Finite element methods for a singularly perturbed transmission problem

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Abstract — We consider a one-dimensional singularly perturbed transmission problem with two different diffusion coefficients. The solution will contain boundary layers only in the part of the domain where the diffusion coefficient is high. We derive and analyze various finite element approaches for the approximation of the solution and conduct numerical computations that show the robustness (or lack thereof) of each approach.

Keywords: Transmission problems, boundary layers, finite element method

1. Introduction

The approximation of singularly perturbed problems has retained the attention of many authors in recent years. Let us mention [7,8,6] and the references quoted there, for different $h$ version finite element methods (FEMs) that lead to algebraic rates of convergence, and [10,5,9] for $hp$ FEMs that lead to robust exponential convergence (if the data of the problem is piecewise analytic). But, as far as we know, such analyses were not performed for differential operators with piecewise constant or piecewise smooth coefficients. On the other hand, in many real life applications, the differential operators have such piecewise coefficients that may have a very large discrepancy. In that case, the solution of the problem will contain boundary layers near the exterior boundary (as usual) but will also contain boundary layers along the interface where the coefficients have a large jump. We refer to [4] for the description of this phenomenon in one and two dimensions.

The results from [4] provide a decomposition of the solution into some explicit terms (outer expansion and layers) and a remainder that is itself a solution of a boundary value problem. This suggests the use an indirect approach that consists of an approximation (by a finite element method for instance) of this remainder.

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The goal of the present paper is to set up the $h$ version FEM for this indirect approach and to compare it with various standard direct approaches. By analyzing the indirect method proposed in [4], we discover that it is not the optimal one, and therefore we construct two alternative indirect approaches. Finally in order to perform the error analysis for some direct approaches we give a general decomposition of the solution of our transmission problem in the spirit of [5]. This allows us to show that the $p$ version FEM on a uniform mesh (resp. on a non-uniform one) leads to a non robust (resp. robust) exponential convergence rate if the data of the problem is analytic.

Finally, all these methods are compared numerically. From these tests we can conclude that the $p$ version on a non-uniform mesh is robust with respect to the small parameter and is the best one among the direct approaches, while the third indirect approach is the most attractive one. We further note that if one uses the $h$ version for the direct and indirect approaches, then the indirect ones are more competitive. We also perform a numerical test for the $p$ version on a uniform mesh using the third indirect approach (with a remainder without layers), and we observe a robust exponential rate of convergence. Hence its theoretical analysis should be developed.

Even if we restrict ourselves to the case of a model problem in one dimension we believe that it provides the main ideas how to proceed in higher dimensions. This will be the goal of a forthcoming paper.

Throughout the paper the spaces $H^s(I)$, with $s \geq 0$, are the standard Sobolev spaces on the interval $I = (a,b)$, with norm $\| \cdot \|_{s,I}$ and semi-norm $| \cdot |_{s,I}$. The space $H^1_0(I)$ is defined, as usual, by $H^1_0(I) := \{ v \in H^1(I) : v(a) = v(b) = 0 \}$. $L^p(I)$, $p > 1$, are the usual Lebesgue spaces with norm $\| \cdot \|_{0,p,I}$ (we drop the index $p$ for $p = 2$). Finally, the notation $A \lesssim B$ means the existence of a positive constant $C$, which is independent of the quantities $A$ and $B$ under consideration and of the parameter $\varepsilon$, such that $A \leq CB$.

The rest of the paper is organized as follows: In Section 2 we present the one-dimensional singularly perturbed transmission problem and describe the typical phenomena. Section 3 is devoted to the development of finite element methods based on a direct approach and Section 4 contains three finite element methods based on indirect approaches. Finally, in Section 5 we show the results of extensive numerical computations for all the methods considered.

2. The model problem and its regularity

Let $\varepsilon \in (0,1)$ be a fixed parameter and $f$ a given sufficiently smooth function, e.g. $f \in H^2(-1,0)$. Consider the following transmission problem in $(-1,1)$: Find $u_\varepsilon$
such that

\[
\begin{cases}
-\varepsilon^2 (u^-)'' + u^- = f & \text{in } (-1, 0), \\
-(u^+)' + u^+ = 0 & \text{in } (0, 1), \\
u^+(-1) = u^-(1) = 0, \\
\varepsilon (u^-)'(0) - (u^+)'(0) = 0,
\end{cases}
\]

(2.1)

where \( u^- \) (resp. \( u^+ \)) means the restriction of \( u \) to \((-1, 0)\) (resp. \((0, 1)\)). Note that in this problem the small parameter \( \varepsilon \) appears only on \((-1, 0)\). Consequently the formal limit problem is the following non standard transmission problem:

\[
\begin{cases}
u_0 - = f & \text{in } (-1, 0), \\
-(u_0^+)' + u_0^+ = 0 & \text{in } (0, 1), \\
u_0^-(-1) = u_0^+(1) = 0, \\
u_0^- (0) - u_0^+(0) = 0, \\
(u_0^+)'(0) = 0.
\end{cases}
\]

(2.2)

This limit problem has a solution \( u_0^- \equiv 0 \), but has no solution \( u_0^+ \) since \( u_0^- = f \) does not, in general, satisfy the boundary condition \( u_0^- (-1) = 0 \) and the transmission condition \( u_0^- (0) - u_0^+(0) = 0 \). Therefore we may expect that \( u \) will develop boundary layers at 0 (transmission layer) and at -1 (standard boundary layer). This was explained in full detail in [4] for \( f = 1 \). We also refer to Figure 1 for an illustration of this fact.

![Exact solution with f(x) = 1 and \( \varepsilon = 0.05 \).](attachment:exact_solution.png)

Figure 1. The exact solution \( u \) with \( f(x) = 1 \) and \( \varepsilon = 0.05 \).
We now extend the results from [4] to the case of an arbitrary smooth function $f$. As in [4] the transmission layer at 0 may be seen as a (Dirichlet) boundary layer. Nevertheless in numerical tests, we shall see that this point of view is too limited, hence we shall describe alternative decompositions that yield better numerical results.

We first recall the variational formulation of problem (2.1): On $H^1_0(-1,1)$, we introduce the bilinear form

$$a_\epsilon(u,v) = \int_{-\epsilon}^{0} e^2 u' v' \, dx + \int_{0}^{1} u' v' \, dx + \int_{-\epsilon}^{0} uv \, dx + \int_{0}^{1} uv \, dx, \forall u, v \in H^1_0(-1,1).$$

For further use we also introduce the associated energy norm

$$\|u\|^2 = a_\epsilon(u,u), \forall u \in H^1_0(-1,1).$$

Then the weak formulation of (2.1) consists of looking for $u_\epsilon \in H^1_0(-1,1)$ such that

$$a_\epsilon(u_\epsilon,v) = \int_{-\epsilon}^{0} f v \, dx, \forall v \in H^1_0(-1,1). \quad (2.3)$$

This formulation is obtained by multiplying (2.1) by a test function $v$, taking the sum and integrating by parts in $(-1,0)$ and $(0,1)$.

The existence and uniqueness of a solution to (2.3) follows directly from the Lax-Milgram lemma. Furthermore, by integration by parts with $v \in \mathcal{D}(0,1)$, $v \in \mathcal{D}(-1,0)$ and then with $v \in \mathcal{D}(-1,1)$, we see that $u_\epsilon^- \in H^2(-1,0)$, $u_\epsilon^+ \in H^2(0,1)$ and $u_\epsilon$ is indeed a solution of (2.1) (Here, $\mathcal{D}(a,b)$ denotes the space of smooth functions with compact support in the interval $(a,b)$). In particular, we obtain

$$\|u_\epsilon\|_{0,(-1,0)} \lesssim \epsilon^{-2} \|f\|_{0,(-1,0)}, \|u_\epsilon\|_{0,(0,1)} \lesssim \|f\|_{0,(0,1)}. \quad (2.4)$$

The following theorem is a generalization of Theorem 2.1 from [4].

**Theorem 2.1** For any $\epsilon \in (0,1)$, the unique solution $u_\epsilon$ of (2.1) satisfies

$$u_\epsilon(x) = f(x) - \chi^b(x)f(-1)e^{-\frac{|x-\epsilon|}{\epsilon^2}} - \chi^i(x)f(0)e^\frac{|x|}{\epsilon} + r_\epsilon(x), \forall x \in (-1,0), \quad (2.5)$$

where $\chi^b$ and $\chi^i$ are the two following cut-off functions:

$$\begin{cases} 
\chi^b = 1 \text{ on } (-1, -1+\eta), \\
\chi^i = 1 \text{ on } (-\eta, \eta), \\
supp \chi^b \cap supp \chi^i = \emptyset.
\end{cases}$$

Moreover the next estimate holds:

$$\epsilon \|r_\epsilon\|_{0,(-1,0)} + \|r_\epsilon\|_{0,(-1,0)} + \|u_\epsilon\|_{1,(0,1)} \lesssim \epsilon |f(0)| \quad (2.6)$$

$$+ \epsilon^2 (\|f''\|_{0,(-1,0)} + |f'(0)|) + (|f(-1)| + |f(0)|)\epsilon e^{-\frac{\eta}{\epsilon^2}}.$$
Proof. Let us define the functions \( v^b : x \mapsto e^{-\frac{x}{\varepsilon}} \) and \( v^f : x \mapsto e^{\frac{i}{\varepsilon}} \), which are, respectively, the solutions of

\[
\begin{aligned}
\begin{cases}
-\varepsilon^2 (v^b)'' + v^b = 0 & \text{in } (-1, \infty), \\
v^b(-1) = 1, & \\
v^b(+\infty) = 0,
\end{cases}
\end{aligned}
\quad \begin{aligned}
\begin{cases}
-\varepsilon^2 (v^f)'' + v^f = 0 & \text{in } (-\infty, 0), \\
v^f(0) = 1, & \\
v^f(-\infty) = 0.
\end{cases}
\end{aligned}
\]

