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The Singular Function Boundary Integral Method for Laplacian problems with boundary singularities in two and three–dimensions

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Abstract

We present a Singular Function Boundary Integral Method (SFBIM) for solving elliptic problems with a boundary singularity. In this method the solution is approximated by the leading terms of the asymptotic solution expansion, which exists near the singular point and is known for many benchmark problems. The unknowns to be calculated are the singular coefficients, i.e. the coefficients in the asymptotic expansion, also called (generalized) stress intensity factors. The discretized Galerkin equations are reduced to boundary integrals by means of Green’s theorem and the Dirichlet boundary conditions are weakly enforced by means of Lagrange multipliers, the values of which are introduced as additional unknowns in the resulting linear system. The method is described for two–dimensional Laplacian problems for which the analysis establishes exponential rates of convergence as the number of terms in the asymptotic expansion is increased. We also discuss the extension of the method to three–dimensional Laplacian problems with exhibits edge singularities.

Keywords: boundary singularities; Lagrange multipliers; stress intensity factors; boundary approximation methods

1. Introduction

In the past few decades several methods for treating elliptic boundary value problems with boundary singularities, have been proposed. Among them one finds the so-called hybrid methods [1] which incorporate, directly or indirectly, the form of the local asymptotic expansion for the solution \( u \), in the approximation scheme. This expansion is known in many occasions and has the following form:

\[
    u(r, \theta) = \sum_{j=1}^{\infty} \alpha_j W_j(r, \theta),
\]

where \((r, \theta)\) are polar coordinates centered at the singular point, \( \alpha_j \in \mathbb{R} \) are the singular coefficients and

\[
    W_j(r, \theta) = r^{\omega j}, \quad f_j(\theta),
\]

where \( \omega_j \) is a constant, \( j \in \mathbb{N} \).

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are the singular functions, with \( \mu_j, f_j \) being the eigenvalues and eigenfunctions of the problem which are uniquely determined by the geometry and the boundary conditions along the boundaries sharing the singularity. It is important to note that the singular functions \( W_j \) satisfy the governing partial differential equation (PDE) and the boundary conditions (BCs) along the boundary parts sharing the singularity.

Knowledge of the singular coefficients is of great importance in many engineering fields, such as fracture mechanics. Many methods have been proposed in the literature for their effective and efficient approximation, including high order p/hp finite element methods with post-processing [2, 3] and Trefftz methods [1]. In the former, the solution is first approximated on a refined grid designed especially to capture the singularity and the coefficients are obtained by an extraction formula which uses the computed solution. Methods that do not require any post-processing and/or include information about the exact solution in the approximation scheme, such as Trefftz methods, are more attractive if the approximation of the coefficients is the main objective. The SFBIM, which was developed by Georgiou and co-workers [4] for Laplacian problems with a boundary singularity, also has this trait. In the SFBIM the solution is approximated by the leading terms of the expansion (1), i.e.

\[
 u_j = \sum_{i=1}^{N} \alpha_j^N W_j (r, \theta), 
\]

where \( \alpha_j^N \) are the approximate singular coefficients. The method has been successfully applied to a number of benchmark problems [5, 6]. In [7] it was shown theoretically that its convergence rate is exponential as \( N \to \infty \). The method has also been extended to biharmonic problems in two-dimensions arising from solid and fluid mechanics [8, 9].

The main advantages of the SFBIM are:

- The dimension of the problem is reduced by one, leading to considerable computational savings
- The singular coefficients are calculated directly, hence avoiding the need for post-processing

In this work we present the method and its properties for a model two-dimensional Laplacian problem with a boundary singularity, including numerical experiments for illustration purposes. We then discuss its extensions to three-dimensional Laplacian problems with an edge singularity.

