}
\end{figure}

**Theorem.** Let \( u^\varepsilon \) satisfy (2.2) and assume \( f \in H^{4M+2}(\Omega) \) for some \( M \in \mathbb{N} \). Then

\[ u^\varepsilon = w^\varepsilon_M + \chi u^{\text{BL}}_M + r^\varepsilon_M, \quad (2.6) \]

where \( \chi \in C^\infty([0,\infty)) \) is a cutoff function such that

\[ \chi(r) = \begin{cases} 1 & \text{for } 0 < r \leq \rho_0/3 \\ 0 & \text{for } r \geq 2\rho_0/3 \end{cases} \quad (2.7) \]

with \( |\chi^{(m)}(r)| \leq C(\rho_0, m), m = 0, 1, \ldots \),

\[ w^\varepsilon_M(x, y) = \sum_{i=0}^{M} \varepsilon^{2i} \Delta^{(i)} f(x, y), \quad (2.8) \]
and in $\Omega_0$,
\[
    u^{BL}_M = \sum_{i=0}^{M} \varepsilon^i \sum_{k=0}^{i} C_{k,i}(\theta) \rho_k^{i} e^{\rho_k^{i}} = \sum_{i=0}^{M} C_{i}(\theta) \rho^{i} e^{\rho^{i}},
\]
with $c \in \mathbb{R}$ a constant independent of $\varepsilon$ and $C_{k,i}(\theta)$ smooth and independent of $\varepsilon$.

\[\|r^\varepsilon_M\|_{k,\Omega} \leq c\varepsilon^{M+3/2} k, \quad 0 \leq k \leq M+3/2\]

3. The Finite Element Method

The finite element approximation of (2.2) consists of finding $u^N \in S^N$ such that
\[
    B_\varepsilon (u^N, v) = F(v) \quad \forall v \in S^N,
\]
where $S^N \subset H^1_0 (\Omega)$, is a finite dimensional subspace of dimension $N$. For every $\varepsilon \in (0,1]$, there exists a unique solution $u^N \in S^N$ to (3.1) satisfying
\[
    \|u^\varepsilon - u^N\|_{\varepsilon,\Omega} = \inf_{v \in S^N} \|u^\varepsilon - v\|_{\varepsilon,\Omega},
\]
where the energy norm is defined as
\[
    \|u\|^2 = B_\varepsilon (u, u) = \varepsilon^2 \|u\|^2_{1,\Omega} + \|u\|^2_{0,\Omega}.
\]

Theorem gives a representation for the solution $u^\varepsilon$ to (2.1), so that the design of a finite element space to approximate $u^\varepsilon$ will consist of separate spaces for the approximation of $u^\varepsilon_M$, $u^{BL}_M$, $r^\varepsilon_M$ given by (2.8), (2.9) and (2.10) respectively. Thus, we will design specific finite element spaces $S^N_{BL}, S^N_w, S^N_r$ for the approximation of $u^\varepsilon_M$, $u^\varepsilon_w, r^\varepsilon_M$, so that by (2.6) we will have an approximation for $u^\varepsilon$. Then we have for any $v_1 \in S^N_{BL}, v_2 \in S^N_w, v_3 \in S^N_r$ such that $v = (v_1 + v_2 + v_3) \in S^N$,
\[
    \|u^\varepsilon - v\|_{\varepsilon,\Omega} \leq \|u^\varepsilon_M - v_1\|_{\varepsilon,\Omega} + \|\chi u^\varepsilon_M - v_2\|_{\varepsilon,\Omega} + \|r^\varepsilon_M - v_3\|_{\varepsilon,\Omega}.
\]

Note that $M \in \mathbb{N}$ is chosen so that $f \in H^{4M+2} (\Omega)$. If $f \in \Pi_p (\Omega)$, then we could choose $M = p$. In particular, if $f$ is constant, we choose $M = 0$, which gives $w^\varepsilon_M = u^\varepsilon_C = f(x,y)$ and
\[
    u^{BL}_M = u^{BL}_0 = C_0 (\theta) e^{\rho^0},
\]
which is precisely the function (1.2).

4. The Boundary Layer Approximation

First, we design the space $S^N_{BL}$ for the approximation of $\chi u^{BL}_M$, with $u^{BL}_M$ given by (2.9), over $\Omega \subset \mathbb{R}^2$ with $\partial\Omega$ analytic. To this end, let $\Omega_0$ be given by (2.5) and set $\Omega_1 = \Omega \setminus \Omega_0$, as seen in Figure 2. In practice, $\Omega_0$ could be selected in the following way. Divide $\partial\Omega$ into subintervals $(\theta_{j}, \theta_{j+1})$, $j = 1,...,m - 1$, $\theta \in \partial\Omega$, and draw the inward normal at $\theta_j$, $j = 1,...,m$, of length $\rho_0$. Then, connect each
point \((\rho_j, \theta_j) = (\rho_0, \theta_j)\) using the curve \(\rho = \rho_0\) =constant. Further, divide each \(\Omega^j_i = \{(\rho, \theta) : 0 \leq \rho \leq \rho_0, \theta_j \leq \theta \leq \theta_{j+1}\}, j = 1, \ldots, m\) into \(\Omega^j_{i,1}, \Omega^j_{i,2}\), where

\[
\Omega^j_{i,1} = \left\{(\rho, \theta) : \theta_j \leq \theta \leq \theta_{j+1}, 0 \leq \rho \leq \frac{1}{2} \rho_0 \kappa \varepsilon \right\}, \\
\Omega^j_{i,2} = \Omega^j_i \setminus \Omega^j_{i,1}.
\]

In the above construction, \(\kappa \in \mathbb{R}\) is a fixed constant, \(p\) is the degree of the approximating polynomial, and we assume that \(\kappa \varepsilon < 1/2\). This will define a mesh \(\Delta_0 = \{\Omega^j_{i,1}, \Omega^j_{i,2}\}_{j=1}^m\) over \(\Omega_0\). Next, let \(\{\Omega_i^j\}_{i=1}^n\) be some subdivision of \(\Omega_1\), that is compatible with \(\Delta_0\), and define the mesh \(\Delta = \{\Omega^j_{i,1}, \Omega^j_{i,2}, \Omega_i^j, j = 1, \ldots, m, i = 1, \ldots, n\}\). (4.3)

\[\xi = \frac{2}{L}\theta - 1, \ \eta = \frac{2}{\rho_0}\rho - 1, \quad (4.4)\]

Next, we consider a typical element \(\Omega^j_i \subset \Omega_0\), and map it to the master element \(J = (-1,1)^2\), as follows. Recall that \(\rho \in [0, \rho_0], \theta \in [0, L]\) for some \(\rho_0, L \in \mathbb{R}\). Then, the linear mapping (in \(\rho\) and \(\theta\))

\[u_M^{RL} (\xi, \eta) = e^\frac{\rho_0 (\eta + 1)}{\varepsilon^k} \sum_{i=0}^M \varepsilon^i \sum_{k=0}^i C_{ik} \left(\frac{L}{2} (\xi + 1)\right) \left(\frac{\rho_0 (\eta + 1)}{\varepsilon^k}\right)^k \]

Under the mapping (4.4), \(u_M^{RL}\) defined by (2.9) becomes

\[u_M^{RL} (\xi, \eta) = e^\frac{\rho_0 (\eta + 1)}{\varepsilon^k} \sum_{i=0}^M \varepsilon^i \sum_{k=0}^i C_{ik} (\xi) \frac{\pi_k (\eta)}{\varepsilon^k}, \]
where \( \pi_k(\eta) \) is a polynomial of degree \( k \) in \( \eta \) and

\[
\tilde{C}_{ki}(\xi) = \sum_{j=0}^{i} \sum_{\ell=0}^{j} \alpha_{ikj\ell} \left( \frac{L}{2} (\xi + 1) \right) \left( \left[ \Delta^{(j)} f \right]_{\partial \Omega} \right),
\]

with \( \alpha_{ikj\ell}(\xi) \) smooth. In the case when \( f \) is constant, i.e. \( M = 0 \), we have

\[
\tilde{u}_C^{BL}(\xi,\eta) = \tilde{C}_C(\xi) e^{\frac{\mu_1(\eta+1)}{2}},
\]

and in the general case of \( f \in H^{2M+2}(\Omega) \), we have

\[
\tilde{u}_M^{BL}(\xi,\eta) = e^{\frac{\mu_1(\eta+1)}{2}} \sum_{i=k=0}^{M} \sum_{i} \tilde{C}_{ik}(\xi) \varepsilon^i k \pi_k(\eta) =: e^{\frac{\mu_1(\eta+1)}{2}} C(\xi,\eta).
\]

The finite element space must be chosen so that both the exponential term \( \exp(-\rho_{0}(\eta+1)/(2\varepsilon)) \) and \( C(\xi,\eta) \) are approximated at a sufficiently fast rate. For the approximation of \( \tilde{C}_{ik}(\xi) \) we shall assume that

\[
\tilde{C}_{ik}(\xi) \text{ is analytic with all derivatives bounded independently of } \varepsilon.
\]

Then, by the results of \(^3\), we have that there exist \( K_2, K_3 \in \mathbb{R} \) and \( 0 < \beta < 1 \), such that

\[
\left\| \tilde{C}_{ik}(\xi) \right\|_{\varepsilon, I} \leq K_2, \ C^p_\varepsilon(I) \leq K_3 \beta^p,
\]

where \( I = (-1,1) \),

\[
C^p_\varepsilon(I) := \left\| \frac{d^i}{d\xi^i} \left( \tilde{C}_{ki}(\xi) - \tilde{r}(\xi) \right) \right\|_{\varepsilon, I}, i = 0,1,
\]

and \( \tilde{r}(\xi) \) is the \( H^1 \) projection of \( \tilde{C}_{ki}(\xi) \) onto \( \Pi_p(I) \).

