Research Article

Philippos Constantinou and Christos Xenophontos

Finite Element Analysis of an Exponentially Graded Mesh for Singularly Perturbed Problems

Abstract: We present the mathematical analysis for the convergence of an h version Finite Element Method (FEM) with piecewise polynomials of degree $p \geq 1$, defined on an *exponentially graded* mesh. The analysis is presented for a singularly perturbed reaction-diffusion and a convection-diffusion equation in one dimension. We prove convergence of optimal order and independent of the singular perturbation parameter, when the error is measured in the natural energy norm associated with each problem. Numerical results comparing the exponential mesh with the Bakhvalov–Shishkin mesh from the literature are also presented.

Keywords: Boundary Layers, Finite Element Method, Exponentially Graded Mesh

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Philippos Constantinou, Christos Xenophontos: Department of Mathematics and Statistics, University of Cyprus, P.O. Box 20537, Nicosia 1678, Cyprus, e-mail: constantinouphilippos@gmail.com, xenophontos@ucy.ac.cy

1 Introduction

The numerical solution of singularly perturbed problems has been studied extensively over the last few decades (see, e.g., the books [3, 4, 6] and the references therein). Besides the question of stability of discretizations (e.g., in the treatment of convection-dominated problems), a main difficulty in these problems is the presence of *boundary layers* in the solution, whose accurate approximation, independently of the singular perturbation parameter(s), is of great importance for the overall quality of the approximate solution. In the context of the Finite Element Method (FEM), the robust approximation of boundary layers requires either the use of the h version on non-uniform meshes (such as the Shishkin [10] or Bakhvalov [1] mesh), or the use of the high order p and hp versions on specially designed (variable) meshes [9]. In both cases, the a priori knowledge of the position of the layers is taken into account, and mesh-degree combinations can be chosen for which uniform error estimates can be established [9]. Among the many meshes that have been proposed over the last few decades, one finds the non-uniform, *exponentially graded* mesh, first appearing in [9] and [11] and then in [12]. The mesh was constructed by optimizing certain upper bounds on the error (see [12] for details) and was used in the conjunction with the FEM for polynomials of degree $p \geq 1$. Since then, there has been no further analysis of the FEM on this mesh, even though numerical experiments illustrated its superiority over existing meshes, such as the Shishkin and Bakhvalov meshes, at least for reaction-diffusion problems [13]. In the present article, we prove, for the first time, the robustness of the h version of the FEM on the exponential mesh as well as its optimal convergence rate in the energy norm. This is achieved by casting the mesh in the framework of [5] and utilizing the results obtained there. We present the analysis for the standard Galerkin FEM applied to one-dimensional singularly perturbed problems, but the mesh is suitable for *any* appropriate numerical method applied to singularly perturbed boundary value problems with exponential boundary layers.

The rest of the paper is organized as follows: In Section 2 we present the exponentially graded mesh, which is expressed via a generating function. The interpolation error analysis is presented in Section 3 and it utilizes the recent results of [7]. In Sections 4 and 5, we apply the interpolation error estimates to a reaction-diffusion and a convection-diffusion problem, respectively. Finally, in Section 6 we show the results of some numerical computations.

We will utilize the usual Sobolev space notation $H^k(I)$ to denote the space of functions on I with up to order k generalized derivatives in $L^2(I)$, equipped with the norm and seminorm $\|\cdot\|_{k,I}$ and $|\cdot|_{k,I}$, respectively. We will also use the space

$$H_0^1(I) = \{u \in H^1(I) : u|_{\partial I} = 0\},$$

where ∂I denotes the boundary of I. The norm of the space $L^\infty(I)$ of essentially bounded functions is denoted by $\|\cdot\|_{\infty,I}$. Finally, the notation "$a \lesssim b$" means "$a \le Cb$" with C a generic positive constant, independent of any discretization or singular perturbation parameters.

2 The Exponentially Graded Mesh

As already mentioned, the exponentially graded mesh first appeared in [9] and [11] and then in [12]. It was constructed specifically for the approximation of the typical boundary layer function

$$u_{BL}(x) := e^{-x/\varepsilon}, \quad x \in I := [0,1],\ \varepsilon \in (0,1]. \tag{1}$$

To fix ideas, let $N > 2$ be even and divide the interval $I = [0,1]$ into N subintervals I_j, using the mesh $\Delta = \{x_j\}_{j=0}^N$ and set $h_j = |I_j| = x_j - x_{j-1}$, $j = 1,\dots,N$. With $\mathbb{P}_p(\alpha,\beta)$ the space of polynomials of degree $\le p$ on the interval (α,β), we define the subspace $V_h \subset H_0^1(I)$ as

$$V_h = \{u \in H_0^1(I) : u|_{I_j} \in \mathbb{P}_p(I_j),\ j = 1,\dots,N\}. \tag{2}$$