Using these two problems and by substitution of (2.5) in (2.1), we see that \((r_\varepsilon, u^+_\varepsilon)\) is the solution of

\[
\begin{aligned}
\begin{cases}
-\varepsilon^2 r''_\varepsilon + r_\varepsilon = g_\varepsilon & \text{in } (-1, 0), \\
u^+_\varepsilon - (u^+_\varepsilon)^b = 0 & \text{in } (0, 1), \\
r_\varepsilon(-1) = 0, & \\
u^+_\varepsilon(1) = 0, & \\
r_\varepsilon(0) = u^+_\varepsilon(0), & \\
e^2 u'_\varepsilon(0) - (u^+_\varepsilon)'(0) = \alpha,
\end{cases}
\end{aligned}
\]  

(2.7)

where \( \alpha = e f(0) - e^2 f'(0) \) and

\[
g_\varepsilon := e^2 \left( f'' - f(-1)[\chi^b, \frac{d^2}{dx^2}]v^b - f(0)[\chi^f, \frac{d^2}{dx^2}]v^f \right),
\]

(2.8)

with the bracket \([\chi^b, \frac{d^2}{dx^2}]\) defined as usual by

\[
[\chi^b, \frac{d^2}{dx^2}] h = \frac{d^2}{dx^2} (\chi^b h) - \chi^b \frac{d^2}{dx^2} h = h \frac{d^2}{dx^2} \chi^b + 2 \frac{d}{dx} \chi^b \frac{d}{dx} h.
\]

The transmission condition \( r_\varepsilon(0) = u^+_\varepsilon(0) \) suggests to extend the remainder \( r_\varepsilon \) in the whole domain \((-1, 1)\) by setting:

\[
r_\varepsilon = r_\varepsilon^+ = u^+_\varepsilon.
\]

Then the variational formulation of problem (2.7) reads: find \( r_\varepsilon \in H^1_0(-1, 1) \) such that

\[
a_\varepsilon(r_\varepsilon, w) = \int_{-1}^0 g_\varepsilon w \, dx + \alpha w(0), \; \forall w \in H^1_0(-1, 1).
\]

(2.9)

Since the left-hand side of (2.9) is trivially coercive on \( H^1_0(-1, 1) \), the Lax-Milgram lemma guarantees that this problem has a unique solution \( r_\varepsilon \in H^1(-1, 1) \). Moreover by using \( w = r_\varepsilon \) as a test function in (2.9) we get

\[
e^2 ||r_\varepsilon'||^2_{0,(-1,0)} + ||r_\varepsilon||^2_{0,(-1,0)} + ||u^+_\varepsilon||^2_{0,(0,1)} + ||u^e||^2_{0,(0,1)} \leq ||g_\varepsilon||_{0,(-1,0)} ||r_\varepsilon||_{0,(-1,0)} + |\alpha| |u_\varepsilon(0)|.
\]

(2.10)
To estimate the $L^2$-norm of $g_{\varepsilon}$, note that
\[
\|g_{\varepsilon}\|_{0,(-1,0)} \lesssim \varepsilon^2 \left( \|f''\|_{0,(-1,0)} + |f(-1)| \|\chi^b; \frac{d^2}{dx^2}\|_{0,(-1,0)} + |f(0)| \|\chi^i; \frac{d^2}{dx^2}\|_{0,(-1,0)} \right).
\]

By the properties of $\chi^b$ and $\chi^i$ we have that the last two terms above have support on the interval $[-1+\eta, -\eta]$. Since on this interval $e^{-\frac{|u_{\varepsilon}|}{\varepsilon^2}} \leq e^{-\frac{\eta}{\varepsilon_0}}$ and $e^\frac{\eta}{\varepsilon_0} \leq e^{-\frac{\eta}{\varepsilon^2}}$, we obtain
\[
\|g_{\varepsilon}\|_{0,(-1,0)} \lesssim (|f(-1)| + |f(0)|) \varepsilon e^{-\frac{\eta}{\varepsilon^2}} + \varepsilon^2 \|f''\|_{0,(-1,0)}.
\]

(2.11)

Now using a standard trace theorem, we have
\[
|u_{\varepsilon}(0)| \lesssim \|u_{\varepsilon}\|_{1,(0,1)}.
\]

Using this estimate in (2.10), we get
\[
\varepsilon^2 \|r_{\varepsilon}'\|_{0,(-1,0)}^2 + \|r_{\varepsilon}\|_{0,(-1,0)}^2 + \|u_{\varepsilon}\|_{1,(0,1)}^2 \
\lesssim \|g_{\varepsilon}\|_{0,(-1,0)} \|r_{\varepsilon}\|_{0,(-1,0)} + |\alpha| \|u_{\varepsilon}\|_{1,(0,1)}.
\]

By skipping the first term of the left hand side and using Cauchy-Schwarz’s inequality (in $\mathbb{R}^2$) we obtain
\[
\|r_{\varepsilon}\|_{0,(-1,0)}^2 + \|u_{\varepsilon}\|_{1,(0,1)}^2 \lesssim \|g_{\varepsilon}\|_{0,(-1,0)}^2 + \alpha^2.
\]

This estimate in the previous one leads to
\[
\varepsilon^2 \|r_{\varepsilon}'\|_{0,(-1,0)}^2 + \|r_{\varepsilon}\|_{0,(-1,0)}^2 + \|u_{\varepsilon}\|_{1,(0,1)}^2 \lesssim \|g_{\varepsilon}\|_{0,(-1,0)}^2 + \alpha^2.
\]

We conclude using the estimate (2.11). □

Note that at least in the case $f = 1$, the estimate (2.6) is optimal because direct calculations yield $\|u_{\varepsilon}\|_{0,(0,1)} \sim \varepsilon$ (cf. [4]).

The above theorem gives an explicit expansion of $u_{\varepsilon}$, which also shows that $u_{\varepsilon}$ has two layers (at $0$ and $-1$). It further says that the natural energy norm of the remainder $r_{\varepsilon}$ is of order $\varepsilon$. Finally it says that $u_{\varepsilon}$ has no layer and that its natural energy norm is of order $\varepsilon$.

This theorem also allows us to obtain an a priori estimate in the $H^2$ semi-norm:

**Corollary 2.2** Under the assumptions of the previous theorem, we have
\[
\varepsilon^2 \|r_{\varepsilon}'\|_{0,(-1,0)} + \|u_{\varepsilon}\|_{0,(0,1)} \lesssim \varepsilon |f(0)| + \varepsilon^2 \left( \|f''\|_{0,(-1,0)} + |f'(0)| + (|f(-1)| + |f(0)|) \right) e^{-\frac{\eta}{\varepsilon^2}}.
\]

(2.12)
\textbf{Proof.} Using (2.1), we see that
\[ u''_\varepsilon = u_\varepsilon \quad \text{in} \ (0, 1). \]
Hence using (2.6), we directly get
\[ \|u''_\varepsilon\|_{0, (0, 1)} \leq \varepsilon |f(0)| + \varepsilon^2 (\|f''\|_{0, (-1, 0)} + |f'(0)|) + (|f(-1)| + |f(0)|) \varepsilon e^{-\frac{\eta}{\varepsilon}}. \]

For the remainder \( r_\varepsilon \) in \((-1, 0)\), by (2.7) we notice that
\[ \varepsilon^2 r''_\varepsilon = r_\varepsilon - g_\varepsilon \quad \text{in} \ (-1, 0). \]
Therefore
\[ \varepsilon^2 \|r''_\varepsilon\|_{0, (-1, 0)} \leq \|r_\varepsilon\|_{0, (-1, 0)} + \|g_\varepsilon\|_{0, (-1, 0)}, \]
and by (2.6) and (2.11), we conclude that
\[ \varepsilon^2 \|r''_\varepsilon\|_{0, (-1, 0)} \lesssim \varepsilon |f(0)| + \varepsilon^2 (\|f''\|_{0, (-1, 0)} + |f'(0)|) + (|f(-1)| + |f(0)|) \varepsilon e^{-\frac{\eta}{\varepsilon}}. \]

\[ \square \]

The estimate (2.12) is not fully satisfactory because it yields the estimate
\[ \|r''_\varepsilon\|_{0, (-1, 0)} \lesssim \varepsilon^{-1} |f(0)| + \varepsilon^2 (\|f''\|_{0, (-1, 0)} + |f'(0)|) + \varepsilon^{-1} e^{-\frac{\eta}{\varepsilon}}, \]
which is not uniform in \( \varepsilon \) due to the first term of this right hand side. By analyzing carefully the proof of Theorem 2.1 we can see that this first term appears since the interface layer, extended by zero in \((0, 1)\), actually does not satisfy the interface condition at 0. Therefore an alternative approach is to introduce the following new interface layer \( w^i \) defined by
\[ w^i : x \mapsto \begin{cases} w^i_-(x) = \frac{1}{1 + \varepsilon} e^{\frac{\varepsilon}{1 + \varepsilon} x} & \text{if } x < 0, \\ w^i_+(x) = -\frac{\varepsilon}{1 + \varepsilon} e^{-x} & \text{if } x > 0, \end{cases} \]
which is the solution of
\[ \begin{cases} -\varepsilon^2 \left( w^i \right)' + w^i = 0, & \text{in } (-\infty, 0), \\ -\left( w^i \right)' + w^i = 0, & \text{in } (0, \infty), \\ w^i(0-) - w^i(0+) = 1, & \\ \varepsilon^2 \left( w^i \right)'(0) - \left( w^i \right)'(0) = 0, & \\ w^i(-\infty) = w^i(+\infty) = 0. & \end{cases} \]

We notice that the restriction \( w^i_- \) of \( w^i \) to \((-\infty, 0)\) is very close to \( v^i \), while the restriction \( w^i_+ \) of \( w^i \) to \((0, \infty)\) is exponentially decaying and is bounded by \( \varepsilon \). An illustration of \( w^i \) is shown in figure 2 below.