We begin the construction of the space \( S_{BL}^N \) by noting that since \( \chi = 0 \) in \( \Omega \setminus \Omega_0 \),

\[
\left\| \chi u_M^{BL} - v_2 \right\|_{\varepsilon, \Omega} = \left\| \chi u_M^{BL} - v_2 \right\|_{\varepsilon, \Omega_0} + \left\| \chi u_M^{BL} - v_2 \right\|_{\varepsilon, \Omega \setminus \Omega_0} = \left\| \chi u_M^{BL} - v_2 \right\|_{\varepsilon, \Omega_0},
\]

provided \( v_2 = 0 \) in \( \Omega \setminus \Omega_0 \). Thus, we define the space

\[
S_{BL}^N(\Sigma) = \left\{ v \in H^1(\Omega) : v|_{\Omega_0 \setminus \Gamma} \in \Pi_p(\Omega_0 \setminus \Gamma), v|_{\Gamma} = 0, \right\}
\]

(4.10)

\[
\Sigma = (\Delta, \bar{p}), \Delta = \left( \Omega_0^j, \Omega_1^j, j = 1, \ldots, m, i = 1, \ldots, n, \ell = 1, 2 \right), \bar{p} = \{ p, p, \ldots, p \},
\]

where \( \bigcup_{j=1}^{m} \Omega_0^j = \bigcup_{j=1}^{m} \left( \Omega_0^j \cup \Omega_0^j \right) = \Omega_0, \bigcup_{i=1}^{n} \Omega_1^i = \Omega_1 \), as given by (4.3),(2.5). In (4.10), we define \( \Pi_p \) as follows:

\[
\Pi_p(\Omega_0) = \left\{ \pi(\rho, \theta) : \pi(\rho, \theta) = r(\rho) q(\theta) \text{ with } r, q \text{ polynomials of degree } \leq p \text{ over } \Omega_0 \text{ and } p > 2M \right\}.
\]

We have the following result.
Lemma 4.1 Let \( u^BL_M \) be given by (4.6) satisfy (4.7) and let \( S^N_{BL} (\Sigma) \) be the space defined by (4.10). Then for any \( s > 0 \), there exists \( v_2 \in S^N_{BL} \) such that
\[
\| \chi u^BL_M - v_2 \|_{C, \Omega} \leq C \varepsilon^p s^p,
\]
where \( C = C(s) \in \mathbb{R} \) is independent of \( p \) and \( \varepsilon \), but depends on \( s \).

**Proof.** Note that by (4.9), we have for any \( v \in S^N_{BL} \)
\[
\| \chi u^BL_M - v \|_{C, \Omega} = \| \chi u^BL_M - v \|_{C, \Omega_0} + \| u^BL_M - v \|_{C, \Omega_0 \setminus \Omega_{C, \chi}}.
\]

Consider \( u^BL_M \) given by (2.9), over an element \( \Omega_\ell \subset \Delta \), \( \Omega_\ell \subset \Omega_0 \setminus \Omega_{C, \chi} \), and note that under the linear mapping (4.4), \( u^BL_M \) becomes \( u^BL_M (\xi, \eta) \) as given by (4.6), over the square \( J = (-1,1)^2 \). Let \( v_2 \in S^N_{BL} \) be such that \( v_2 (\rho, \theta) = r(\rho)q(\theta) \), with \( q(\theta) = \sum_{k=1}^M d_k(\theta) q_k^2(\theta) \), and \( q_k^2(\theta) \) are polynomials of degree \( \leq p/2 \) in \( \theta \). Under the mapping (4.4), \( v_2 (\rho, \theta) \) becomes \( \bar{v}_2 (\xi, \eta) \) and we have
\[
\| u^BL_M - v_2 \|_{C, \Omega_\ell} = \| \bar{u}^BL_M - \bar{v}_2 \|_{C, J} = \| C (\xi, \eta) e^{\frac{\rho(\eta)}{2}} - \bar{r}(\xi) q(\eta) \|_{C, J}.
\]

Set \( U_k(\eta) : = e^{\frac{\rho(\eta)}{2}} k \pi_k(\eta) \) and \( \bar{q}_k(\eta) : = q_k(\eta) q_k^2(\eta) \). Then,
\[
\| e^{\frac{\rho(\eta)}{2}} C_{ik} (\xi) e^{\frac{\rho(\eta)}{2}} k \pi_k(\eta) - \bar{r}(\xi) q_k^2(\eta) \|_{C, J} =
\]
\[
= \| \bar{C}_{ik}(\xi) U_k(\eta) - \bar{r}(\xi) \bar{q}_k(\eta) \|_{C, J}
\]
\[
= \| \bar{C}_{ik}(\xi) U_k(\eta) - \bar{C}_{ik}(\xi) \bar{q}_k(\eta) + \bar{C}_{ik}(\xi) \bar{q}_k(\eta) - \bar{r}(\xi) \bar{q}_k(\eta) \|_{C, J}
\]
\[
\leq \| \bar{C}_{ik}(\xi) U_k(\eta) - \bar{C}_{ik}(\xi) \bar{q}_k(\eta) \|_{C, J} + \| \bar{C}_{ik}(\xi) \bar{q}_k(\eta) - \bar{r}(\xi) \bar{q}_k(\eta) \|_{C, J}
\]
\[
\leq \| \bar{C}_{ik}(\xi) \|_{C, J} \| U_k(\eta) - \bar{q}_k(\eta) \|_{C, J} + \| \bar{q}_k(\eta) \|_{C, J} \| \bar{C}_{ik}(\xi) - \bar{r}(\xi) \|_{C, J}
\]
\[
\leq K \left\{ \| \bar{C}_{ik}(\xi) \|_{C, J} \| U_k(\eta) - \bar{q}_k(\eta) \|_{C, J} + \| \bar{q}_k(\eta) \|_{C, J} \| \bar{C}_{ik}(\xi) - \bar{r}(\xi) \|_{C, J} \right\}
\]
where \( K \in \mathbb{R} \) and \( I = (-1,1) \). Since
\[
\| \bar{q}_k(\eta) \|_{C, J} \leq \| U_k(\eta) - \bar{q}_k(\eta) \|_{C, J} + \| U_k(\eta) \|_{C, J},
\]
and
\[
\| \bar{C}_{ik}(\xi) - \bar{r}(\xi) \|_{C, J} \approx \varepsilon \| \bar{C}_{ik}(\xi) - \bar{r}(\xi) \|_{1, I} + \| \bar{C}_{ik}(\xi) - \bar{r}(\xi) \|_{C, I}
\]
\[
\approx \varepsilon C^p_\rho I + C^p_\rho I
\]
we further obtain
\[
\left\| \bar{C}_{ik} (\xi) U_{ik}(\eta) - \bar{r} (\xi) \bar{q}_{ik}(\eta) \right\|_{\infty, I} \leq K \left\{ \left\| \bar{C}_{ik} (\xi) \right\| \left\| U_{ik}(\eta) - \bar{q}_{ik}(\eta) \right\|_{\infty, I} + \right.
\]
\[
+ \left( \left\| U_{ik}(\eta) - \bar{q}_{ik}(\eta) \right\|_{\infty, I} + \left\| U_{ik}(\eta) \right\|_{\infty, I} \right) (\varepsilon C_{p}^p (I) + C_{q}^q (I)) \right\}
\]
\[
\leq K \left\{ \left\| \bar{C}_{ik} (\xi) \right\| \left( C_{p}^p (I) + \varepsilon C_{p}^p (I) \right) \left\| U_{ik}(\eta) - \bar{q}_{ik}(\eta) \right\|_{\infty, I} + \right.
\]
\[
+ C_{q}^q (I) \left\| U_{ik}(\eta) \right\|_{\infty, I} + \varepsilon C_{p}^p (I) \left\| \bar{C}_{ik} (\xi) \right\| \left\| U_{ik}(\eta) \right\|_{\infty, I} \right\}.
\]

By (4.8) we have
\[
\left\| \bar{C}_{ik} (\xi) U_{ik}(\eta) - \bar{r} (\xi) \bar{q}_{ik}(\eta) \right\|_{\infty, I} \leq (K_2 + K_3 \beta^p + \varepsilon K_3 \beta^p ) \left\| U_{ik}(\eta) - \bar{q}_{ik}(\eta) \right\|_{\infty, I} +
\]
\[
+ K_3 \beta^p \left\| U_{ik}(\eta) \right\|_{\infty, I} + \varepsilon K_3 \beta^p \left\| U_{ik}(\eta) \right\|_{\infty, I},
\]

where \(0 < \beta < 1\) and \(K_2, K_3 \in \mathbb{R}\) are independent of \(p\) and \(\varepsilon\). Next consider
\[
\left\| U_{ik}(\eta) - \bar{q}_{ik}(\eta) \right\|_{\infty, I} = \left\| \varepsilon^i k \pi_k (\eta) e^{\frac{\gamma(1+i)}{\alpha}} - \bar{q}_{ik}(\eta) \right\|_{\infty, I}
\]
\[
= \left\| \varepsilon^i k \pi_k (\eta) e^{\frac{\gamma(1+i)}{\alpha}} - \varepsilon^i k \pi_k (\eta) \bar{q}_{ik}(\eta) + \varepsilon^i k \pi_k (\eta) \bar{q}_{ik}(\eta) - \bar{q}_{ik}(\eta) \bar{q}_{ik}(\eta) \right\|_{\infty, I}
\]
\[
\leq \left\| \varepsilon^i k \pi_k (\eta) \right\|_{\infty, I} \left\| e^{\frac{\gamma(1+i)}{\alpha}} - \bar{q}_{ik}(\eta) \right\|_{\infty, I} + \left\| \bar{q}_{ik}(\eta) \right\|_{\infty, I} \left\| \varepsilon^i k \pi_k (\eta) - \bar{q}_{ik}(\eta) \right\|_{\infty, I}.
\]