2. The SFBIM

2.1. Description of the method

For concreteness, we consider the following Laplacian problem (see also Figure 1 below): Find \( u \) such that

\[
 \nabla^2 u = 0 \quad \text{in } \Omega, \quad \frac{\partial u}{\partial n} = 0 \quad \text{on } S_1, \quad u = 0 \quad \text{on } S_2, \quad u = f \quad \text{on } S_3, \quad \frac{\partial u}{\partial n} = g \quad \text{on } S_4. 
\]

\[ (4) \]

Fig. 1. A planar Laplacian problem with a boundary singularity at \( O \).
It is assumed that the data of the problem, \( f \) and \( g \), are smooth and such that no other singularities arise. Now, by applying Galerkin’s principle we have
\[
\int_{\Omega} \nabla^2 u_j = 0, \ j = 1, \ldots, N.
\]
A double application of Green’s second identity, reduces the above integral to
\[
\int_{\Omega} W_j \frac{\partial u_j}{\partial n} - \int_{\Omega} u_j \frac{\partial W_j}{\partial n} + \int_{\Omega} u_j \nabla^2 W_j = 0, \ j = 1, \ldots, N.
\]
Taking into account the boundary conditions of the problem, as well as the fact that the singular functions satisfy the governing PDE, we further obtain
\[
\int_{\partial \Omega} u_j \frac{\partial W_j}{\partial n} - \int_{\partial \Omega} W_j \frac{\partial u_j}{\partial n} = \int_{\partial \Omega} W_j g, \ j = 1, \ldots, N.
\]
Letting \( \lambda_M = (\partial u_j / \partial n) \), and assuming that
\[
\lambda_M = \sum_{\ell=1}^M \lambda_{\ell} \Phi_{\ell}, \ \lambda_{\ell} \in \mathbb{R},
\]
where \( \Phi_{\ell}, \ \ell = 1, \ldots, M \) are piecewise polynomials of degree \( p \) defined on a subdivision of \( S_1 \) in elements of mesh width \( h \); \( M \) is then proportional to \( 1/h \). In the case when the \( \Phi_{\ell} \)'s are the typical (Lagrange based) basis functions used in the Finite Element Method, we have that the constants \( \lambda_{\ell} \) in (6) are the nodal values of \( \lambda_M \). We then impose the boundary condition on \( S_1 \) weakly, by means of Lagrange multipliers, i.e. we require that
\[
\int_{S_1} \Phi_{\ell} (u_N - f) = 0, \ \ell = 1, \ldots, M.
\]
Equations (5) and (7) yield the following block system of \( (N + M) \times (N + M) \) equations
\[
\begin{bmatrix}
K_1 & K_2 \\
K_2^T & 0
\end{bmatrix}
\begin{bmatrix}
\alpha \\
\lambda
\end{bmatrix}
= \begin{bmatrix}
\tilde{F}_1 \\
\tilde{F}_2
\end{bmatrix},
\]
in which \( \alpha = [\alpha_1^N, \ldots, \alpha_N^N]^T, \ \tilde{\lambda} = [\lambda_1, \ldots, \lambda_M]^T \),
\[
[K_1]_{ij} = \int_{\partial \Omega} W_i \frac{\partial W_j}{\partial n}, \ i, j = 1, \ldots, N
\]
\[
[K_2]_{\ell i} = \int_{\partial \Omega} W_i \Phi_{\ell}, \ i = 1, \ldots, N, \ \ell = 1, \ldots, M
\]
\[
[\tilde{F}_1]_i = \int_{\partial \Omega} W_i g, \ i = 1, \ldots, N
\]
\[
[\tilde{F}_2]_{\ell} = \int_{\partial \Omega} f \Phi_{\ell}, \ \ell = 1, \ldots, M
\]
Solving the system given by eq. (8) will produce the approximate singular coefficients as well as the discrete Lagrange multipliers. We emphasize that all integrals appearing above are one dimensional (thus the dimension of the problem is reduced by one) and are carried out on portions of the domain away from the singularity; therefore they can be evaluated by standard techniques, such as Gaussian quadrature. We should also mention that in the above system, the coefficient matrix \( K \) becomes singular when \( M > N \), so the number of (discrete) Lagrange multipliers has to be smaller than the number of singular functions.
2.2. Error Analysis