With this new layer, we can build another splitting for \( u_\varepsilon \) and prove the following results:
Figure 2. The interface layer function $w_i$ for $\varepsilon = 0.075$.

**Theorem 2.3** For any $\varepsilon \in (0, 1)$, the unique solution $u_\varepsilon$ of (2.1) can be split up as follows

$$u_\varepsilon(x) = f(x) - \chi^b(x)f(-1)e^{-\frac{x-1}{\varepsilon}} - \chi^i(x)f(0)w^l(x) + \hat{r}_\varepsilon^-(x), \quad \forall x \in (-1, 0),$$

$$u_\varepsilon(x) = -\chi^i(x)f(0)w^l(x) + \hat{r}_\varepsilon^+(x), \quad \forall x \in (0, 1),$$

where $\chi^b$ and $\chi^i$ are the two cut-off functions introduced in Theorem 2.1. Moreover the next estimate holds:

$$|||\hat{r}_\varepsilon|||_{\varepsilon} \lesssim \varepsilon |f(0)| + \varepsilon^2 (|||f''|||_{0,(-1,0)} + |f'(0)|) + (|f(-1)| + |f(0)|)\varepsilon^{-\eta}. \quad (2.15)$$

**Proof.** Using the system satisfied by the boundary layer function $v^b$ and the interface layer function $w^l$, the decomposition (2.13)–(2.14), and finally (2.1), we see that $\hat{r}_\varepsilon$ is the solution of

$$\begin{cases}
-\varepsilon^2 (\hat{r}_\varepsilon^-)'' + \hat{r}_\varepsilon^- = \hat{g}_\varepsilon^- & \text{in } (-1, 0), \\
\hat{r}_\varepsilon^+ - (\hat{r}_\varepsilon^+)' = \hat{g}_\varepsilon^+ & \text{in } (0, 1), \\
\hat{r}_\varepsilon^- (-1) = 0, \\
\hat{r}_\varepsilon^+ (1) = 0, \\
\hat{r}_\varepsilon^- (0) = \hat{r}_\varepsilon^+ (0), \\
\varepsilon^2 (\hat{r}_\varepsilon^-)'(0) - (\hat{r}_\varepsilon^+)'(0) = \varepsilon^2 f'(0),
\end{cases} \quad (2.16)$$

where

$$\hat{g}_\varepsilon^- := \varepsilon^2 \left( f'' - f(-1)|\chi^b| \frac{d^2}{dx^2}v^b - f(0)|\chi^i| \frac{d^2}{dx^2}w^l \right),$$

$$\hat{g}_\varepsilon^+ := -f(0)|\chi^i| \frac{d^2}{dx^2}w^l. \quad (2.17)$$

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The variational formulation of problem (2.16) reads: find $\hat{r}_\varepsilon \in H^1_0(-1,1)$ such that
\[
a_\varepsilon(\hat{r}_\varepsilon, w) = \int_{-1}^\varepsilon \hat{g}_\varepsilon^- w \, dx + \int_0^1 \hat{g}_\varepsilon^+ w \, dx + \varepsilon^2 f'(0)w(0), \forall w \in H^1_0(-1,1). \tag{2.19}
\]
The Lax-Milgram lemma guarantees that this problem has a unique solution $\hat{r}_\varepsilon \in H^1(-1,1)$. Moreover by taking $w = \hat{r}_\varepsilon$ as a test function in (2.19) and using Cauchy-Schwarz’s inequality we get
\[
|||\hat{r}_\varepsilon||| \lesssim ||\hat{g}_\varepsilon^-||_{0,(-1,0)} + ||\hat{g}_\varepsilon^+||_{0,(0,1)} + \varepsilon^2 |f'(0)|. \tag{2.20}
\]
The $L^2$-norm of $\hat{g}_\varepsilon^-$ and $\hat{g}_\varepsilon^+$ are estimated like in Theorem 2.1 and we obtain
\[
||\hat{g}_\varepsilon^-||_{0,(-1,0)} \lesssim \varepsilon(|f(1)| + |f(0)|)e^{\varepsilon k} + \varepsilon^2 ||f'||_{0,(-1,0)},
\]
\[
||\hat{g}_\varepsilon^+||_{0,(0,1)} \lesssim \varepsilon|f(0)|.
\]
The use of these estimates in (2.20) leads to the conclusion. □

In this theorem the term $\varepsilon |f(0)|$ has not yet disappeared hence the same proof as the one of Corollary 2.2 leads to the $H^2$ estimate:
\[
\varepsilon^2 ||\hat{r}_\varepsilon'||_{0,(-1,0)} + ||\hat{r}_\varepsilon''||_{0,(0,1)} \lesssim \varepsilon |f(0)| + \varepsilon^2 (||f''||_{0,(-1,0)} + |f'(0)|) + (|f(1)| + |f(0)|)ke^{\varepsilon k}. \tag{2.21}
\]
Here we see that the term $\varepsilon |f(0)|$ comes from the term $||\hat{g}_\varepsilon^+||_{0,(0,1)}$ and hence due to the use of the cut-off function $\chi^\varepsilon$. Therefore we should avoid the use of such cut-off functions. To do so we need to introduce layers fulfilling the homogeneous starting transmission problem except at $-1$ and $0$. This leads to the third approach. More precisely, we denote by $\tilde{\varphi}_0^\varepsilon, \tilde{\varphi}_1^\varepsilon$ the unique solutions of
\[
\begin{align*}
-\varepsilon^2 (\tilde{\varphi}_0^\varepsilon)'' + \tilde{\varphi}_0^\varepsilon &= 0, &\text{in } (-1,0), \\
-(\tilde{\varphi}_0^\varepsilon)' + \tilde{\varphi}_0^\varepsilon &= 0, &\text{in } (0,1), \\
\tilde{\varphi}_0^\varepsilon(0) - \tilde{\varphi}_0^\varepsilon(0) &= 0, \\
\varepsilon^2 (\tilde{\varphi}_0^\varepsilon)'(0) - (\tilde{\varphi}_0^\varepsilon)'(0) &= 0, \\
\tilde{\varphi}_0^\varepsilon(-1) &= 1, \\
\tilde{\varphi}_0^\varepsilon(1) &= 0,
\end{align*}
\]
and
\[
\begin{align*}
-\varepsilon^2 (\tilde{\varphi}_1^\varepsilon)'' + \tilde{\varphi}_1^\varepsilon &= 0, &\text{in } (-1,0), \\
-(\tilde{\varphi}_1^\varepsilon)' + \tilde{\varphi}_1^\varepsilon &= 0, &\text{in } (0,1), \\
\tilde{\varphi}_1^\varepsilon(0) - \tilde{\varphi}_1^\varepsilon(0) &= 1, \\
\varepsilon^2 (\tilde{\varphi}_1^\varepsilon)'(0) - (\tilde{\varphi}_1^\varepsilon)'(0) &= 0, \\
\tilde{\varphi}_1^\varepsilon(-1) &= 0, \\
\tilde{\varphi}_1^\varepsilon(1) &= 0.
\end{align*}
\]
These solutions can be computed analytically and are given by

\[
\begin{aligned}
  \tilde{v}_b(x) &= a_1 e^{-x} + a_2 e^x + \varepsilon \text{ for } x < 0, \\
  \tilde{v}_b(x) &= b_1 e^{-x} + b_2 e^x + \varepsilon \text{ for } x > 0,
\end{aligned}
\]

(2.22)

where

\[
\begin{aligned}
a_1 &= (k\varepsilon + 1)(1 - e^{-\frac{x}{\varepsilon}} + \varepsilon k(1 + e^{-\frac{x}{\varepsilon}}))^{-1}, \\
b_1 &= \varepsilon e^{-\frac{x}{\epsilon}}(\sinh 1)^{-1}(1 - e^{-\frac{x}{\varepsilon}} + \varepsilon k(1 + e^{-\frac{x}{\varepsilon}}))^{-1},
\end{aligned}
\]

with \( k = \frac{1 - \varepsilon^2}{1 + \varepsilon^2} \).

\[
\begin{aligned}
  \tilde{v}_i^-(x) &= c_1 e^x + c_2 e^{-x} \text{ if } x < 0, \\
  \tilde{v}_i^+(x) &= d_1 e^{-x} + d_2 e^x \text{ if } x > 0,
\end{aligned}
\]

(2.23)

where

\[
\begin{aligned}
c_1 &= (1 - e^{-\frac{x}{\varepsilon}} + \varepsilon k(1 + e^{-\frac{x}{\varepsilon}}))^{-1}, \\
c_2 &= -c_1 e^{-2}, \\
d_1 &= -c_2 (1 + e^{-\frac{x}{\varepsilon}})(1 + e^{-2})^{-1}, \\
d_2 &= -d_1 e^{-2}.
\end{aligned}
\]

Again we can remark that \( \tilde{v}^i \) is very close to \( v^i \), while \( \tilde{v}^b \) is very close to \( v^b \). An illustration of the functions \( \tilde{v}^i \) and \( \tilde{v}^b \) is shown in figure 3 below.