Recall that by construction, \(\bar{q}_{ik}(\eta)\) is a polynomial of degree less than or equal to \(p/2\) with \(p > 2M\). Since \(\varepsilon^i k \pi_k (\eta)\) is also a polynomial of degree \(\leq k\) with \(k = 0, 1, \ldots, i, i = 0, \ldots, M\), we take \(\bar{q}_{ik}(\eta)\) to be equal to \(\varepsilon^i k \pi_k (\eta)\), so that
\[
\left\| \varepsilon^i k \pi_k (\eta) - \bar{q}_{ik}(\eta) \right\|_{\infty, I} = 0 \quad \text{and}
\]
\[
\left\| U_{ik}(\eta) - \bar{q}_{ik}(\eta) \right\|_{\infty, I} \leq \left\| \varepsilon^i k \pi_k (\eta) \right\|_{\infty, I} \left\| e^{\frac{\gamma(1+i)}{\alpha}} - \bar{q}_{ik}(\eta) \right\|_{\infty, I}.
\]

Combining we get
\[
\left\| u^{PL}_{M} - u_{2} \right\|_{\infty, \mathbb{R}^n} \leq
\]
\[
\leq \sum_{i=0}^{M} \sum_{k=0}^{i} \left\{ K_3 \beta^p \left\| U_{ik}(\eta) \right\|_{\infty, I} + K_2 \left\| \varepsilon^i k \pi_k (\eta) \right\|_{\infty, I} \left\| e^{\frac{\gamma(1+i)}{\alpha}} - \bar{q}_{ik}(\eta) \right\|_{\infty, I} \right\}.
\]

Using Theorem 5.1 of \(^{13}\), we have
\[
\left\| e^{\frac{\gamma(1+i)}{\alpha}} - \bar{q}_{ik}(\eta) \right\|_{\infty, I} \leq K \varepsilon^{1/2} \alpha^{p+1/2},
\]
with $\alpha < 1$ and $K \in \mathbb{R}$ independent of $p$ and $\varepsilon$. (We actually get $\alpha^{p+1/2}$ but we can adjust the value of $\alpha$ to get $\alpha^{p+1/2}$ as stated above). Also
\[
\|U_{ik}(\eta)\|_{\varepsilon,I} = \left\| e^{i k \pi_k(\eta)} e^{\frac{\alpha/(\alpha+1)}{\varepsilon^t}} \right\|_{\varepsilon,I} \leq \max_I \{\pi_k(\eta)\} \left\| e^{i k \pi_k} e^{\frac{\alpha/(\alpha+1)}{\varepsilon^t}} \right\|_{\varepsilon,I}
\]
and (4.13) becomes
\[
\left\| u_{M}^{BL} - v_2 \right\|_{\varepsilon,\Omega} \leq \sum_{i=0}^{M} \left\{ K_3 \beta^i \varepsilon^{1/2} \max_I \{\pi_k(\eta)\} + K \varepsilon^{1/2} \alpha^{p+1/2} \left\| e^{i k \pi_k(\eta)} \right\|_{\varepsilon,I} \right\}
\]
(4.14)
with $\bar{\beta} := \max \{\beta^i, \alpha^{p+1/2}\}$, $0 < \alpha, \beta < 1$ and $K_M \in \mathbb{R}$ independent of $p$ and $\varepsilon$. Then,
\[
\left\| u_{M}^{BL} - v_2 \right\|_{\varepsilon,\Omega \setminus \Omega_{0,\varepsilon}} = \sum_{\Omega_i \subset \Omega \setminus \Omega_{0,\varepsilon}} \left\| u_{M}^{BL} - v_2 \right\|_{\varepsilon,\Omega_i}
\]
(4.15)
with $K \in \mathbb{R}$ independent of $p$ and $\varepsilon$.

Next, we consider $\left\| \chi u_{M}^{BL} - v_2 \right\|_{\varepsilon,\Omega_0,\varepsilon}$. Using the standard $p$ version estimate (c.f. 4), we obtain
\[
\left\| \chi u_{M}^{BL} - v_2 \right\|_{1,\Omega_0,\varepsilon} \leq p^{s+1} \left\| \chi u_{M}^{BL} \right\|_{s,\Omega_0,\varepsilon}.
\]
By assumption, $|\chi(r)| \leq C(\rho, s)$, $C \in \mathbb{R}$ independent of $p$ and $\varepsilon$. Thus,
\[
\left\| \chi u_{M}^{BL} - v_2 \right\|_{1,\Omega_0,\varepsilon} \leq C(\rho, s) p^{s+1} \exp(-\rho / \varepsilon) \varepsilon^t
\]
(4.16)
for any $t > 0$. Combining (4.12), (4.14) and (4.16), we obtain
\[
\left\| \chi u_{M}^{BL} - v \right\|_{\varepsilon,\Omega} = \left\| \chi u_{M}^{BL} - v \right\|_{\varepsilon,\Omega_0,\varepsilon} + \left\| u_{M}^{BL} - v \right\|_{\varepsilon,\Omega \setminus \Omega_{0,\varepsilon}}
\]
\[
\leq C(\rho, s) \varepsilon^t p^{s+1} + K \varepsilon^{1/2} \max_I \{\pi_k(\eta)\} \bar{\beta}^p
\]
\[
\leq C \varepsilon^{1/2} p^{s+1},
\]
with $s > 0$ arbitrary and $C = C(s) \in \mathbb{R}$ independent of $p$ and $\varepsilon$. $\blacksquare$
Remark 1 Note that in the above proof, \( q^2_{ik}(\eta) \) is equal to \( e^{i k \pi_k(\eta)} \) and \( q^1_{ik}(\eta) \) matches \( \exp(-\rho_0(\eta + 1)/2\varepsilon) \) at \( \eta = -1 \), so that \( q_{ik}(-1) = U_{ik}(-1) \). Hence, on \( \partial \Omega \)
\[
u_{M}^{B} - v_2 = C_{ik}(\xi) U_{ik}(-1) - \tilde{r}(\xi) q_{ik}(-1) = \tilde{C}_{ik}(\xi) - \tilde{r}(\xi),
\]
where \( \tilde{r}(\xi) \) is the \( H^1 \) projection of \( \tilde{C}_{ik}(\xi) \).

Remark 2 We point out that the assumption (4.7) in Lemma 4.1 can be weakened to
\[
C_{ik}(\theta) \in C^\infty(\partial \Omega)
\]
since we only need arbitrarily high, algebraic rates of convergence for the boundary layer functions in the proof of Lemma 4.1, rather than the exponential rate implied by the analyticity of \( \tilde{C}_{ik}(\theta) \).

5. The Approximation of the Smooth Part and the Remainder

In this section we discuss the approximation of the smooth part \( w_{M}^{s} \), and the remainder \( r_{M}^{s} \), of the weak solution \( u^{s} \) to (2.1). We will assume that the right hand side \( f \) in (2.1) is analytic over \( \Omega \), so that \( w_{M}^{s}(x,y) = \sum_{i=0}^{M} e^{2i} \Delta_i f(x,y) \) and \( r_{M}^{s} \) will also be analytic over \( \Omega \), with \( M \) arbitrarily high. With \( \{\Omega_i\}_{i=1}^{n} \) some subdivision of \( \Omega \), we define \( S_{r,w}^{N} \) as the standard finite element space of piecewise polynomials,
\[
S_{r,w}^{N}(\Sigma) = \{ v \in H^1(\Omega) : v|_{\Omega_i} \in \Pi_p(\Omega_i), \Omega_i \in \Delta \},
\]
\[
\Sigma = (\Delta, \overline{\omega}), \Delta = \{\Omega_i\}_{i=1}^{n}, p = \{p, \ldots, p\}
\]
to be used for the finite element approximation of \( w_{M}^{s} \) and \( r_{M}^{s} \). Let \( v_1, v_2 \) be the \( H^2 \) projections of \( w_{M}^{s}, r_{M}^{s} \) respectively onto \( S_{r,w}^{N} \cap C^1(\Omega) \), with \( S_{r,w}^{N} \) given by (5.1) above. Then we have the standard \( p \) version estimate (c.f. 4),
\[
\|v_1 - w_{M}^{s}\|_{\ell,\Omega} \leq C p^{-s+\ell} \|w_{M}^{s}\|_{s,\Omega},
\]
\[
\|v_2 - r_{M}^{s}\|_{\ell,\Omega} \leq C p^{-s+\ell} \|r_{M}^{s}\|_{s,\Omega}
\]
for \( \ell = 0, 1, 2 \). Also,
\[
\|w_{M}^{s}\|_{s,\Omega} = \left\| \sum_{i=0}^{M} e^{2i} \Delta_i f \right\|_{s,\Omega} \leq \sum_{i=0}^{M} e^{2i} \left\| \Delta_i f \right\|_{s,\Omega} \leq C(M),
\]
and by (2.10),
\[
\|r_{M}^{s}\|_{s,\Omega} \leq C \varepsilon^{M+3/2 - s},
\]
so that
\[
\|v_1 - w_{M}^{s}\|_{\epsilon,\Omega} \leq \|v_1 - w_{M}^{s}\|_{1,\Omega} \leq C(M) p^{-s+1},
\]
\[
\|v_2 - r_{M}^{s}\|_{\epsilon,\Omega} \leq \|v_2 - r_{M}^{s}\|_{1,\Omega} \leq C p^{-s+1} \varepsilon^{M+3/2 - s}.
\]
6. The Approximation of $u^\varepsilon$

We now address the approximation of the solution $u^\varepsilon = u^\varepsilon_M + \chi u^\varepsilon_{BL} + r^\varepsilon_M$, given by (2.6). Define the space $\mathcal{S}_N \subset H^1(\Omega)$ as

$$\mathcal{S}_N = \{ v = v_1^N + v_2^N + v_3^N : v_1 \in S_{BL}^N, v_2, v_3 \in S_{r,w}^N \},$$

(6.1)

and note that $v^N = v_1^N + v_2^N + v_3^N \in \mathcal{S}_N$, satisfies (4.11), (5.2) and (5.3). Also,

$$\| v - u^\varepsilon \|_{x,\Omega} \leq \| v_1^N - w^\varepsilon_M \|_{x,\Omega} + \| v_2^N - \chi u^\varepsilon_{BL} \|_{x,\Omega} + \| v_3^N - r^\varepsilon_M \|_{x,\Omega}. \quad \text{Using Theorem 4.1, (5.4) and (5.5) we obtain the following result.}

**Lemma 6.2** Let $u^\varepsilon$ be the solution to (2.2) as given by (2.6) and let $\mathcal{S}_N$ be the space defined by (6.1). Then for any $\bar{s} > 0$, there exists $v^N \in \mathcal{S}_N$ such that

$$\| v^N - u^\varepsilon \|_{x,\Omega} \leq C \bar{s}, \quad \text{(6.2)}$$

where $C = C(\bar{s}) \in \mathbb{R}$ is independent of $p$ and $\varepsilon$, but depends on $\bar{s}$.