The SFBIM has been analyzed in the context of Laplacian problems and in this subsection we summarize the main results from [7]. Let \( H^1(\Omega) \) denote the usual Sobolev space of functions on \( \Omega \) whose (generalized) partial derivatives of order 0, 1, \ldots, \( k \) are square integrable, and let \( \| \cdot \|_{k,\Omega} \) denote the associated norms. We then define

\[ H^k_\Omega = \left\{ w \in H^1(\Omega) : w|_{\partial \Omega} = 0 \right\}, \quad H^{1/2}(\partial \Omega) = \left\{ w \in H^1(\Omega) : w|_{\partial \Omega} \in H^0(\partial \Omega) \right\}, \]

and \( H^{-1/2}(\partial \Omega) \) as the closure of \( H^0(\partial \Omega) \) with respect to the norm

\[ \|w\|_{-1/2,\partial \Omega} = \sup_{\phi \in H^{1/2}(\partial \Omega)} \left\{ \frac{\langle w, \phi \rangle_{\partial \Omega}}{\|\phi\|_{1/2,\partial \Omega}} \right\}. \]

Now, for any function \( w \) such that

\[ w = \sum_{j=1}^{N} a_j W_j, \quad (10) \]

we can always write \( w = w_N + r_N \), where

\[ w_N = \sum_{j=1}^{N} a_j W_j, \quad r_N = \sum_{j=N+1}^{\infty} a_j W_j. \]

(11)

Under the assumption that there exist positive constants \( C_1, C_2 \) and \( \beta \in (0, 1) \) such that

\[ |r_N| \leq C_1 \beta^N, \quad \left| \frac{\partial r_N}{\partial n} \right| \leq C_2 \sqrt{N} \beta^N, \]

(12)

it was shown in [7] that

\[ \left\| u - u_N \right\|_{1,2,\Omega} + \frac{\| \partial u - \lambda_N \|_{1,2,\partial S_i}}{\| \partial n \|_{1,2,\partial S_i}} \leq C \left\{ \inf_{v \in V_N^M} \left\| u - v \right\|_{1,2,\Omega} + \inf_{\mu \in M^\Omega} \left\| \frac{\partial u}{\partial n} - \mu \right\|_{1,2,\partial S_i} \right\}, \]

(13)

where \( V_N^M = \text{span} \left\{ W_{j,i} \right\}_{j=1}^{N} \) and the space \( V_M^i \) is defined as follows: Let \( S_i \) be divided into sections \( \Gamma_i, \ i = 1, \ldots, n \) such that

\[ S_i = \bigcup_{j=1}^{\infty} \Gamma_j, \ h_i = |\Gamma_j|, \ h = \max_{1 \leq i \leq n} h_i. \]

Then, with \( \vartheta_p(I) \) the set of polynomials of degree \( \leq p \) on \( I \), we set

\[ V^M_i = \left\{ \lambda : \lambda|_{I_i} \in \vartheta_p \left( \Gamma_i \right), i = 1, \ldots, n \right\}. \]

(14)

(We have \( N = \dim(V_N^M), M = M(p,n) = \dim(V^M_i) \). Using (13) it was also shown in [7] that if \( \partial u / \partial n \in H^{k}(\partial \Omega) \) for some \( k \geq 1 \), then there exists a positive constant \( C \), independent of \( N \) and \( M \), such that

\[ \left\| u - u_N \right\|_{1,2,\Omega} + \frac{\| \partial u - \lambda_N \|_{1,2,\partial S_i}}{\| \partial n \|_{1,2,\partial S_i}} \leq C \left\{ \sqrt{N} \beta^N + h^{n-p+1} \right\}, \]

(15)

where \( m = \min \{k,p+1\} \). Moreover, since \( |a_j - a^N_j| \leq \left\| u - u_N \right\|_{1,2,\Omega} \), it follows that

\[ |a_j - a^N_j| \leq C \beta^N, \]

(16)

which shows that the method approximates the singular coefficients at an exponential rate as \( N \to \infty \).
3. Numerical results

We now present the results of numerical computations for the test problem depicted graphically in Figure 2 below.

The function \( f \) is taken as \( f = \frac{\theta - \theta^2}{2\alpha \pi} \), with \( \alpha \pi \) being the angle of the circular sector. The local solution is given by

\[
\psi_j = \sum_{j=1}^{\infty} \alpha_j r^{2j+1} \sin \mu_j \theta, \quad \mu_j = \frac{2j-1}{2\alpha}. \tag{17}
\]

Since this problem can be solved analytically, we have that the exact singular coefficients are given by

\[
\alpha_j = \frac{16\alpha}{\pi^2 R^2 \left(2j - 1\right)^3}, \tag{18}
\]

where \( R \) is the radius of the sector. The numerical results that follow correspond to \( \alpha = R = 1 \) and our goal is to illustrate the method and its convergence rate.