With these new layers, we can build a third splitting for \( u_\varepsilon \) and prove the following results.

**Theorem 2.4** For any \( \varepsilon \in (0, 1) \), the unique solution \( u_\varepsilon \) of (2.1) can be split up as follows

\[
u_\varepsilon = \tilde{f} - f(-1)\tilde{v}^b - f(0)\tilde{v}^i + \tilde{r}_\varepsilon,
\]

(2.24)

where \( \tilde{f} \) is the extension of \( f \) by zero outside \((-1, 0)\). Moreover the next estimate holds:

\[
|||\tilde{r}_\varepsilon||| \lesssim \varepsilon^2 (\sqrt{\varepsilon} f''(0, (-1, 0)) + |f'(0)|).
\]

(2.25)

**Proof.** Using the system satisfied by the boundary layer function \( \tilde{v}^b \) and the interface layer function \( \tilde{v}^i \), the decomposition (2.24), and finally (2.1), we see that
The functions $\tilde{v}^b$ (top) and $\tilde{v}^i$ (bottom), for $\epsilon = 0.075$.

$\tilde{r}_\epsilon$ is the solution of
\begin{align*}
-\epsilon^2 (\tilde{r}_\epsilon^-)''' + \tilde{r}_\epsilon^- = \epsilon^2 f'' & \quad \text{in } (-1,0), \\
\tilde{r}_\epsilon^+ - (\tilde{r}_\epsilon^+)' &= 0 & \quad \text{in } (0,1), \\
\tilde{r}_\epsilon^-(-1) &= 0, \\
\tilde{r}_\epsilon^+(1) &= 0, \\
\tilde{r}_\epsilon^-(0) &= \tilde{r}_\epsilon^+(0), \\
\epsilon^2 (\tilde{r}_\epsilon^-)'(0) - (\tilde{r}_\epsilon^+)'(0) &= \epsilon^2 f'(0). 
\end{align*}

(2.26)

The variational formulation of problem (2.26) reads: find $\tilde{r}_\epsilon \in H_0^1(-1,1)$ such that
\begin{align*}
a_\epsilon(\tilde{r}_\epsilon, w) = \epsilon^2 \int_{-1}^{0} f'' w \, dx + \epsilon^2 f'(0)w(0), \forall w \in H_0^1(-1,1). 
\end{align*}

(2.27)

The Lax-Milgram lemma guarantees that this problem has a unique solution $\tilde{r}_\epsilon \in H^1(-1,1)$, while (2.25) follows from the same arguments as the ones used in Theorem 2.1.
In this theorem the factor $\varepsilon |f(0)|$ has disappeared and the same proof as the one of Corollary 2.2 leads to the following improved $H^2$ estimate.

**Corollary 2.5** Under the assumptions of the previous theorem, we have

$$\varepsilon^2 \| \tilde{r}_\varepsilon'' \|_{0,(-1,0)} + \| \tilde{r}_\varepsilon'' \|_{0,(0,1)} \lesssim \varepsilon^2 (\| f'' \|_{0,-1,0} + |f'(0)|). \quad (2.28)$$

**Remark 2.6** From (2.26), we can see that the right-hand side is a multiple of $\varepsilon^2$. Hence if we set $s_{\varepsilon} = \varepsilon^{-2} \tilde{r}_\varepsilon$, we see that $s_{\varepsilon}$ is solution of

$$\begin{cases}
-\varepsilon^2 (s_{\varepsilon}'')'' + s_{\varepsilon}'' = f'' & \text{in } (-1,0), \\
 s_{\varepsilon}'' - (s_{\varepsilon})'' = 0 & \text{in } (0,1), \\
 s_{\varepsilon}(-1) = 0, \\
 s_{\varepsilon}(1) = 0, \\
 s_{\varepsilon}(0) = s_{\varepsilon}(0), \\
 \varepsilon^2 (s_{\varepsilon}'(0) - (s_{\varepsilon})'(0) = f'(0).
\end{cases}$$

But the solution of this problem has boundary layers since the formal limit $s$ of this problem should be the solution of

$$\begin{cases}
 s_- = f'' & \text{in } (-1,0), \\
 s_+ - s_-' = 0 & \text{in } (0,1), \\
 s_-(-1) = 0, \\
 s_+(1) = 0, \\
 s_-(0-) = s_+(0+), \\
 -s_+'(0+) = f'(0).
\end{cases}$$

The differential equation in $(0,1)$ and the boundary conditions at $-1$ and 0 imply that

$$s_+ = -f'(0)(x-1) \text{ in } (0,1),$$

while we recall that the first identity means that

$$s_- = f'' \text{ in } (-1,0).$$

But then the boundary condition at $-1$ is not satisfied (at least for $f''(-1) \neq 0$) and the interface condition $s_-(0-) = s_+(0+)$ is also not satisfied (if $f''(0) \neq f'(0)$). These two facts will be responsible for the layer of $s_{\varepsilon}$ at $-1$ and at 0, and since $\tilde{r}_\varepsilon = \varepsilon^2 s_{\varepsilon}$, the same phenomena appear at a lower level for $\tilde{r}_\varepsilon$ (this can be observed in numerical tests). Obviously this problem will be avoided if $f$ is such that $f''(-1) = 0$ and $f''(0) = f'(0)$ (see Section 5).

In the sections that follow we will consider different finite element approximations for $u_\varepsilon$ based on the various decompositions obtained in this section. To this end, we consider a final decomposition which is based on the one found in [5].
A calculation shows that
\[
-\varepsilon^2 \left( (u^c)^{''} - w^c_M \right) + \left[ u^c - w^c_M \right] = \varepsilon^{2M+2} f^{(2M+2)}.
\]

Then, with \( \bar{\psi}^b, \bar{\psi}^c \) as in Theorem 2.4, we have the following decomposition for \( u^c \):
\[
u^c(x) = w_M(x) - w_M(-1) \bar{\psi}^b(x) - w_M(0) \bar{\psi}^c(x) + \tau^c(x),
\]
where \( \tau^c(x) \) is the solution of
\[
\begin{align*}
-\varepsilon^2 (\tau^{c^+})'' + \tau^{c^-} &= \varepsilon^{2M+2} f^{(2M+2)} \quad \text{in } (-1,0), \\
\tau^{c^+}(-1) &= 0, \\
\tau^{c^+}(0) &= \tau^{c^-}(0), \\
\varepsilon^2 (\tau^{c^+})'(0) - (\tau^{c^-})'(0) &= \varepsilon^2 w_M' (0).
\end{align*}
\]

The function \( \tau^c(x) \) can be further decomposed as
\[
\tau^c(x) = \varepsilon^2 \left( \sum_{j=0}^{M-1} \varepsilon^{2j+1} f^{(2j+1)}(0) \right) \rho(x) + s^c(x),
\]
where \( \rho(x) \) is the solution of
\[
\begin{align*}
-\varepsilon^2 (\rho^-)'' + \rho^- &= 0, \quad \text{in } (-1,0), \\
-\varepsilon^2 (\rho^+)' + \rho^+ &= 0, \quad \text{in } (0,1), \\
\rho^-(0) &= -\rho^+(0), \\
\varepsilon^2 (\rho^-)'(0) - (\rho^+)'(0) &= 1, \\
\rho^-(1) &= \rho^+(1) = 0,
\end{align*}
\]
and \( s^c \) is the solution of
\[
\begin{align*}
-\varepsilon^2 (s^-_c)'' + s^-_c &= \varepsilon^{2M+2} f^{(2M+2)}, \quad \text{in } (-1,0), \\
-\varepsilon^2 (s^+_c)' + s^+_c &= 0, \quad \text{in } (0,1), \\
s^-_c(0) - s^+_c(0) &= 0, \\
\varepsilon^2 (s^-_c)'(0) - (s^+_c)'(0) &= \varepsilon^{2M+2} f^{(2M+1)}(0), \\
s^-_c(-1) &= s^+_c(1) = 0.
\end{align*}
\]

The variational formulation of (2.34) reads: find \( s^c \in H^1_0(-1,1) \) such that \( \forall w \in H^1_0(-1,1), \)
\[
a^c(s^c, w) = \varepsilon^{2M+2} \int_{-1}^{0} f^{(2M+2)} w dx + \varepsilon^{2M+2} f^{(2M+1)}(0) w(0).
\]
from which we obtain, in the same fashion as before, the following estimate:

\[ ||s_\varepsilon||_\varepsilon \lesssim \varepsilon^{2M+2} \|f^{(2M+2)}\|_{0,(-1,0)} + \varepsilon^{2M+2} |f^{(2M+1)}(0)|. \] (2.36)

Note that the solution of (2.33) can be obtained analytically as

\[
\rho(x) = \begin{cases} 
\alpha_1 e^{\frac{x}{\varepsilon^2}} + \alpha_2 e^{\frac{-x}{\varepsilon^2}} & \text{if } x < 0, \\
\beta_1 e^{1-x} + \beta_2 e^{x-1} & \text{if } x > 0,
\end{cases}
\] (2.37)

where

\[
\alpha_1 = -\beta_1 \frac{\sinh 1}{\sinh \left(\frac{1}{\varepsilon}\right)}, \\
\alpha_2 = -\alpha_1,
\] 
\[
\beta_1 = \frac{\sinh \left(\frac{1}{\varepsilon}\right)}{2 (\cosh 1 \sinh \left(\frac{1}{\varepsilon}\right) + \varepsilon \sinh 1 \cosh \left(\frac{1}{\varepsilon}\right))},
\] 
\[
\beta_2 = -\beta_1.
\]

Hence, the decomposition (2.30) becomes

\[ u_\varepsilon = w_M - w_M(-1)\tilde{v}^b - w_M(0)\tilde{v}^d + \varepsilon^2 \left( \sum_{j=0}^{M-1} e^{2j/\varepsilon^2} f^{(2j+1)}(0) \right) \rho + s_\varepsilon, \] (2.38)

with \(s_\varepsilon\) satisfying (2.36) and \(\rho\) given by (2.37).