**Proof.** Theorem 4.1, (5.4) and (5.5) give

$$\| v^N - u^\varepsilon \|_{x,\Omega} \leq C(M)p^{s+1} + C\varepsilon^{1/2}p^{s} + C\bar{s}^{s+1}\varepsilon^{M+3/2} \quad \text{and (6.2) follows with } \bar{s} = s - 1. \quad \square$$

Note that $\mathcal{S}_N$ defined by (6.1) is not a subspace of $H^1_0$, thus the best approximation result (3.2) has not been established yet for this space. We need to adjust the approximating polynomial $v^N$ so that $v^N|_{\partial\Omega} = 0$ and (6.2) still holds, i.e. such that $v^N \in \mathcal{S}_N \cap H^1_0(\Omega)$. Note that for $\delta > 0$ small, and $C \in \mathbb{R}$ independent of $p$ and $\varepsilon$, we have

$$\| v_1^N - w^\varepsilon_M \|_{0,\partial\Omega} \leq \| v_1^N - w^\varepsilon_M \|_{1/2+\delta,\partial\Omega} \leq C(\delta)p^{s+1},$$

$$\| v_1^N - w^\varepsilon_M \|_{1,\partial\Omega} \leq \| v_1^N - w^\varepsilon_M \|_{3/2+\delta,\partial\Omega} \leq C(\delta)p^{s+2},$$

with similar estimates holding for $v_3^N - r^\varepsilon_M$. Also, by Remark 1 and (4.8),

$$\| v_2^N - u^\varepsilon_{BL} \|_{0,\partial\Omega} = C^p(I) \leq C\beta,$$

$$\| v_2^N - u^\varepsilon_{BL} \|_{1,\partial\Omega} = C^p(I) \leq C\beta,$$

with $0 < \beta < 1$ and $C \in \mathbb{R}$ is independent of $p$ and $\varepsilon$. Combining, we get (since $u^\varepsilon|_{\partial\Omega} = 0$)

$$\| v^N \|_{0,\partial\Omega} \leq \| v^N - u^\varepsilon \|_{0,\partial\Omega} \leq C_p^{s+1}, \quad \text{(6.3)}$$

$$\| v^N \|_{1,\partial\Omega} \leq \| v^N - u^\varepsilon \|_{1,\partial\Omega} \leq C_p^{s+2}. \quad \text{(6.4)}$$

The following Lemma gives us the tool for adjusting the nonzero trace of $v^N \in \mathcal{S}_N$ on the boundary $\partial\Omega$. (For a proof see Lemma 3.6.2 of [16]).
Lemma 6.3 Let \( S = (h_1, h_2) \times (0, k) \) be a rectangular element. Let \( z(x) \) be a polynomial of degree \( \leq p \) on \( \gamma = (h_1, h_2) \), and define \( V(x, y) = z(x) L(y) \in \Pi_p(S) \), with \( L(y) \) linear, \( L(0) = 1 \), and \( L(k) = 0 \). Then, \( V \) satisfies \( \| V \|_{0, \gamma} = z \|_{0, \gamma} \), \( V = 0 \) on the side opposite \( \gamma \) and

\[
\| V \|_{r, S} \leq C \left( \varepsilon k^{1/2} |z|_{1, \gamma} + k^{1/2} \| z \|_{0, \gamma} + \varepsilon k^{1/2} \| z \|_{0, \gamma} \right),
\]

where \( C \in \mathbb{R} \) is independent of \( p \) and \( \varepsilon \).

Using Lemma 6.3, we obtain the following result.

Lemma 6.4 Let \( v^N \in \bar{S}^N \) be as in Lemma 6.2. Then, there exists \( V^N \in \bar{S}^N \) such that \( V^N \big|_{\partial k} = v^N \big|_{\partial k} \) and

\[
\| V^N \|_{r, \Omega} \leq K \varepsilon^{1/2} p \bar{s},
\]

where \( \bar{s} > 0 \) is arbitrary and \( K \in \mathbb{R} \) is independent of \( p \) and \( \varepsilon \).

Proof. Define \( V^N(\rho, \theta) = L(\rho) z(\theta), 0 \leq \rho \leq \frac{h_1}{\kappa p}, 0 \leq \theta \leq \theta_0 \) with \( L(\rho) \) linear, \( L(0) = 1, L(\frac{h_1}{\kappa p}, \theta_0) = 0 \) and \( z(\theta) = v^N(\theta) = v^N \big|_{\partial k} \). Let \( \Omega_t \) be any element of the mesh \( \Delta \), with \( \partial \Omega_t \cap \partial \Omega \neq \emptyset \). Then, using the mapping (4.4), we map \( \Omega_t \) to the rectangle \( J = (-1, 1) \times (-1, -1 + \kappa p) \) and \( V^N(\rho, \theta) \) to \( V^N(\eta, \xi) = \tilde{L}(\eta) \tilde{z}(\xi) = \tilde{L}(\eta) \tilde{v}^N(\xi) \). Applying Lemma 6.3, we obtain

\[
\| V^N \|_{r, J} \leq C \left\{ \varepsilon \left( \kappa p \right)^{1/2} \| \tilde{z} \|_{1, \partial J} + \left( \kappa p \right)^{1/2} \| \tilde{z} \|_{0, \partial J} \right\}
\]

\[
\leq \bar{C} \left\{ \varepsilon^{1/2} p \left( k e \right)^{1/2} \| \tilde{v}^N \|_{1, \partial J} + \left( pe \right)^{1/2} \| \tilde{v}^N \|_{0, \partial J} \right\}.
\]

Mapping back to the \( \rho - \theta \) plane, we get

\[
\| V^N \|_{r, \Omega_t} \leq C \left\{ \varepsilon^{1/2} p \left( k e \right)^{1/2} \| v^N \|_{1, \partial k} + \left( pe \right)^{1/2} \| v^N \|_{0, \partial k} \right\}.
\]

Using (6.3) and (6.4), we have

\[
\| V^N \|_{r, \Omega_t} \leq C \left\{ \varepsilon^{1/2} p \left( k e \right)^{1/2} \| v^N \|_{1, \partial k} + \left( pe \right)^{1/2} \| v^N \|_{0, \partial k} \right\} \leq K \varepsilon^{1/2} p \bar{s}^{3/2}.
\]

Since,

\[
\| V^N \|_{r, \Omega_t} = \sum_{\Omega_t} \| V^N \|_{r, \Omega_t} \leq \sum_{\Omega_t} K \varepsilon^{1/2} p \bar{s}^{3/2},
\]

(6.6) follows, with \( \bar{s} = s - 3/2 \). □

We now present our main result.

Theorem 6.1 Let \( u^\varepsilon \) be the solution to (2.2) and \( u^N \) the solution to (3.1). Then, for any \( s > 0 \),

\[
\| u^\varepsilon - u^N \|_{r, \Omega} \leq C(s) p \bar{s}^s
\]

where \( C(s) \in \mathbb{R} \) is independent of \( p \) and \( \varepsilon \).
Proof. By Lemma 6.2 there exists \( v^N \in S^N \) such that
\[
\|v^N - u^\varepsilon\|_{\varepsilon,\Omega} \leq C p^{-s},
\]
with \( C \in \mathbb{R} \) independent of \( p \) and \( \varepsilon \). Let \( \tilde{V}^N \) be constructed as in Lemma 6.4, and define \( u^N = v^N - \tilde{V}^N \). Then \( u^N \in S^N = S^N \cap H^1_0(\Omega) \) and
\[
\|u^\varepsilon - u^N\|_{\varepsilon,\Omega} \leq \|u^\varepsilon - \tilde{V}^N\|_{\varepsilon,\Omega} \leq C p^{-s} + K \varepsilon^{1/2} p^{-s}.
\]
Since by (3.2) \( u^N \) is the best approximation, the assertion follows. \( \square \)

7. Numerical Results

In this section we present the results of numerical computations for the model boundary value problem (2.1), in the case where \( \Omega \) is the unit disk. That is, we consider the problem
\[
-\varepsilon^2 \Delta u + u = 1 \quad \text{in } \Omega,
\]
\[
u = 0 \quad \text{on } \partial\Omega,
\]
where \( \Omega = \{(r, \vartheta) : 0 \leq r \leq 1, 0 \leq \vartheta \leq 2\pi\} \), in polar coordinates, and \( \varepsilon \in (0, 1] \), as before. The exact solution to (7.1) is a function \( u(r, \vartheta) \), defined in polar coordinates by
\[
u(r, \vartheta) = u(r) = 1 - \frac{I_0(r/\varepsilon)}{I_0(1/\varepsilon)},
\]
where \( I_0(z) \) is the modified Bessel function of order zero. From (7.2), we see that \( u = u^\varepsilon + u^{BL} \) where \( u^\varepsilon = 1 \) and \( u^{BL}(r, \vartheta) = u^{BL}(r) = \frac{I_0(r/\varepsilon)}{I_0(1/\varepsilon)} \). Even though \( u^{BL}(r) \) is not the typical boundary layer function, it behaves like \( \exp(-r/\varepsilon) \), c.f. 15.