In figure 3 we show the percentage relative error in the solution \( u \) versus \( N \), in a semi-log scale, for different values of the polynomial degree \( p \) used to approximate the Lagrange multipliers. We see that, independently of \( p \), the method converges at an exponential rate, as predicted by equation (13).
In figure 4 we show the percentage relative error in the first four singular coefficients versus $N$, in a semi-log scale, for the case when $p = 1$ (other values of $p$ gave similar results). The exponential convergence, as predicted by equation (14), is again clearly visible.

Fig. 4. Error in the first four singular coefficients for $p = 1$.

Finally, in figure 5 we show the error in $\partial u/\partial n$ versus $M$, in a log-log scale, when $\partial u/\partial n$ on the curved side of our domain is discretized by piece-wise polynomials of (fixed) degree $p$ on a mesh with width $h$, i.e. $M \sim p/h$. The error estimate (13) states that the convergence rate is algebraic of order $p$, and indeed this is what figure 5 shows.

Fig. 5. Error in $\partial u/\partial n$ for different values of $p$. 
4. Extension to three-dimensions

In this section we discuss the extension of the SFBIM to three-dimensional Laplacian problems, an area that, to our knowledge, still possesses several important open questions. In particular we consider the following: Find $u$ such that

$$
\begin{align*}
\nabla^2 u = 0, & \quad \text{in } \Omega \\
 u = g_1 & \quad \text{on } S_1 \\
u = g_2 & \quad \text{on } S_2 \\
v = g_3 & \quad \text{on } S_3 \\
\frac{\partial u}{\partial n} = 0 & \quad \text{on } S_4 \cup S_5 
\end{align*}
$$

(19)

where the functions $g_i, i = 1, 2, 3$ are given and the domain $\Omega$ is shown in figure 6 below.

![Fig. 6. Domain $\Omega$ for the 3-D Laplace equation given by (19).](image)

A singularity will arise along the edge AB, and we assume that the given data is smooth enough such that no other singularities are present. The difference between two and three-dimensions is substantial, since now the local solution is given by

$$
u = \sum_{j=1}^{\infty} \left( r^{\mu_j} f_j(\theta) \left( \alpha_j(x_3) + \sum_{k=1}^{\infty} c_{j,k}^{2i} \left( \alpha_j(x_3) \right) \left( \frac{r^{2i} (-1)^i}{\prod_{n=1}^{\infty} n(\mu_j + n)} \right) \right) \right),$$

(20)

where $f_j$ and $\mu_j$ are the eigenfunctions and eigenvalues, respectively, of the two-dimensional problem (posed on the face $S_i$). Moreover, the singular coefficients $\alpha_j$ are no longer constants but now they are functions of the third coordinate $x_3$; for this reason they are called Edge Stress Intensity Functions (ESIFs) [10, 11]. Nevertheless, they are known to be smooth functions [11], hence they can be approximated by polynomials, of say degree $N$, as

$$
\alpha_j(x_3) = \sum_{k=1}^{N+1} a_{j,k} x_3^{k-1}
$$

(21)

where the coefficients $a_{j,k}$ must be determined for each $j$. Once again, the solution $u$ is approximated by the leading
terms of the asymptotic expansion (20), as in the two-dimensional case:

\[
U_N = \sum_{j=1}^{N} r^{\mu_j} f_j (\theta) \left( a^N_j(x_3) + \sum_{i=1}^{\infty} \frac{\partial^{2i}}{\partial x_i^2} \left( a^N_j(x_3) \right) \frac{r^{2i} \left( -\frac{1}{4} \right)^i}{\prod_{n=1}^{i} n(\mu_j + n)} \right) .
\]  

(22)

We note that the infinite sum in (22) will terminate after a finite number of terms, since \( a^N_j(x_3) \) is a polynomial.