3. Direct approaches

In this section we consider the finite element approximation of (2.1), using the variational formulation (2.3). The discrete problem reads: Find \(u_N^\varepsilon \in V_N \subset H^1_0(-1,1)\) such that

\[ a_\varepsilon(u_N^\varepsilon, v) = \int_{-1}^{0} fv dx, \forall v \in V_N. \] (3.1)

Since \(u_\varepsilon \in H^1_0(-1,1)\) (resp. \(u_N^\varepsilon \in V_N \subset H^1_0(-1,1)\)) is the solution of (2.3) (resp. (3.1)), by Céa’s lemma, we have that

\[ |||u_\varepsilon - u_N^\varepsilon|||_\varepsilon \lesssim \inf_{v_N \in V_N} |||u_\varepsilon - v_N|||_\varepsilon. \] (3.2)

The discrete subspace \(V_N\) is defined as follows: Let \(\Delta = \{-1 = x_0 < x_1 < \ldots < x_M = 1\}\) be an arbitrary partition of \([-1,1]\) which includes 0 as a node, and set

\[ I_j = (x_{j-1}, x_j), \quad h_j = x_j - x_{j-1}, \quad j = 1, \ldots, M. \]
Also, define the master (or standard) element $I_{ST} = (-1, 1)$, and note that it can be mapped onto the $j^{th}$ element $I_j$ by the linear mapping

$$x = Q_j(t) = \frac{1}{2} (1-t)x_{j-1} + \frac{1}{2} (1+t)x_j.$$  

With $\Pi_p(I_{ST})$ the space of polynomials of degree $\leq p$ on $I_{ST}$, we define $V_N$ as

$$V_N = \left\{ u \in H^1_0(-1,1) : u(Q_j(t)) \in \Pi_{p_j}(I_{ST}) , j = 1, \ldots, M \right\},$$

where $\overrightarrow{p} = (p_1, ..., p_M)$ is the vector of polynomial degrees assigned to the elements. The dimension $N$ of $V_N$, referred to as the number of degrees of freedom, is given by

$$N = \sum_{i=1}^{M} p_i - 1.$$

This rather general definition of $V_N$ will allow us to consider both the $h$-version FEM, in which $p_i$ is kept fixed $\forall i$ and $h = \max_i h_i \to 0$, and the $p$-version FEM, in which $h$ is kept fixed and $p_i \to \infty$. We note that for the $h$ version we have $N = O(1/h)$, while for the $p$ version we have $N = O(p)$.

We re-iterate that in all the methods considered we must have 0 as a node, something that will be assumed from this point forward.

### 3.1. The $h$ version with piecewise linear elements on a uniform mesh

This is the most straightforward, but at the same time most naive approach for this type of problems, because the mesh does not account for the presence of the boundary layers. We have

$$h_i = \frac{2}{M}, \ p_i = 1 \ \forall \ i = 1, \ldots, M,$$

and as already mentioned, $N = \dim(V_N) = O(1/h)$. We have the following result.

**Theorem 3.1** Let $\varepsilon \in (0,1)$, $u_\varepsilon$ be the solution of (2.3) and $u_\varepsilon^N \in V_N$ be the solution of (3.1), where $V_N$ is built using a uniform mesh of size $h$ and piecewise linear elements. Then,

$$||u_\varepsilon - u_\varepsilon^N||_\varepsilon \lesssim \left(1 + \frac{1}{\varepsilon} + \frac{h}{\varepsilon^2}\right) ||f||_{0,(-1,0)}.$$  

(3.3)

**Proof.** Since $u_\varepsilon$ is piecewise smooth, we use (3.2) and take $v^N$ to be the Lagrange interpolant $I^N u_\varepsilon$ of $u_\varepsilon$. Hence, by standard interpolation error estimates (see
for instance [3]), we get
\[\|u^\epsilon_\epsilon - (I^N u^\epsilon_\epsilon)\|_{0,(-1,0)} \lesssim h\|u^\epsilon_\epsilon\|_{0,(-1,0)},\]
\[\|u^\epsilon_\epsilon - I^N u^\epsilon_\epsilon\|_{0,(-1,0)} \lesssim h^2\|u^\epsilon_\epsilon\|_{0,(-1,0)},\]
\[\|\epsilon u^\epsilon_\epsilon - (I^N u^\epsilon_\epsilon)\|_{0,(0,1)} \lesssim h\|u^\epsilon_\epsilon\|_{0,(0,1)},\]
\[\|\epsilon u^\epsilon_\epsilon - I^N u^\epsilon_\epsilon\|_{0,(0,1)} \lesssim h^2\|u^\epsilon_\epsilon\|_{0,(0,1)}\]
Inserting these estimates in (3.2) we obtain
\[\|\|\epsilon u_\epsilon - u^N_\epsilon\|\|_\epsilon \lesssim (\epsilon h + h^2)\|u^\epsilon_\epsilon\|_{0,(-1,0)} + h\|u^\epsilon_\epsilon\|_{0,(0,1)},\]
from which (3.3) follows once we use (2.4).

The above theorem shows that this choice of \(V_N\) does not yield a robust approximation, something that is well known for non-transmission problems. There are, of course, wiser choices one can make for \(V_N\), and in particular for the mesh \(\Delta\), as is done in non-transmission problems. For example, one could use the popular Shishkin mesh [11], which is piecewise uniform. Other examples include the Bakhvalov mesh [2] and the exponentially graded mesh from [12], which are non-uniform and include more elements in the layer regions. In Section 5 ahead, we will consider some of the aforementioned methods in our computational comparisons.

### 3.2. The \(p\) version on a uniform (fixed) mesh

High order \(p\) FEMs (or spectral methods) have always been an attractive choice for problems whose solutions are smooth (e.g. analytic), mainly due to their fast (exponential) convergence rates. Here we consider a \(p\)-version FEM on a uniform (fixed) mesh with \(p_i = p\), \forall i, and assume that the right hand side function \(f\) in (2.1) is analytic and satisfies
\[\|f^{(n)}\|_{\infty,(-1,0)} \lesssim \gamma^n n!, \forall n = 0, 1, 2, \ldots.\] (3.4)
We have the following lemma.

**Lemma 3.2** Let \(\epsilon \in (0, 1)\) and \(u^\epsilon_\epsilon\) be the solution of (2.1), with \(f\) satisfying (3.4). Then, there exists a constant \(K \geq 1\) independent of \(\epsilon\) such that
\[\|\(u^\epsilon_\epsilon\)^{(n)}\|_{0,(-1,0)} \lesssim K^n \max\{\epsilon^{-1}, n\}^n, \forall n = 0, 1, 2, \ldots,\] (3.5)
\[\|\(u^\epsilon_\epsilon\)^{(n)}\|_{0,(0,1)} \lesssim \|f\|_{0,(-1,0)}, \forall n = 0, 1, 2, \ldots.\] (3.6)

**Proof.** Using (2.3) with \(v = u^\epsilon_\epsilon\), and the Cauchy-Schwarz inequality, we obtain
\[\epsilon^2 \|\(u^\epsilon_\epsilon\)^{'}\|_{0,(-1,0)} + \|u^\epsilon_\epsilon\|_{0,(-1,0)}^2 + \|\(u^\epsilon_\epsilon\)^{'}\|_{0,(0,1)}^2 + \|u^\epsilon_\epsilon\|_{0,(0,1)}^2 \lesssim \|f\|_{0,(-1,0)} \|u^\epsilon_\epsilon\|_{0,(-1,0)},\]
from which (3.5) and (3.6) follow for \( n = 0, 1 \). To establish (3.6) for all \( n \geq 2 \), we use (2.1) and differentiate the differential equation satisfied by \( u_\varepsilon^- \), obtaining the desired result by induction on \( n \).

For the inductive argument that gives the bound (3.5) for all \( n \geq 2 \), see the proof of Theorem 1 in [5]. □

Next, we state an approximation result from [9].