We will consider two finite element schemes for this model problem, and we will compare the relative errors in the energy norm,
\[
E = \frac{\|u^{EX} - u^{FEM}\|_{\varepsilon,\Omega}}{\|u^{EX}\|_{\varepsilon,\Omega}},
\]
versus the number of degrees of freedom, \( N \). Figures 3, 4 show the mesh design for the two methods considered. The first scheme consists of 8 elements, with the inner circle taken at 1/2 (see Figure 3). The second scheme consists of the same number of elements but with the inner circle taken at \( (1 - \varrho \varepsilon) \), i.e. an \( O(\varrho \varepsilon) \) layer of elements along the edge of the boundary (see Figure 4).

We performed experiments for various values of \( \varepsilon \), using the software package STRESSCHECK\textsuperscript{TM}, for \( p = 1, \ldots, 8 \). Due to the fact that STRESSCHECK\textsuperscript{TM} is a \( p \) version software package, we are not able to perform “true” \( h_p \) version experiments.
Fig. 3. The mesh for Scheme 1.

Fig. 4. The mesh for Scheme 2.
Scheme 2 is designed to represent an $hp$ version, given the software at hand. Moreover the maximum allowed polynomial degree is $p = 8$, thus the difference between the two methods we are considering lies only in the position of the “inner-circle” element. That is, for the first scheme the inner circle has radius $1/2$ and for the second scheme the inner circle has radius $1 - p_{\text{max}}\epsilon = 1 - 8\epsilon$. This type of experiment will illustrate the necessity of an $O(p\epsilon)$ layer of elements at the edge of the boundary, in order to uniformly approximate boundary layers at a sufficiently fast rate.

Figures 5 and 6 show the relative error in the energy norm $E$, versus the number of degrees of freedom $N$, in a log-log scale for $\epsilon = 0.01$ and $\epsilon = 0.001$, respectively. As can be clearly seen from this comparison, the first scheme deteriorates as $\epsilon \to 0$, while the second scheme demonstrates the expected robustness and spectral convergence rate. In particular, this numerical experiment suggests that the second scheme converges at a robust exponential rate. This was not established in our analysis due to technicalities in the proofs. The observations made here establish the superiority of this $hp$ method for problems of this type.

![Energy norm comparison of the two schemes.](image.png)

We have also considered the pointwise error in the derivatives of $u^{EX}$ and $u^{FEM}$ at the point $(1,0)$ on the boundary of $\Omega$. Strictly speaking, the previous analysis does not cover the convergence of pointwise derivatives, but we address this issue here computationally. Figures 7 and 8 show the comparison of the two finite element
Fig. 6. Energy norm comparison for the two schemes.

Fig. 7. Derivative comparison for the two schemes.
schemes for

\[ E' = \left[ \frac{du^E}{dr} - \frac{du^E_M}{dr} \right]_{r=1}, \]

versus the number of degrees of freedom, \( N \), in a log-log scale. The same behavior as in the energy norm error is observed, with the first scheme deteriorating completely as \( \varepsilon \to 0 \).

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Appendix

Here we give a proof of Theorem which we restate for convenience:

Let \( u^\varepsilon \) satisfy (2.2) and assume \( f \in H^{4M+2}(\Omega) \) for some \( M \in \mathbb{N} \). Then

\[ u^\varepsilon = w_M^\xi + \chi u^BL_M + r^\varepsilon_M, \]

where \( \chi \in C^\infty ([0, \infty)) \) is a cutoff function such that

\[ \chi(r) = \begin{cases} 1 & \text{for } 0 \leq r \leq \rho_0/3 \\ 0 & \text{for } r \geq 2\rho_0/3 \end{cases} \]
with $|\chi^{(m)}(r)| \leq C(\rho_0, m)$, $m = 0, 1, \ldots$, 

$$w^\varepsilon_M(x, y) = \sum_{i=0}^{M} \varepsilon^{2i} \Delta^{(i)} f(x, y),$$

and in $\Omega_0$,

$$u^BL_M = \sum_{i=0}^{M} \varepsilon^{i} \sum_{k=0}^{i} C_{ki}(\theta) \rho^k \varepsilon^{k} = \sum_{i=0}^{M} C_i(\theta) \rho^i \varepsilon^{i},$$

$$\|r^\varepsilon_M\|_{k, \Omega} \leq c \varepsilon^{M+3/2} \quad 0 \leq k \leq M + 3/2$$

with $c \in \mathbb{R}$ a constant independent of $\varepsilon$ and $C_{ki}(\theta)$ smooth and independent of $\varepsilon$.

**Proof.** Define

$$w^\varepsilon_M(x, y) = \sum_{i=0}^{M} \varepsilon^{2i} \Delta^{(i)} f(x, y).$$

Then, using (2.1) we see that $R^\varepsilon_M := u^\varepsilon - w^\varepsilon_M$ satisfies

$$L_\varepsilon R^\varepsilon_M = f - L_\varepsilon w^\varepsilon_M = \varepsilon^{2M+2} \Delta^{(M+1)} f =: g(x, y). \quad (A.1)$$

For $M$ sufficiently large, $w^\varepsilon_M$ will satisfy (2.1) up to the correction factor $g(x, y)$. However, $w^\varepsilon_M$ will not satisfy, in general, the boundary conditions. To correct this we introduce appropriate boundary layer terms as follows. Define $u^BL$ formally to be a function satisfying

$$L_\varepsilon u^BL = 0, \; \text{in} \; \Omega_0, \quad (A.2)$$

$$u^BL|_{\partial \Omega} = - \sum_{i=0}^{\infty} \varepsilon^{2i} \left[ \Delta^{(i)} f \right]|_{\partial \Omega}, \quad \text{(formally)}$$

where, for any function $G$ defined on $\overline{\Omega}$, $[G]|_{\partial \Omega}$ denotes $G$ restricted to $\partial \Omega$. We solve (A.2) via series expansion, but since $u^BL$ is defined only in $\Omega_0$, we consider (A.2) in boundary fitted coordinates. Let $\kappa(\theta)$ be the curvature of $\partial \Omega$ and set

$$\sigma(\rho, \theta) = \frac{1}{1 - \kappa(\theta) \rho} \quad (A.3)$$

We have, see $^1$, an expression for the Laplacian in boundary fitted coordinates as

$$\Delta u^BL(\rho, \theta) = \frac{\partial^2 u^BL}{\partial \rho^2} - \kappa(\theta) \sigma(\rho, \theta) \frac{\partial u^BL}{\partial \rho} + \sigma^2(\rho, \theta) \frac{\partial^2 u^BL}{\partial \theta^2} + \quad (A.4)$$

$$+ \rho \kappa' (\theta) \sigma^3(\rho, \theta) \frac{\partial u^BL}{\partial \theta}.$$

So, $u^BL$ is obtained by solving

$$-\varepsilon^2 \Delta u^BL + u^BL = 0, \; \text{in} \; \Omega_0, \quad (A.5)$$
where $\Delta u^{BL}$ is given by (A.4) and $\Omega_0$ is given by (2.5). Using a geometric series, expand $\sigma(\rho, \theta)$, given by (A.3) as

$$\sigma(\rho, \theta) = \sum_{j=0}^{\infty} |\kappa(\theta)| \rho^j = \sum_{j=0}^{\infty} |\kappa(\theta)\varepsilon\tilde{\rho}|^j,$$

where $\tilde{\rho} = \rho/\varepsilon$ denotes a stretched variable. Then, (A.5) becomes

$$-\varepsilon^2 \left\{ \frac{\partial^2 u^{BL}}{\partial \tilde{\rho}^2} + \sum_{j=0}^{\infty} \rho^j \left( a_1^j \frac{\partial u^{BL}}{\partial \rho} + a_2^j \frac{\partial^2 u^{BL}}{\partial \theta^2} + a_3^j \frac{\partial u^{BL}}{\partial \theta} \right) \right\} + u^{BL} = 0, \text{ in } \Omega_0,$$

with

$$a_1^j = -|\kappa(\theta)|^{j+1}, \quad a_2^j = (j+1)|\kappa(\theta)|^j, \quad a_3^j = \frac{j(j+1)}{2}|\kappa(\theta)|^j 1^{1'} \theta.$$

Switching to the stretched variable notation, (A.7) becomes

$$-\frac{\partial^2 u^{BL}}{\partial \tilde{\rho}^2} - \varepsilon^2 \sum_{j=0}^{\infty} (\varepsilon\tilde{\rho})^j \left( a_1^j \frac{\partial u^{BL}}{\partial \rho} + a_2^j \frac{\partial^2 u^{BL}}{\partial \theta^2} + a_3^j \frac{\partial u^{BL}}{\partial \theta} \right) + u^{BL} = 0, \text{ in } \hat{\Omega}_0,$$

where $\hat{\Omega}_0$ denotes the image of $\Omega_0$ under the stretched variable transformation.