Consequently, the number of unknowns in the above expression is \( N \times (N + 1) \). To determine them we weigh the governing PDE by

\[
\hat{W}^i_j(r, \theta, x_3) = r^{\mu_i} f_j (\theta) \left( \beta^i_j(x_3) + \sum_{i=1}^{\infty} \frac{\partial^{2i}}{\partial x_i^2} \left( \beta^i_j(x_3) \right) \frac{r^{2i} \left( -\frac{1}{4} \right)^i}{\prod_{n=1}^{i} n(\mu_j + n)} \right) ,
\]  

(23)

where the functions \( \beta^i_j(x_3) \) are at our disposal – we may choose them as polynomials. It is easy to show that \( \hat{W}^i_j \) satisfies the governing PDE and the boundary conditions on either side of the singular edge, independently of the choice for \( \beta^i_j(x_3) \). As a result we have

\[
\int_{\Omega} \hat{W}^i_j \nabla^2 u_N = 0 , \ j, l = 1, ..., N .
\]  

(24)

Using Green’s second identity twice, we get

\[
\int_{\Omega} u_N \nabla^2 \hat{W}^i_j + \int_{\Omega} \hat{W}^i_j \frac{\partial u_N}{\partial n} - \int_{\Omega} u_N \frac{\partial \hat{W}^i_j}{\partial n} = 0 , \ j, l = 1, ..., N .
\]  

(25)

Since the functions \( \hat{W}^i_j \) satisfy the PDE and the boundary conditions along \( S_4 \) and \( S_5 \), we have

\[
\int_{S_4 \cup S_5 \cup S_i} \left( \frac{\partial u_N}{\partial n} \hat{W}^i_j - u_N \frac{\partial \hat{W}^i_j}{\partial n} \right) = 0 , \ j, l = 1, ..., N ,
\]  

(26)

which is the analog of our 2-D method (cf. eq. (5)). The next step is to represent \( \partial u_N / \partial n \) by a Lagrange multiplier function \( \lambda^j_M \) on each face \( S_i , \ i = 1, 2, 3 \) (expanded as a sum of polynomial basis functions \( \Phi_i \)) and to impose the Dirichlet conditions weakly, viz.

\[
\lambda^j_M = \sum_{i=1}^{M} \lambda^j_i \Phi_i , \ \lambda^j_i \in \mathbb{R} , \ i = 1, 2, 3 ,
\]  

(27)

\[
\int_{S_i} \Phi_i \left( g_i - u_N \right) = 0 , \ \ell = 1, ..., M .
\]  

(28)

Then, eq. (26) becomes

\[
\sum_{i=1}^{3} \int_{S_i} \left( \lambda^j_M \hat{W}^i_j - u_N \frac{\partial \hat{W}^i_j}{\partial n} \right) = 0 , \ j, l = 1, ..., N ,
\]  

(29)
which along with (27) yield a linear system analogous to (8), with unknown vector \( \tilde{a}_1, \tilde{a}_2, \ldots, \tilde{a}_N, \tilde{\lambda}_1, \tilde{\lambda}_2, \tilde{\lambda}_3 \)T, where

\[
\tilde{a}_1 = [a_{1,1}, a_{1,2}, \ldots, a_{1,N+1}], \quad \tilde{a}_2 = [a_{2,1}, a_{2,2}, \ldots, a_{2,N+1}], \ldots, \tilde{a}_N = [a_{N,1}, a_{N,2}, \ldots, a_{N,N+1}],
\]

\[
\tilde{\lambda}_1 = [\lambda_1^1, \lambda_2^1, \ldots, \lambda_M^1], \quad \tilde{\lambda}_2 = [\lambda_1^2, \lambda_2^2, \ldots, \lambda_M^2], \quad \tilde{\lambda}_3 = [\lambda_1^3, \lambda_2^3, \ldots, \lambda_M^3].
\]

Solving the system will produce the coefficients for the functions \( \alpha_j^N(x_j) \) (cf. (21)), as well as for the Lagrange multipliers \( \lambda_M^j \) (cf. (27)).

Currently we are investigating possible choices for the polynomials \( \beta_j^f(x_j) \) and \( \Phi_j^f \). The results of this study and the implementation of the method will be reported in a future communication.

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**References**