With N a multiple of 2, we split our interval into

$$[0, x_{N/2-1}],\ [x_{N/2-1}, 1]$$

and on $[x_{N/2-1}, 1]$ we choose an equidistant mesh with $N/2+1$ elements. For the other subinterval the mesh will be *exponentially graded* with $N/2-1$ elements. In particular, on $[0, x_{N/2-1}]$ the mesh is given by a continuous, monotonically increasing, piecewise continuously differentiable, generating function ϕ with $\phi(0) = 0$. Then, the nodal points in our mesh are given by

$$x_j = \begin{cases} (p+1)\varepsilon\phi(\frac{j}{N}) & \text{if } j = 0,1,\dots,N/2-1, \\ x_{N/2-1} + (\frac{1-x_{N/2-1}}{N/2+1})(j - N/2 + 1) & \text{if } j = N/2,\dots,N, \end{cases} \tag{3}$$

with

$$\phi(t) = -\ln[1 - 2C_{p,\varepsilon}t], \quad t \in [0, 1/2 - 1/N], \tag{4}$$

where

$$C_{p,\varepsilon} = 1 - \exp\Big(-\frac{1}{(p+1)\varepsilon}\Big) \in \mathbb{R}^+. \tag{5}$$

An example of this mesh is shown in Figure 1.

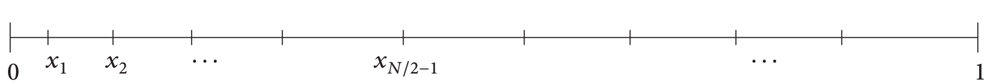

Figure 1. Example of the exponential mesh.

It is worth noting that we do not identify a "transition point", as is done in almost all other (non-uniform, layer-adapted) meshes designed for singularly perturbed problems (see [2]). We simply generate the exponentially graded mesh on $(0,1)$ and think of $x_{N/2-1}$ as our "transition point".

The mesh width in the interval $[0, x_{N/2-1}]$ satisfies

$$h_j = x_j - x_{j-1} = (p+1)\varepsilon\Big[\phi\Big(\frac{j}{N}\Big) - \phi\Big(\frac{j-1}{N}\Big)\Big] \le (p+1)\varepsilon N^{-1} \max_{I_j} \phi', \quad j = 1,\dots,N/2-1. \tag{6}$$

For $j = N/2 - 1$ we have

$$x_{N/2-1} = (p+1)\varepsilon\phi\Big(1/2 - \frac{1}{N}\Big) = -(p+1)\varepsilon \ln\Big[1 - C_{p,\varepsilon} + 2\frac{C_{p,\varepsilon}}{N}\Big],$$

hence, assuming

$$\varepsilon(p+1)\ln(N-2) < 1, \tag{7}$$

we have

$$e^{-x_{N/2-1}/\varepsilon} = e^{(p+1)\ln[1-C_{p,\varepsilon}+2\frac{C_{p,\varepsilon}}{N}]} \lesssim e^{(p+1)\ln[\frac{1}{N}]} \lesssim N^{-(p+1)}. \tag{8}$$

Assumption (7) simply dictates that ε is small and we are in the singularly perturbed case. The analysis in [5, 7] is carried out under the assumption

$$\max \phi' \lesssim N. \tag{9}$$

For our choice of ϕ we have

$$\max_{t\in[0,1/2-1/N]} \phi'(t) = \max_{t\in[0,1/2-1/N]} \frac{2C_{p,\varepsilon}}{1-2C_{p,\varepsilon}t} \le \frac{2\big(1-e^{\frac{-1}{(p+1)\varepsilon}}\big)}{1-2C_{p,\varepsilon}(1/2-1/N)} \lesssim N,$$

hence (9) holds. We also define the function ψ by

$$\phi = -\ln\psi,$$

which is monotonically decreasing with $\psi(0) = 1$. In our case

$$\psi(t) = 1 - 2C_{p,\varepsilon}t,$$

and since $\psi'(t) = -2C_{p,\varepsilon} \in \mathbb{R}^-$, the mesh generating function is of 'optimal' type in the terminology of [7].

3 Interpolation Error Estimates

Definition 3.1. Let $\{x_j\}_{j=1}^N$ be the nodes (3) obtained with the mesh generating function ϕ given by (4). For a given function v on I, we define its interpolant $v^I \in \mathbb{P}_p(0,1)$ as follows: $v^I|_{I_j} \in \mathbb{P}_p(I_j)$ with

$$v^I\Big(x_{j-1} + \frac{kh_j}{p}\Big) = v\Big(x_{j-1} + \frac{kh_j}{p}\Big), \quad k = 0,\dots,p,\ j = 1,\dots,N.$$

Using the above interpolant, we have the following lemma.