Now, we formally write $u^{BL} = \sum_{i=0}^{\infty} \varepsilon^i \hat{U}_i(\tilde{\rho}, \theta)$, with $\hat{U}_i(\tilde{\rho}, \theta)$ to be determined, and we insert it into (A.9). Equating like powers of $\varepsilon$, we get

$$-\frac{\partial^2 \hat{U}_i}{\partial \tilde{\rho}^2} + \hat{U}_i = \hat{F}_i(\tilde{\rho}, \theta), \quad i = 0, 1, 2, \ldots$$

where

$$\hat{F}_i(\tilde{\rho}, \theta) := \sum_{j=0}^{i} \tilde{\rho}^j \left( a_1^j \frac{\partial \hat{U}_i}{\partial \rho} + a_2^j \frac{\partial^2 \hat{U}_i}{\partial \theta^2} + a_3^j \frac{\partial \hat{U}_i}{\partial \theta} \right).$$

(We set $\hat{U}_j \equiv 0$ for $j < 0$). We shall consider solutions which satisfy

$$\lim_{\tilde{\rho} \to \infty} \hat{U}_i = 0,$$

to ensure that each $\hat{U}_i$ decays exponentially with $\tilde{\rho}$ and is therefore negligible outside $\hat{\Omega}_0$. As a boundary condition for (A.10) we will consider

$$[\hat{U}_i]_{\partial \hat{\Omega}} = \left\{ \begin{array}{ll} -[\Delta^{[i/2]} f]_{\partial \hat{\Omega}}, & \text{if } i \text{ is even}, \\ 0, & \text{if } i \text{ is odd}. \end{array} \right.$$

Let us find $\hat{U}_i(\tilde{\rho}, \theta)$ using (A.10) and (A.13). First, consider the case of $i = 0$. We have to solve

$$-\frac{\partial^2 \hat{U}_0}{\partial \tilde{\rho}^2} + \hat{U}_0 = 0 \text{ in } \hat{\Omega}_0,$$

$$[\hat{U}_0]_{\partial \hat{\Omega}} = [-f]_{\partial \Omega} \text{ on } \partial \Omega.$$
Letting \( C \), Assume us with \( k /; j /; \)` and this establishes \( (2/./9/)/ \). We will now obtain bounds on \( \rho /\); function is of type \( \induction \) on for \( 0 /\( \), Note that by \( /(A/./1/0/) /-/(A/./1/3/)/ \), using standard techniques such as the method of undetermined coefficients. Since the right hand side of \( /(A/./1/0/) \), function in differentiation is with respect to \( \). In particular, for \( i \in \mathbb{N} \),

\[
\hat{U}_i(\rho, \theta) = e^\rho \sum_{k=0}^i \sum_{j=0}^i \sum_{\ell=0}^j \alpha_{ikj} \theta^k \frac{\partial^\ell f}{\partial \theta^\ell} \left( \left[ \Delta^\ell f \right]_{\theta=\rho} \right),
\]

where \( \alpha_{ikj} \) are smooth functions of \( \theta \), depending only on \( \Omega \). We will say that a function is of type \( (m, n) \) if it is a sum of terms of the form

\[
\alpha(\theta) \theta^k \frac{\partial^\ell f}{\partial \theta^\ell} \left( \left[ \Delta^\ell f \right]_{\theta=\rho} \right) e^\rho,
\]

with \( k, j, \ell \in \mathbb{N} \), satisfying \( k \leq m, j + \ell \leq n \), and \( \alpha(\theta) \) smooth depending only on \( \Omega \). We wish to show that \( \hat{U}_i(\rho, \theta) \) is of type \( (i, i) \) for any \( i \in \mathbb{N} \). The proof is by induction on \( i \). The result is known for \( \hat{U}_0 \) from (A.15). We assume the result holds for \( 0, 1, 2, ..., i - 1 \) and establish it for \( \hat{U}_i \). Note that \( \hat{F}_i \) defined by (A.11) is of type \( (i - 1, i) \). To obtain \( \hat{U}_i(\rho, \theta) \), we solve the ordinary differential equation defined by (A.10) - (A.13), using standard techniques such as the method of undetermined coefficients. Since the right hand side of (A.10), \( \hat{F}_i \), is of type \( (i - 1, i) \) and the differentiation is with respect to \( \rho \), the solution \( \hat{U}_i \) will be of the form \( \sum_{n=0}^i a_n \rho^n \), with \( a_n(\theta) \) the undetermined coefficients. It is then easy to see that \( \hat{U}_i(\rho, \theta) \) is of type \( (i, i) \) as desired. This establishes (A.16).

Next, define

\[
u^{BL}_M = \sum_{i=0}^M \varepsilon^i \hat{U}_i.
\]

By (A.16), we have

\[
u^{BL}_M = e^{\rho/\varepsilon} \sum_{i=0}^M \varepsilon^i \sum_{k=0}^i \sum_{j=0}^i \sum_{\ell=0}^j \alpha_{ikj} \theta^k \frac{\partial^\ell f}{\partial \theta^\ell} \left( \left[ \Delta^\ell f \right]_{\theta=\rho} \right).
\]

Letting \( C_{ki}(\theta) := \sum_{j=0}^i \sum_{\ell=0}^j \alpha_{ikj} \theta^k \frac{\partial^\ell f}{\partial \theta^\ell} \left( \left[ \Delta^\ell f \right]_{\theta=\rho} \right) \), we get

\[
u^{BL}_M = \sum_{i=0}^M \varepsilon^i \sum_{k=0}^i \sum_{\ell=0}^j C_{ki}(\theta) \frac{\partial^\ell f}{\partial \theta^\ell} \rho^{\ell/\varepsilon},
\]

and this establishes (2.9). We will now obtain bounds on \( \nu^{BL}_M \). To this end define the region

\[
\Omega_{\rho, \lambda} = \{ \rho - \rho_0^2 : \rho \in \partial \Omega, \rho/3 < \rho_0 \}.
\]

Assume \( \hat{U}_0 = C_0(\theta) G_0(\hat{\rho}) \). Then, (A.14) gives \( -C_0(\theta) G_0''(\hat{\rho}) + C_0(\theta) G_0(\hat{\rho}) = 0 \), thus \( G_0(\hat{\rho}) = K e^\hat{\rho} \). The boundary condition gives \( K = \int_{\partial \Omega} G_0(\hat{\rho}) d\Omega \), so that

\[
\hat{U}_0 = \left[ \int_{\partial \Omega} G_0(\hat{\rho}) d\Omega \right] e^\hat{\rho} = [\int_{\partial \Omega} e^\rho] e^{\rho/\varepsilon}.
\]
and note that \( \chi(r) \) defined by (2.7), satisfies \( \chi \equiv 1 \) on \( \Omega_0 \setminus \Omega_0 \chi \), where \( \Omega_0 \) was defined by (2.5).

Also let

\[
\mathcal{U}_{ijkt} = \alpha_{ikjt}(\theta) \frac{\partial^t}{\partial t} \left( \left[ \Delta (j) f \right]_{\partial \Omega} \right) e^t, \tag{A.19}
\]

so that

\[
\hat{U}(\hat{\rho}, \theta) = \sum_{k=0}^{i} \sum_{j=0}^{i} \sum_{t=0}^{j} \hat{\rho}^k \mathcal{U}_{ijkt}.
\]

Lemma 4.6 of \(^1\) states that there exists a constant \( K \) depending on \( \Omega \) such that for any \( s = 0, 1, \ldots \),

\[
\left\| \hat{\rho}^k \frac{\partial^{a+b} \mathcal{U}_{ijkt}}{\partial \rho^a \theta^b} \right\|_{s, \Omega_0} \leq K \varepsilon^{1/2} s \sum_{n=0}^{s} \varepsilon^n \left\| \mathcal{U}_{ijkt} \right\|_{n+\ell, \partial \Omega}, \tag{A.20}
\]

and for any \( t > 0 \)

\[
\left\| \hat{\rho}^k \frac{\partial^{a+b} \mathcal{U}_{ijkt}}{\partial \rho^a \theta^b} \right\|_{s, \Omega_0, \chi} \leq K \varepsilon^{1/2} s \sum_{n=0}^{s} \varepsilon^n \left\| \mathcal{U}_{ijkt} \right\|_{n+\ell, \partial \Omega}. \tag{A.21}
\]

Using (A.20) with \( a = b = 0 \) and \( \mathcal{U}_{ijkt} \) defined by (A.19), we obtain

\[
\left\| \hat{\rho}^k \mathcal{U}_{ijkt} \right\|_{s, \Omega_0} \leq K \varepsilon^{1/2} s \sum_{n=0}^{s} \varepsilon^n \left\| \alpha_{ikjt}(\theta) \frac{\partial^t}{\partial t} \left( \left[ \Delta (j) f \right]_{\partial \Omega} \right) \right\|_{n, \partial \Omega}
\]

\[
\leq K \varepsilon^{1/2} s \sum_{n=0}^{s} \varepsilon^n \left\| \left[ \Delta (j) f \right]_{\partial \Omega} \right\|_{n+\ell, \partial \Omega}
\]

Thus,

\[
\left\| \hat{U}(\hat{\rho}, \theta) \right\|_{s, \Omega_0} \leq \sum_{k=0}^{i} \sum_{j=0}^{i} \sum_{t=0}^{j} \left\| \hat{\rho}^k \mathcal{U}_{ijkt} \right\|_{s, \Omega_0} \tag{A.22}
\]

\[
\leq K \varepsilon^{1/2} s \sum_{n=0}^{s} \varepsilon^n \sum_{j=0}^{i} \sum_{t=0}^{j} \left\| \left[ \Delta (j) f \right]_{\partial \Omega} \right\|_{n+\ell, \partial \Omega}
\]

\[
\leq K \varepsilon^{1/2} s \sum_{n=0}^{s} \varepsilon^n \sum_{j=0}^{i} \left\| \left[ \Delta (j) f \right]_{\partial \Omega} \right\|_{n+i, \partial \Omega}
\]

\[
\leq K \sum_{j=0}^{i} \left\{ \varepsilon^{1/2} s \left\| \left[ \Delta (j) f \right]_{\partial \Omega} \right\|_{i, \partial \Omega} + \varepsilon^{1/2} s \left\| \left[ \Delta (j) f \right]_{\partial \Omega} \right\|_{s+1, \partial \Omega} \right\}
\]