**Theorem 3.3** For any \( u \in C^\infty (I_{ST}) \) there exists \( \mathcal{J}_p u \in \Pi_p (I_{ST}) \) such that

\[
\begin{align*}
uu{u(\pm 1) = \mathcal{J}_p u(\pm 1),} \\
\left\lVert u - \mathcal{J}_p u \right\rVert_{0,I_{ST}}^2 & \leq \frac{1}{p^2} \left( \frac{(p-s)!}{(p+s)!} \right)^2 \left\lVert u^{(s+1)} \right\rVert_{0,I_{ST}}^2, \forall s = 0, 1, \ldots, p, \\
\left\lVert (u - \mathcal{J}_p u)' \right\rVert_{0,I_{ST}}^2 & \leq \frac{(p-s)!}{(p+s)!} \left\lVert u^{(s+1)} \right\rVert_{0,I_{ST}}^2, \forall s = 0, 1, \ldots, p.
\end{align*}
\] (3.7)

The lemma that follows can be easily proved using Stirling’s formula.

**Lemma 3.4** Let \( p \in \mathbb{N}, \lambda \in (0, 1] \). Then

\[
\frac{(p-\lambda p)!}{(p+\lambda p)!} \leq \left[ \frac{(1-\lambda)(1+\lambda)}{(1+\lambda)^2} \right]^p e^{-2\lambda p}.
\] (3.8)

Using the above results, we can prove the following.

**Theorem 3.5** Let \( \varepsilon \in (0, 1) \), \( u_\varepsilon^- \) be the solution of (2.3) and \( u_\varepsilon^N \in V_N \) be the solution of (3.1), where \( V_N \) corresponds to the \( p \) version on a uniform (fixed) mesh with uniform polynomial degree \( p \) satisfying \( \varepsilon p \geq 1/2 \) for some constant \( \varepsilon > 0 \). (The value of \( \varepsilon \) is determined in the proof.) Then,

\[
||| u_\varepsilon^- - u_\varepsilon^N |||_1 \lesssim e^{-\sigma p},
\] (3.9)

with \( \sigma \) a positive constant independent of \( p \) and \( \varepsilon \).

**Proof.** From Theorem 3.3 we have that there exists \( \mathcal{J}_p u_\varepsilon^- \in \Pi_p (-1, 0) \) such that

\[
\mathcal{J}_p u_\varepsilon^- (-1) = u_\varepsilon^- (-1), \mathcal{J}_p u_\varepsilon^- (0) = u_\varepsilon^- (0)
\]

and

\[
\left\lVert u_\varepsilon^- - \mathcal{J}_p u_\varepsilon^- \right\rVert_{0,(-1,0)}^2 \leq \frac{1}{p^2} \left( \frac{(p-s)!}{(p+s)!} \right)^2 \left\lVert (u_\varepsilon^-)^{(s+1)} \right\rVert_{0,(-1,0)}^2, \forall s = 0, 1, \ldots, p
\] (3.10)
\[ \|(u^-_e - \mathcal{I}_p u^-_e)'\|^2_{0,(-1,0)} \leq \frac{(p-s)^1}{(p+s)^1} \|(u^-_e)^{(s+1)}\|^2_{0,(-1,0)}, \forall s = 0, 1, \ldots, p, \quad (3.12) \]

while Lemma 3.2 gives
\[ \|(u^-_e)^{(s+1)}\|^2_{0,(-1,0)} \leq K^{2(s+1)} \max \{ \varepsilon^{-1}, s+1 \}^{2(s+1)}. \]

Now, choose \( s = \lambda p \) for some rational number \( \lambda \in (0, 1] \) to be fixed shortly. Then, since \( \varepsilon p \varepsilon \geq 1 \), we have
\[ \max \{ \varepsilon^{-1}, s+1 \}^{2(s+1)} = \max \{ \varepsilon^{-1}, \lambda p + 1 \}^{2(\lambda p + 1)} = (\lambda p + 1)^{2(\lambda p + 1)}, \]
provided \( \varepsilon \leq \lambda / 2 \). Hence, from (3.12) and Lemma 3.4 we get
\[
\|(u^-_e - \mathcal{I}_p u^-_e)'\|^2_{0,(-1,0)} \leq \left[ \frac{(1 - \lambda)(1 - \lambda)}{(1 + \lambda)(1 + \lambda)} \right]^p \lambda^{2\lambda p} (Ke)^{2\lambda p} (\lambda p + 1)^{2(\lambda p + 1)} \\
\leq \left[ \frac{(1 - \lambda)(1 - \lambda)}{(1 + \lambda)(1 + \lambda)} \right]^p (Ke)^{2\lambda p} \left( \frac{1 + \lambda p}{p} \right)^{2\lambda p} \\
\leq \lambda^{2\lambda p} \left( 1 + \frac{1}{\lambda p} \right)^{2\lambda p} \\
\leq e^{2\lambda^{2\lambda p}}, \]

so if we choose \( \lambda = (Ke)^{-1} \in (0, 1) \) we obtain
\[ \|(u^-_e - \mathcal{I}_p u^-_e)'\|^2_{0,(-1,0)} \leq p^2 e^{-\sigma p}, \quad (3.13) \]

where \( \sigma = |\ln q|, q = \frac{(1 - \lambda)(1 - \lambda)}{(1 + \lambda)(1 + \lambda)} < 1 \). This means that the constant \( \varepsilon \) must satisfy \( \varepsilon \leq (2eK)^{-1} \), with \( K \) the constant from Lemma 3.2.

Repeating the above argument for the \( L^2 \) norm of \( (u^-_e - \mathcal{I}_p u^-_e) \) we get
\[ \|(u^-_e - \mathcal{I}_p u^-_e)\|^2_{0,(-1,0)} \leq e^{-\sigma p}. \quad (3.14) \]

Now, \( u^-_e \) is even smoother than \( u^-_e \), as is given in Lemma 3.2, hence its approximation at an exponential rate follows from standard results (see, e.g., [1]). Therefore, combining (3.13), (3.14) and the analogous bounds for \( u^+_e \), we obtain (3.10).
The above theorem says that in the asymptotic range of $p$ (or equivalently, for $\varepsilon$ large) the $p$ version will produce exponential convergence. For the pre-asymptotic range of $p$ (or when $\varepsilon$ is small) we have the following result.

**Theorem 3.6** Let $\varepsilon \in (0,1)$, $u_\varepsilon$ be the solution of (2.3) and $u_N^\varepsilon \in V_N$ be the solution of (3.1), where $V_N$ corresponds to the $p$ version on a uniform (fixed) mesh with uniform polynomial degree $p$, satisfying $\varepsilon p < 1/2$, where the constant $\varepsilon$ is the same as in Theorem 3.5. Then,

$$||| u_\varepsilon - u_N^\varepsilon |||_\varepsilon \lesssim p^{-1} \sqrt{\ln p}.$$  \hspace{1cm} (3.15)

**Proof.** We decompose $u_\varepsilon$ as in (2.38),

$$u_\varepsilon = w_M - w_M(-1)^{v^p} - w_M(0)^{v^q} + \varepsilon^2 \left( \sum_{j=0}^{M-1} \varepsilon^{2j} f^{(2j+1)}(0) \right) \rho + s_\varepsilon,$$

and we choose the order $M$ of the expansion as the integer part of $\mu \kappa^p / 2 - 1$, where $\mu > 0$ is a fixed constant satisfying $\mu \gamma < 1$, with $\gamma$ the constant from (3.4). Then, since $\varepsilon p < 1/2$ we have by (2.36) and (3.4),

$$||| s_\varepsilon |||_\varepsilon \lesssim \varepsilon^{2M+2} ||| f^{(2M+2)} |||_{0,(−1,0)} + \varepsilon^{2M} ||| f^{(2M+1)}(0) |||$$

$$\lesssim \varepsilon^{2M+2} \gamma^{2M+2} (2M + 2)!$$

$$\lesssim \varepsilon^{2M+2} \gamma^{2M+2} (2M + 2)^{(2M+2)}$$

$$\lesssim (\varepsilon \gamma \mu \varepsilon p)^{\mu \varepsilon p}$$

$$\lesssim \left( \frac{\mu}{2} \right)^{\mu \varepsilon p},$$

(3.16)

which shows that the term $s_\varepsilon$ is small and goes to 0 at an exponential rate as $p$ increases.