Using (A.21) with \( a = b = 0 \), we get for any \( t > 0 \)

\[
\left\| \hat{\rho}^k \mathcal{U}_{ijkt} \right\|_{s, \Omega_0, \chi} \leq K \varepsilon^{1/2} s \sum_{n=0}^{s} \varepsilon^n \left\| \alpha_{ikjt}(\theta) \frac{\partial^t}{\partial t} \left( \left[ \Delta (j) f \right]_{\partial \Omega} \right) \right\|_{n, \partial \Omega}
\]

\[
\leq K \varepsilon^{1/2} s \sum_{n=0}^{s} \varepsilon^n \left\| \left[ \Delta (j) f \right]_{\partial \Omega} \right\|_{n+\ell, \partial \Omega}.
\]
This gives
\[
\| \hat{\mathbf{u}} \|_{s, \Omega_0, \chi} \leq K \varepsilon^{1/2+t} \sum_{n=0}^{s} \varepsilon^n \sum_{j=0}^{i} \sum_{l=0}^{j} \left\| [\Delta^{(j)} f]_{\partial \Omega} \right\|_{n+i, \partial \Omega}
\]
(A.23)
\[
\leq K \varepsilon^{1/2+t} \sum_{n=0}^{s} \varepsilon^n \sum_{j=0}^{i} \left\| [\Delta^{(j)} f]_{\partial \Omega} \right\|_{n+i, \partial \Omega} + \varepsilon^{1/2+t} \left\| [\Delta^{(j)} f]_{\partial \Omega} \right\|_{n+i, \partial \Omega}
\]
\[
\leq K \sum_{j=0}^{i} \left\{ \varepsilon^{1/2+t} \sum_{n=0}^{s} \varepsilon^n \left\| [\Delta^{(j)} f]_{\partial \Omega} \right\|_{n+i, \partial \Omega} + \varepsilon^{1/2+t} \left\| [\Delta^{(j)} f]_{\partial \Omega} \right\|_{n+i, \partial \Omega} \right\},
\]
Hence
\[
\| \mathbf{u}_{BL} \|_{s, \Omega_0, \chi} \leq K \sum_{i=0}^{M} \left\{ \varepsilon^{i+1+1/2} \sum_{n=0}^{s} \varepsilon^n \left\| [\Delta^{(i)} f]_{\partial \Omega} \right\|_{n+i, \partial \Omega} + \varepsilon^{i+1+1/2} \left\| [\Delta^{(i)} f]_{\partial \Omega} \right\|_{n+i, \partial \Omega} \right\},
\]
(A.24)
for any \( t > 0 \).

It remains to establish the desired bounds on the remainder, which is defined by
\[
r^*_{BL} = u - v_{BL} - \chi u_{BL}.
\]
Note that \( r^*_{BL} \) satisfies the boundary value problem
\[
L_{BL} r^*_{BL} = \begin{cases} g & \text{in } \Omega, \\ 0 & \text{on } \partial \Omega. \end{cases}
\]
(A.26)

This gives
\[
\| r^*_{BL} \|_{L_2, \Omega} = B_r (r^*_{BL}, r^*_{BL}) = (g, r^*_{BL}) - (L_{BL} (\chi u_{BL}), r^*_{BL})
\]
where \( (\cdot, \cdot) \) denotes the usual \( L_2 (\Omega) \) inner product. Since
\[
\chi = \begin{cases} 1 & \text{on } \Omega \setminus \Omega_0, \\ 0 & \text{on } \Omega \setminus \Omega_0 \end{cases}
\]
we have
\[
\| r^*_{BL} \|_{L_2, \Omega} \leq \| g \|_{L_2, \Omega} + \| u_{BL} \|_{L_2, \Omega} + \| L_{BL} (\chi u_{BL}) \|_{L_2, \Omega},
\]
which gives
\[
\| r^*_{BL} \|_{L_2, \Omega} \leq \| g \|_{L_2, \Omega} + \| L_{BL} u_{BL} \|_{L_2, \Omega} + \| L_{BL} \chi u_{BL} \|_{L_2, \Omega}
\]
(A.27)
Let us estimate \( \| L_{e} u_{M}^{BL} \|_{s}, s = 0, 1, \ldots \). We have

\[
L_{e} u_{M}^{BL} = -\varepsilon^2 \Delta u_{M}^{BL} + u_{M}^{BL}
\]

\[
= -\varepsilon^2 \sum_{i=0}^{M} \varepsilon^i \left\{ a_i \partial_{\hat{\rho}} \hat{U}_i + \kappa(\theta) \sigma (\hat{\rho}, \theta) \frac{\partial \hat{U}_i}{\partial \rho} + \sigma^2(\hat{\rho}, \theta) \frac{\partial^2 \hat{U}_i}{\partial \theta^2} + \right. \\
+ \rho \kappa(\theta) \sigma (\hat{\rho}, \theta) \frac{\partial \hat{U}_i}{\partial \theta} \right\} + \sum_{i=0}^{M} \varepsilon^i \hat{U}_i.
\]

Write

\[
\kappa(\theta) \sigma (\rho, \theta) = \sum_{j=0}^{k} a_j^i \rho^j + \rho^{k+1} \kappa(\theta) \sigma^{k+1}(\rho, \theta),
\]

\[
\sigma^2(\rho, \theta) = \sum_{j=0}^{k} a_j^i \rho^j + \rho^{k+1} \kappa(\theta) (\sigma^2)^{k+1}(\rho, \theta),
\]

\[
\sigma^3(\rho, \theta) = \sum_{j=0}^{k} a_j^i \rho^j + \rho^{k+1} \left( \rho \kappa(\theta) (\sigma^3)^{k+1}(\rho, \theta) \right),
\]

where \( a_j^i, i = 1, 2, 3 \) are given by (A.8), and for \( \ell = 1, 2, 3 \)

\[
(\sigma^\ell)^{k+1}(\rho, \theta) = \left\{ \begin{array}{ll}
1 & \text{for } k \geq 0 \\
\sigma^\ell(\rho, \theta) & \text{for } k = -1.
\end{array} \right.
\]

Put \( k = M - i \) and substitute in (A.28) to get

\[
L_{e} u_{M}^{BL} = -\sum_{i=0}^{M} \varepsilon^i \left\{ a_i \partial_{\hat{\rho}} \hat{U}_i \right\} - \sum_{i=0}^{M} \varepsilon^i \left( \sum_{j=0}^{i} a_j^i \partial_{\hat{\rho}} \hat{U}_i \right) - \\
- \sum_{i=0}^{M} \varepsilon^i \left( \sum_{j=0}^{i} a_j^i \partial_{\hat{\rho}} \hat{U}_i \right) - \sum_{i=0}^{M} \varepsilon^i \left( \sum_{j=0}^{i} a_j^i \partial_{\hat{\rho}} \hat{U}_i \right) + \\
+ \sum_{i=0}^{M} \varepsilon^i \hat{U}_i - \varepsilon^{M+1} \kappa(\theta) \sigma (\hat{\rho}, \theta) \partial \hat{U}_M - \varepsilon^{M+1} \sigma^2(\hat{\rho}, \theta) \partial^2 \hat{U}_M - \\
- \varepsilon^{M+2} \rho \kappa(\theta) \sigma^3(\hat{\rho}, \theta) \partial^3 \hat{U}_M - \varepsilon^{M+2} \sigma^3(\hat{\rho}, \theta) \partial^3 \hat{U}_M - \\
- \sum_{i=0}^{M} \varepsilon^i \left( \sigma^M \right)^{i+1} \left\{ \left( \sigma^M \right)^{i+1} \varepsilon \kappa(\theta) \partial \hat{U}_i \right\} + \left\{ \left( \sigma^M \right)^{i+1} \varepsilon \kappa(\theta) \partial \hat{U}_i \right\} + \\
+ \varepsilon \hat{\kappa}(\theta) \left( \sigma^M \right)^{i+1} \partial \hat{U}_i \right\}.
\]
Making use of the identity
\[
\sum_{i=0}^{M} \sum_{j=0}^{M} i F(i,j) = \sum_{i=0}^{M} \sum_{j=0}^{M} F(i-j,j),
\]
we get
\[
L_{\varepsilon} u_{M}^{BL} = \sum_{i=0}^{M} \varepsilon^{i} \left[ -\frac{\partial^{2} \hat{U}_{i}}{\partial \hat{\rho}^{2}} - \sum_{j=0}^{i} \left( \frac{\partial \hat{\rho}}{\partial \hat{\rho}} \right) a_{i}^{j} \frac{\partial \hat{U}_{i-j}}{\partial \hat{\rho}} + a_{i}^{j} \frac{\partial^{2} \hat{U}_{i-j}}{\partial \theta^{2}} + \frac{\partial \hat{\rho}^{2}}{\partial \hat{\rho}} \right] + \hat{U}_{i} - \varepsilon^{M+1} \left( \kappa(\hat{\rho},\theta) \frac{\partial \hat{U}_{M}}{\partial \hat{\rho}} + \sigma^{2}(\hat{\rho},\theta) \frac{\partial^{2} \hat{U}_{M}}{\partial \theta^{2}} \right) \right] - 
\]
\[
- \varepsilon^{M+2} \left( \sigma^{2}(\hat{\rho},\theta) \frac{\partial^{2} \hat{U}_{M}}{\partial \theta^{2}} + \rho k'(\theta) \sigma^{3}(\hat{\rho},\theta) \frac{\partial \hat{U}_{M}}{\partial \hat{\rho}} \right) - 
\]
\[
- \sum_{i=0}^{M} \varepsilon^{M+1} \left( \frac{\partial \hat{\rho}}{\partial \hat{\rho}} \right) a_{i}^{M+1} \left[ \frac{\partial \hat{U}_{i-j}}{\partial \hat{\rho}} + \varepsilon^{K}(\hat{\rho}) \frac{\partial^{2} \hat{U}_{i-j}}{\partial \theta^{2}} + \varepsilon k'(\hat{\rho}) \left( \frac{\partial \hat{U}_{i-j}}{\partial \theta} \right)^{2} \right] \right] + 
\]
\[
\frac{\sigma^{2}(\hat{\rho},\theta) \frac{\partial^{2} \hat{U}_{M}}{\partial \theta^{2}} + \rho k'(\theta) \sigma^{3}(\hat{\rho},\theta) \frac{\partial \hat{U}_{M}}{\partial \hat{\rho}}}{\partial \theta^{2}} - 
\]
Note that by (A.10), (A.11) the term in brackets vanishes. Thus,
\[
L_{\varepsilon} u_{M}^{BL} = -\varepsilon^{M+1} \left\{ \kappa(\hat{\rho},\theta) \frac{\partial \hat{U}_{M}}{\partial \hat{\rho}} + \sigma^{2}(\hat{\rho},\theta) \left( \frac{\partial^{2} \hat{U}_{M}}{\partial \theta^{2}} \right) + \right. 
\]
\[
+ \rho k'(\theta) \sigma^{3}(\hat{\rho},\theta) \frac{\partial \hat{U}_{M}}{\partial \hat{\rho}} + \frac{\partial \hat{\rho}^{2}}{\partial \hat{\rho}} \right) + 
\]
\[
+ \sum_{i=0}^{M} \left( \frac{\partial \hat{\rho}}{\partial \hat{\rho}} \right) a_{i}^{M+1} \left[ \frac{\partial \hat{U}_{i-j}}{\partial \hat{\rho}} + \varepsilon^{K}(\hat{\rho}) \frac{\partial^{2} \hat{U}_{i-j}}{\partial \theta^{2}} + \left( \frac{\sigma^{2}}{\partial \theta^{2}} \right) + \right. 
\]
\[
+ \varepsilon k'(\hat{\rho}) \left( \frac{\partial \hat{U}_{i-j}}{\partial \theta} \right)^{2} \right] \}
\]
Using the bounds (A.22), (A.23) on \( \| \hat{U}_{i} \|_{s,\Omega_{0}} \) and the fact that \( a_{i}^{j}, i = 1, 2, 3 \) and \( \sigma^{M+i+1} \) are smooth, we obtain
\[
\| L_{\varepsilon} u_{M}^{BL} \|_{s,\Omega_{0}} \leq K(f, M, s) \varepsilon^{M+3/2}. \quad (A.29)
\]
Then, by (A.27), (A.29) and (A.24) we have \( \forall \ t > 0 \)
\[
\| r_{M}^{\varepsilon} \|_{s,\Omega} \leq \| g \|_{0,\Omega} + \| L_{\varepsilon} u_{M}^{RL} \|_{0,\Omega_{0}} + C \left( \varepsilon^{2} \| u_{M}^{RL} \|_{2,\Omega_{0},\lambda} + C \| u_{M}^{RL} \|_{0,\Omega_{0},\lambda} \right)
\]
\[
K(\Omega, M, f) \left\{ \varepsilon^{2M+2} + \varepsilon^{M+3/2} + \varepsilon^t \right\} \\
\leq K(\Omega, M, f)\varepsilon^{M+3/2}
\]
with \(K(\Omega, M, f)\) independent of \(\varepsilon\). This gives

\[
\|r_M^\varepsilon\|_{0,\Omega} \leq K(\Omega, M, f)\varepsilon^{M+3/2}, \quad (A.30)
\]
\[
\|r_M^\varepsilon\|_{1,\Omega} \leq K(\Omega, M, f)\varepsilon^{M+1/2}. \quad (A.31)
\]

We would like now to obtain bounds on \(r_M^\varepsilon\) in any Sobolev norm \(\|\cdot\|_s, s = 2, 3, \ldots\), similar to (A.30), (A.31). From (A.26) we have

\[
-\varepsilon^2 \Delta r_M^\varepsilon + r_M^\varepsilon = G \text{ in } \Omega,
\]
\[
r_M^\varepsilon = 0 \text{ on } \partial \Omega,
\]
where

\[
G := g + \varepsilon^2 \Delta (\chi u_{BL}^M) - \chi^H_M. \quad (A.33)
\]

Let \(\hat{x} = x/\varepsilon, \hat{y} = y/\varepsilon\) denote stretched variables, so that \(\Delta \hat{r}_M^\varepsilon = \varepsilon^2 \Delta r_M^\varepsilon\). Then (A.32) becomes

\[
-\hat{\Delta} \hat{r}_M^\varepsilon + \hat{r}_M^\varepsilon = \hat{G} \text{ in } \hat{\Omega},
\]
\[
\hat{r}_M^\varepsilon = 0 \text{ on } \hat{\partial} \hat{\Omega},
\]
where \(\hat{\Omega}\) is the image of \(\Omega\) under the stretched variables transformation. Note that (A.34) is an elliptic boundary value problem, thus we have the “shift” estimate

\[
\|\hat{r}_M^\varepsilon\|_{k,\hat{\Omega}} \leq K\|\hat{G}\|_{2,\hat{\Omega}}, \quad k = 2, 3, \ldots \quad (A.35)
\]

with \(K \in \mathbb{R}\). We need to investigate the relationship between \(\|\hat{G}\|_{k,\hat{\Omega}}\) and \(\|G\|_{k,\Omega}\), so that (A.35) can be used in obtaining bounds for \(\|r_M^\varepsilon\|_{k,\Omega}\). We have,

\[
\|\hat{G}\|_{0,\hat{\Omega}}^2 = \int_{\hat{\Omega}} \hat{G}^2(\hat{x}, \hat{y}) d\hat{x} d\hat{y} = \frac{1}{\varepsilon^2} \int_{\Omega} G^2(x, \varepsilon) dx dy,
\]

which gives \(\|\hat{G}\|_{0,\hat{\Omega}} = \frac{1}{\varepsilon} \|G\|_{0,\Omega}\). Similarly,

\[
\|\hat{G}\|_{1,\hat{\Omega}}^2 = \left\| \frac{\partial \hat{G}}{\partial \hat{x}} \right\|_{0,\hat{\Omega}}^2 + \left\| \frac{\partial \hat{G}}{\partial \hat{y}} \right\|_{0,\hat{\Omega}}^2 = \int_{\hat{\Omega}} \left\{ \left( \frac{\partial \hat{G}}{\partial \hat{x}}(\hat{x}, \hat{y}) \right)^2 + \left( \frac{\partial \hat{G}}{\partial \hat{y}}(\hat{x}, \hat{y}) \right)^2 \right\} d\hat{x} d\hat{y}
\]
\[
= \frac{1}{\varepsilon^2} \int_{\Omega} \left\{ \varepsilon^2 \left( \frac{\partial G}{\partial x} \right)^2 + \varepsilon^2 \left( \frac{\partial G}{\partial y} \right)^2 \right\} dx dy,
\]
which gives $|\hat{G}|_{1,\hat{\Omega}} = |G|^2_{1,\hat{\Omega}}$. In general, we get for $k = 0, 1, \ldots$

$$|\hat{G}|_{k,\hat{\Omega}} = \varepsilon^{k-1} |G|_{k,\hat{\Omega}}.$$  \hspace{1cm} (A.36)

Note that $\hat{\gamma}^s_M$ also satisfies (A.36). Now look at

$$|\hat{G}|_{2,\hat{\Omega}} = \sum_{i=0}^{2} |\hat{G}|_{i,\hat{\Omega}} = \sum_{i=0}^{2} \varepsilon^{2(i-1)} |G|^2_{i,\Omega} \leq K \left\{ \varepsilon^2 |G|^2_{0,\Omega} + \varepsilon^{2(k-1)} |G|^2_{k,\Omega} \right\},$$

thus,

$$|\hat{G}|_{2,\hat{\Omega}} \leq K \left\{ \varepsilon^1 |G|_{0,\Omega} + \varepsilon^{(k-1)} |G|_{k,\Omega} \right\}. \hspace{1cm} (A.37)$$

Using (A.37) together with the shift estimate (A.35), we obtain

$$\|\hat{\gamma}^s_M\|_{k,\hat{\Omega}} \leq K \left\{ \varepsilon^1 |G|_{0,\Omega} + \varepsilon^{(k-1)} |G|_{k,\Omega} \right\}.$$  

Since $\hat{\gamma}^s_M$ satisfies (A.36), we further have

$$\|\hat{\gamma}^s_M\|_{k,\Omega} \leq \varepsilon^{(1-k)} K \left\{ \varepsilon^1 |G|_{0,\Omega} + \varepsilon^{(k-1)} |G|_{k,\Omega} \right\}$$

which yields

$$\|\hat{\gamma}^s_M\|_{k,\Omega} \leq K \left\{ \varepsilon^k |G|_{0,\Omega} + \varepsilon^{2} |G|_{k,\Omega} \right\}. \hspace{1cm} (A.38)$$

Recall that by assumption, $|\chi^{(\nu)}(r)| \leq C(\rho, s)$, with $C \in \mathbb{R}$ independent of $p$ and $\varepsilon$. Then, by (A.33) and the above observation, we have

$$|G|_{s,\Omega} \leq \|g - L_x \chi u^BL_M\|_{s,\Omega} \leq \|g\|_{s,\Omega} + \|L_x \chi u^BL_M\|_{s,\Omega}$$

$$\leq \|g\|_{s,\Omega} + \|L_x \chi u^BL_M\|_{s,\Omega} + \|L_x u^BL_M\|_{s,\Omega,0}$$

$$\leq \|g\|_{s,\Omega} + \varepsilon^2 \Delta (\chi u^BL_M)_{s,\Omega,0} + C(s, \Omega) \|u^BL_M\|_{s,\Omega,0} + \|L_x u^BL_M\|_{s,\Omega,0}$$

Using (A.1) and (A.24) we further obtain $\forall t > 0$

$$|G|_{s,\Omega} \leq K(\Omega, M, f, s) \left\{ \varepsilon^{2M+2} + \varepsilon^t s + \varepsilon^{M+3/2} s \right\}, \hspace{1cm} (A.39)$$

$$\leq K(\Omega, M, f, s) \varepsilon^{M+3/2} s$$

with $K \in \mathbb{R}$, independent of $\varepsilon$. Using (A.39) and (A.38) we have

$$\|\hat{\gamma}^s_M\|_{k,\Omega} \leq K(\Omega, M, f, s) \left\{ \varepsilon^k \varepsilon^{M+3/2} + \varepsilon^{2M+3/2} k \right\} \hspace{1cm} (A.40)$$
with $K \in \mathbb{R}$ independent of $\varepsilon$. This establishes (2.10) for $k > 1$. For $k = 1$, (2.10) follows from (A.31). □

References