The remaining terms in the decomposition (2.38) will be approximated individually. First, for $w_M$ we note that by definition, we only need to approximate it on the interval $(-1,0)$. Now, by Lemma 2 and Theorem 3 of [5] we have that

$$||w_M^{(n)}||_{0,(−1,0)} \lesssim K_1^n n! \forall n = 0,1,2,...,$$

(3.17)

for some constant $K_1 \geq 1$ which depends only on the data of the problem. Therefore, we have by Theorem 3.3 that there exists $\mathcal{J}_p w_M \in \Pi_p(-1,0)$ such that

$$\mathcal{J}_p w_M(-1) = w_M(-1), \mathcal{J}_p w_M(0) = w_M(0)$$

and

$$||w_M - \mathcal{J}_p w_M||_{0,(−1,0)} \leq \frac{1}{p^2} \frac{(p-s)!}{(p+s)!} ||w_M^{(s+1)}||^{2}_{0,(−1,0)} \forall s = 0,1,...,p,$$

(3.18)
\[ \|(w_M - \mathcal{J}_p w_M)'\|_{0,(1,0)} \leq \frac{(p-s)!}{(p+s)!} \|w_M^{(s+1)}\|_{0,(1,0)} \quad \forall s = 0, 1, \ldots, p. \quad (3.19) \]

Choose \( s = \lambda_1 p \) for some rational number \( \lambda_1 \in (0, 1) \) to be fixed shortly. Then, from (3.17), (3.19) and Lemma 3.4 we obtain

\[ \|(w_M - \mathcal{J}_p w_M)'\|_{0,(1,0)} \lesssim p^3 e^{-\sigma_1 p}, \quad (3.20) \]

where \( \sigma_1 = |\ln q_1|, \quad q_1 = \frac{(1-\lambda_1)(1-\lambda_1)}{(1+\lambda_1)^{1-\lambda_1} - \lambda_1} < 1. \) Repeating the previous argument for the \( L^2 \) norm of \( (w_M - \mathcal{J}_p w_M) \), we get, using (3.18),

\[ \|w_M - \mathcal{J}_p w_M\|_{0,(1,0)} \lesssim p e^{-\sigma_1 p}. \quad (3.21) \]

Next, we turn our attention to the boundary and interface layers. Note that due to (3.17) we have that \( w_M(-1) \) and \( w_M(0) \), which multiply \( \varphi^b \) and \( \varphi^b \), respectively, are bounded independently of \( \varepsilon \), hence we will only discuss the approximation of the functions \( \varphi^b \) and \( \varphi^b \). From (2.22) and (2.23) we see that both these functions are given in terms of “typical” one dimensional boundary layer functions – the constants appearing in (2.22) and (2.23) are bounded independently of \( \varepsilon \). The approximation of such functions by the \( p \) version of the FEM on a uniform mesh, has been studied rigorously in [10], where it was shown that such functions can be approximated at the rate \( p^{-1} \sqrt{\ln p} \), independently of \( \varepsilon \), when \( p \) satisfies \( \varepsilon p \leq 1. \) Thus, by Theorem 4.4 of [10] we have that there exist functions \( \mathcal{J}_p \varphi^b \) and \( \mathcal{J}_p \varphi^b \), such that

\[ \|\varphi^b - \mathcal{J}_p \varphi^b\|_\varepsilon \lesssim p^{-1} \sqrt{\ln p}, \quad \|\varphi^b - \mathcal{J}_p \varphi^b\|_\varepsilon \lesssim p^{-1} \sqrt{\ln p}. \quad (3.22) \]
The remaining term \( \rho \) in (2.38) also involves “typical” one dimensional layer functions, hence it too can be approximated at the same rate as the boundary and interface layers. (The factor multiplying \( \rho \) in (2.38) can be bounded independently of \( \varepsilon \), due to (3.17).) Therefore, combining (3.16), (3.20), (3.21), (3.22) and the analogous bound for \( \rho \), we obtain (3.15).

By examining the proof of the above theorem we see that the \( p \) version on a uniform mesh fails to capture the layer effects when \( p \) is not in the asymptotic range, due to the fact that the mesh does not account for their presence. This is not surprising, since it is well known (see, e.g., [12]) that meshes that do not incorporate \( \varepsilon \) in their construction will always yield non-robust results.

3.3. The \( p \) version on a non-uniform (variable) mesh

From the results of the previous subsection we see that if \( p \) is large enough then the \( p \) version on a uniform mesh yields exponential convergence rates, independently of \( \varepsilon \). It is possible to obtain these rates for all ranges of \( p \), if instead of a uniform mesh we use a non-uniform mesh which includes elements of size \( \propto p \varepsilon \) in the layer regions, where \( \propto \) is the constant from Theorem 3.5 and is independent of \( p \) and \( \varepsilon \). This was established for non-transmission singularly perturbed problems in [5] and [10]. The mesh is constructed as follows:

\[
\Delta = \begin{cases} 
[-1,0,1] & \text{if } \propto p \varepsilon \geq \frac{1}{2}, \\
[-1, -1 + \propto p \varepsilon, -\propto p \varepsilon, 0,1] & \text{if } \propto p \varepsilon < \frac{1}{2},
\end{cases}
\]

(3.23)

where in both cases the polynomial degree \( p \) is uniform on all the elements and is increased to improve accuracy. We should mention that in practice the constant \( \propto \) may be taken equal to 1 without any loss in the accuracy.

**Theorem 3.7** Let \( \varepsilon \in (0,1) \), \( u_e \) be the solution of (2.3) and \( u_e^N \in V_N \) be the solution of (3.1), where \( V_N \) corresponds to the \( p \) version on the non-uniform (variable) mesh (3.23). Then,

\[
|||u_e - u_e^N|||_\varepsilon \lesssim p^{3/2} e^{-\sigma p},
\]

(3.24)

with \( \sigma \) a positive constant independent of \( p \) and \( \varepsilon \).

**Proof.** By examining the proof of Theorem 3.6, we see that the term \( s_\varepsilon \) is exponentially small, while the term \( w_M \) is approximated at an exponential rate by simply using the \( p \) version on a single element. Hence, the proof of the present theorem is the same as that of Theorem 3.6 except for the arguments giving the approximation of the functions \( \tilde{\psi}, \tilde{\tau} \) and \( n \). For these functions, Theorem 5.1 of [10] gives, with this choice for the mesh, the desired result – see also Theorem 16 of [5]. \( \square \)
4. Indirect approaches

In this section we consider the indirect approaches based on the decompositions given by Theorems 2.1, 2.3 and 2.4. The first is based on Theorem 2.1 and we consider a finite element approach based on the variational formulation (2.9). In particular, we solve the discrete problem: Find \( r^N_\varepsilon \in V_N \subset H^1_0(-1,1) \) such that \( \forall v \in V_N \)

\[
a_\varepsilon(r^N_\varepsilon, v) = \int_{-1}^{0} g_\varepsilon v \, dx + \alpha v(0),
\]

with \( g_\varepsilon \) given by (2.8) and \( \alpha = \varepsilon f(0) - \varepsilon^2 f'(0) \). Once we have the approximation \( r^N_\varepsilon \) to \( r_\varepsilon \), we obtain the approximation \( u^N_\varepsilon \) to \( u_\varepsilon \) via (2.5), i.e.

\[
\begin{align*}
u^N_\varepsilon(x) &= f(x) - \chi^b(x)f(-1)\,e^{-\frac{x}{\varepsilon^2}} - \chi^i(x)f(0)\,e^{\frac{x}{\varepsilon^2}} + r^N_\varepsilon(x) \text{ in } (-1,0), \\
u^N_\varepsilon(x) &= r^N_\varepsilon(x) \text{ in } (0,1).
\end{align*}
\]

Since by Theorem 2.1, \( r_\varepsilon \) is piecewise smooth, we may use the \( h \) version FEM on a uniform mesh with piecewise linear elements. Using Corollary 2.2 and standard interpolation error estimates, we obtain the following.

**Theorem 4.1** Let \( \varepsilon \in (0,1) \) and \( u_\varepsilon \) be the solution of (2.1). Let further \( r^N_\varepsilon \in V_N \) be the solution of (4.1), where \( V_N \) is built using a uniform mesh of size \( h \) and piecewise linear elements. Then we have the error estimate

\[
\|\|u_\varepsilon - u^N_\varepsilon\|\|_\varepsilon \lesssim \left[h(1 + \frac{h}{\varepsilon})\right] \left\{ |f(0)| + \varepsilon(|f''|_{0,(-1,0)} + |f'(0)|) + (|f(-1)| + |f(0)|)|e^{\frac{\varepsilon^2}{2}}\right\},
\]

with \( \eta \) as in Theorem 2.1.

**Proof.** In view of (2.5) and (4.2)–(4.3), it suffices to show that

\[
\|\|r_\varepsilon - r^N_\varepsilon\|\|_\varepsilon \lesssim \left[h(1 + \frac{h}{\varepsilon})\right] \left\{ |f(0)| + \varepsilon(|f''|_{0,(-1,0)} + |f'(0)|) + (|f(-1)| + |f(0)|)|e^{\frac{\varepsilon^2}{2}}\right\},
\]

where \( r_\varepsilon \) was introduced in Theorem 2.1.

Since \( r_\varepsilon \in H^1_0(-1,1) \) (resp. \( r^N_\varepsilon \in V_N \subset H^1_0(-1,1) \)) is the solution of (2.9) (resp. (4.1)), by Céa’s lemma, we have that

\[
\|\|r_\varepsilon - r^N_\varepsilon\|\|_\varepsilon \leq \inf_{\nu^N \in V_N} \|\|r_\varepsilon - \nu^N\|\|_\varepsilon .
\]
Since \( r_\varepsilon \) is piecewise smooth, we take for \( v^N \) its Lagrange interpolant \( I^N r_\varepsilon \). Hence by standard interpolation error estimates (see for instance [3]), we get
\[
\|r'\varepsilon - (I^N r_\varepsilon)'\|_{0,(1,0)} \lesssim h\|r''_\varepsilon\|_{0,(1,0)},
\]
\[
\|r_\varepsilon - I^N r_\varepsilon\|_{0,(1,0)} \lesssim h^2\|r''_\varepsilon\|_{0,(1,0)}.
\]
Inserting these estimates in (4.6), we find that
\[
||| r_\varepsilon - r^N_\varepsilon ||| \lesssim (\varepsilon h + h^2)\|r''_\varepsilon\|_{0,(1,0)} + h\|r''_\varepsilon\|_{0,(1,0)}.
\]
Using (2.12) and the fact that \( r_\varepsilon = u_\varepsilon \) on \((0,1)\), we arrive at (4.5).

The drawback of this first indirect approach is that it does not give an error estimate in \( h \) uniformly in \( \varepsilon \), and this fact is confirmed numerically (see Section 5). The use of the results from Theorem 2.3 yields similar non optimal error estimates. But Theorem 2.4 and Corollary 2.5 suggest to use the alternative indirect approach:

Find \( \tilde{r}^N_\varepsilon \in V_N \subset H^1_0((-1,1)) \) such that
\[
a_\varepsilon(\tilde{r}^N_\varepsilon, v) = \varepsilon^2 \int_{-1}^0 f'' v \, dx + \varepsilon^2 f'(0)v(0). \tag{4.7}
\]
With the help of this approximation \( \tilde{r}^N_\varepsilon \) of \( \tilde{r}_\varepsilon \), we obtain the approximation \( u^N_\varepsilon \) to \( u_\varepsilon \) via (2.24), i.e.
\[
u^N_\varepsilon = \tilde{f} - f(-1) \tilde{v} - f(0) \tilde{v}' + \tilde{r}^N_\varepsilon. \tag{4.8}
\]
Since by Theorem 2.4, \( \tilde{r}_\varepsilon \) is piecewise smooth, we may use the \( h \) version FEM on a uniform mesh with piecewise linear elements. Using Corollary 2.5 and standard interpolation error estimates, we obtain the following result.

**Theorem 4.2** Let \( \varepsilon \in (0,1) \) and \( u_\varepsilon \) be the solution of (2.1). Let further \( \tilde{r}^N_\varepsilon \in V_N \) be the solution of (4.7), where \( V_N \) is built using a uniform mesh of size \( h \) and piecewise linear elements. Then we have the error estimate
\[
|||u_\varepsilon - u^N_\varepsilon||| \leq [\|f''\|_{0,(1,0)} + |f'(0)|] h(\varepsilon + h).
\]

**5. Numerical experiments**

In this section we present the results of numerical computations for the problem (2.1) with \( f = 1 \), for the direct approaches described in Section 3, as well as the
indirect approaches of Section 4. For the latter, we make the following choices for the cut-off functions which appear in (2.5):

\[
\chi^i(x) = \begin{cases} 
1 & \text{on } (-\eta, \eta) \\
\frac{1}{\eta^2} (2x-\eta)(x-2\eta)^2 & \text{on } (\eta, 2\eta) \\
-\frac{1}{\eta^2} (2x+\eta)(x+2\eta)^2 & \text{on } (-2\eta, -\eta) \\
0 & \text{elsewhere}
\end{cases}
\]

\[
\chi^b(x) = \begin{cases} 
1 & \text{on } (-1, -1+\eta) \\
\frac{1}{\eta^2} (2x+2-\eta)(x+1-2\eta)^2 & \text{on } (-1+\eta, -1+2\eta) \\
0 & \text{elsewhere}
\end{cases}
\]

with the positive constant \(\eta\) chosen so that \(\text{supp } \chi^b \cap \chi^i = \emptyset\).

We will be plotting the percentage relative error between the exact and approximate solution, measured in the energy norm \(\|\cdot\|_e\), versus the number of degrees of freedom \(N\) in a log-log scale.

For the direct approaches we will show computations corresponding to the \(h\) version with piecewise linear elements on a uniform mesh, the \(p\) version on a uniform (fixed) mesh, the \(p\) version on the non-uniform (variable) mesh \(\Delta = [-1, -1+p\varepsilon, -p\varepsilon, 0, 1]\), as well as the \(h\) version with piecewise linear elements on two other meshes not analyzed in this paper. The first will be the so-called Shishkin mesh \([11]\), which is piecewise uniform and quite popular for (non transmission) singularly perturbed problems, mainly because of its simplicity and ability to approximate the solution independently of \(\varepsilon\). For our problem this mesh is constructed as follows: The interval \([-1, 1]\) is initially subdivided as \([-1, -1+\tau, -\tau, 0, 1]\), where for some integer \(n\), \(\tau = \min\{1/4, 2\varepsilon \ln(n+1)\}\). Each of the subintervals \((-1, -1+\tau), (-1+\tau, -\tau), (-\tau, 0), (0, 1)\) is then divided further into \(n\) subintervals of equal length, resulting in a piecewise uniform (Shishkin) mesh. For non-transmission singularly perturbed problems, it is known that the approximation error for this choice of \(V_N\) satisfies

\[
\|\|u_e - (u^e_N)\|\|_e \lesssim h \ln(1/h),
\]

which shows that the method is robust and converges at a quasi-optimal rate. As we will see in this section, this method works equally well for singularly perturbed transmission problems.

The second mesh which will be considering in our numerical computations for the \(h\) version direct approach, will be a non-uniform one; non-uniform meshes are also quite popular for non-transmission singularly perturbed problems. Two such examples are the so-called Bakhvalov mesh \([2]\) and an exponentially graded mesh from \([12]\), which is the one we will use here: The interval \((-1, 1)\) is split up into \((-1, 0)\) and \((0, 1)\) and over \((0, 1)\) a uniform mesh with \(n\) subintervals is used. Over \((-1, 0)\) we use \(n\) elements obtained from the nodes \(\{x_j\}_{j=0}^n\), where
\[ x_j = -1 - \frac{1}{2} \ln \left(1 - \frac{c_j}{n}\right), c_j = 1 - \exp\left(-\frac{2\varepsilon}{3}\right). \] This results in more elements within the boundary layer region, as well as a robust convergence rate that is optimal.

Figure 4 shows the performance of the \( h \) version on a uniform mesh with piecewise linear elements. We observe that for \( \varepsilon \) relatively large the method converges at the optimal \( O(h) \) rate, as expected due to the \( H^2 \) regularity and the fact that we are using linear elements. As \( \varepsilon \to 0 \), however, the rate deteriorates and we notice it to be \( O(\sqrt{h}) \).

![Figure 4](image)

**Figure 4.** Convergence of the \( h \) version with piecewise linear elements on a uniform mesh.

In figure 5 we see the robustness of the \( h \) version with piecewise linear elements on a Shishkin mesh. We note that the performance of the method does not deteriorate as \( \varepsilon \to 0 \), and the expected quasi-optimal rate \( O(h \ln(1/h)) \) is observed.

The same robustness is observed in figure 6 for the \( h \) version with piecewise linear elements on the exponentially graded mesh, with the convergence rate being the optimal \( O(h) \).

Figure 7 shows the performance of the \( p \) version on a uniform mesh. For \( \varepsilon \) relatively large, the method yields exponential convergence as \( p \to \infty \), due to the fact that \( p\varepsilon > 1 \). As \( \varepsilon \to 0 \) however, the exponential rate deteriorates to an algebraic one, hence the method is not robust.

The last of the direct approach methods, namely the \( p \) version on the non-uniform (variable) mesh \([-1, -1 + p\varepsilon, -p\varepsilon, 0, 1]\) is shown in figure 8. Indeed, the method converges at an exponential rate as \( p \to \infty \), independently of \( \varepsilon \) and it is clearly the best choice among the direct approach methods.

Figure 9 shows the performance of the first indirect approach (cf. eqs (4.1)–(4.4)) using the \( h \) version with piecewise linear elements on a uniform mesh. We see that for each \( \varepsilon \), the method converges initially at the sub-optimal \( O(\sqrt{h}) \) rate,
but as $h \to 0$, the rate improves to the optimal $O(h)$ one. When we compare the performance of this method with the direct $h$ version with piecewise linear elements on a uniform mesh, we see that even though the observed convergence rate is similar, the indirect approach produces significantly smaller errors, especially for small values of $\varepsilon$. 

Figure 5. Convergence of the $h$ version with piecewise linear elements on a Shishkin mesh.

Figure 6. Convergence of the $h$ version with piecewise linear elements on an exponential mesh.
Figure 7. Convergence of the $p$ version on a uniform mesh.

Figure 8. Convergence of the $p$ version on a non-uniform (variable) mesh.

Even though not shown here, the second indirect approach (cf. Theorem 2.3) yields similar (non-robust) numerical results, hence the last method we will consider in this section is the third indirect approach which is based on (4.7)–(4.8) (and Theorem 2.4). Since for $f = 1$ we have from (4.7) that $\tilde{r}_e = 0$, we choose a different $f$ for this computation, namely $f(x) = x^3 + 3x^2 + 6x(e^2 + 1)/(e^2 - 1)$. This choice
is made in order for $\tilde{r}_k$ not to have boundary layers at $-1$ and $0$, as explained in Remark 2.6. Figure 10 shows the performance of the $h$ version with piecewise linear elements on a uniform mesh, using the third indirect approach. The robustness of this approach is clear, making it the best choice among the indirect methods.

Figure 9. Convergence of the first indirect approach $h$ version with piecewise linear elements on a uniform mesh.

Figure 10. Convergence of the third indirect approach $h$ version with piecewise linear elements on a uniform mesh.
Finally, figure 11 shows the performance of the $p$ version on the fixed mesh $[-1, 0, 1]$ for the third indirect approach. Even though no theory is developed for this case, it is evident that the method converges exponentially as $p \to \infty$, independently of $\varepsilon$. This is due to the fact that, under the assumption of analytic data, the remainder $\tilde{r}_\varepsilon$ will also be analytic, hence the $p$ version yields exponential convergence.

Figure 11. Convergence of the third indirect approach $p$ version on the mesh $[-1,0,1]$.

References