Lemma 3.2. *Let u_{BL} be given by (1) and let $u_{BL}^I \in \mathbb{P}_p(0,1)$ be its interpolant as in Definition 3.1. Then*

$$\|u_{BL} - u_{BL}^I\|_{0,[0,x_{N/2-1}]} \lesssim \varepsilon^{1/2}N^{-(p+1)}, \tag{10}$$

$$\|u_{BL} - u_{BL}^I\|_{\infty,I} \lesssim N^{-(p+1)}, \tag{11}$$

$$|u_{BL} - u_{BL}^I|_{1,I} \lesssim \varepsilon^{-1/2}N^{-p}. \tag{12}$$

Proof. We first show (11). For $I_j \subset [0, x_{N/2-1}]$ we have

$$\|u_{BL} - u_{BL}^I\|_{\infty,I_j} \lesssim h_j^{p+1}\|u_{BL}^{(p+1)}\|_{\infty,I_j}$$

with (cf. (6))

$$h_j \le (p+1)\varepsilon N^{-1}\max_{I_j}\phi' \le (p+1)\varepsilon N^{-1}\max_{I_j}|\psi'|e^{\frac{x_j}{(p+1)\varepsilon}}. \tag{13}$$

Since $\max|\psi'| \leq 2$, inequality (13) allows us to further obtain

$$\begin{aligned}\|u_{BL} - u_{BL}^I\|_{\infty,I_j} &\lesssim ((p+1)\varepsilon N^{-1} e^{\frac{x_j}{(p+1)\varepsilon}})^{p+1} \varepsilon^{-p-1} \|e^{-x/\varepsilon}\|_{\infty,I_j} \\ &\lesssim (p+1)^{p+1} N^{-(p+1)} (e^{x_j/\varepsilon - x_{j-1}/\varepsilon}) \\ &\lesssim N^{-(p+1)} e^{h_j/\varepsilon} \lesssim N^{-(p+1)} e^{(p+1)N^{-1} \max_{I_j} \phi'} \lesssim N^{-(p+1)},\end{aligned}$$

where (9) was used in the last step.

For $x \in [x_{N/2-1}, 1]$ we have, using (8),

$$|(u_{BL} - u_{BL}^I)(x)| \lesssim |u_{BL}(x)| + |u_{BL}^I(x)| \lesssim e^{-x_{N/2-1}/\varepsilon} \lesssim N^{-(p+1)},$$

hence

$$\|u_{BL} - u_{BL}^I\|_{\infty,I} \lesssim N^{-(p+1)},$$

and (11) is established.

We now show (10): using

$$\|u_{BL} - u_{BL}^I\|_{0,I_j} \lesssim h_j^{1/2} \|u_{BL} - u_{BL}^I\|_{\infty,I_j},$$

we see, from (6) and (9), that for $I_j \subset [0, x_{N/2-1}]$ there holds

$$\|u_{BL} - u_{BL}^I\|_{0,I_j} \lesssim (\varepsilon(p+1)N^{-1} \max_{I_j} \phi')^{1/2} N^{-(p+1)},$$

from which the desired result follows.

We finally show (12): using (13), we have for $I_j \subset [0, x_{N/2-1}]$

$$\begin{aligned}|u_{BL} - u_{BL}^I|_{1,I_j}^2 &\lesssim h_j^{2p} \int_{I_j} (u^{(p+1)}(x))^2\, dx \lesssim h_j^{2p} \int_{x_{j-1}}^{x_j} (\varepsilon^{-2(p+1)} e^{-2x/\varepsilon})\, dx \\ &\lesssim \varepsilon^{-2p-1} h_j^{2p} (e^{-2x_{j-1}/\varepsilon} - e^{-2x_j/\varepsilon}) \\ &\lesssim \varepsilon^{-2p-1} ((p+1)\varepsilon N^{-1} e^{\frac{x_j}{(p+1)\varepsilon}})^{2p} (e^{-2x_{j-1}/\varepsilon} - e^{-2x_j/\varepsilon}) \\ &\lesssim (p+1)^{2p} \varepsilon^{-1} N^{-2p} e^{\frac{2px_j}{(p+1)\varepsilon}} (e^{-2x_{j-1}/\varepsilon} - e^{-2x_j/\varepsilon}).\end{aligned}$$

It remains to bound the term $e^{\frac{2px_j}{(p+1)\varepsilon}} (e^{-2x_{j-1}/\varepsilon} - e^{-2x_j/\varepsilon})$. Denoting $x_{j-1/2} = (x_j + x_{j-1})/2$, we have

$$e^{-2x_{j-1}/\varepsilon} - e^{-2x_j/\varepsilon} \lesssim e^{-2x_{j-1/2}/\varepsilon} \sinh \frac{h_j}{\varepsilon},$$

and since

$$\begin{aligned}\sinh \frac{h_j}{\varepsilon} \lesssim \frac{h_j}{\varepsilon} &= (p+1) \int_{(j-1)/N}^{j/N} \phi'(x)\, dx = (p+1) \int_{(j-1)/N}^{j/N} \Big(-\frac{\psi'(x)}{\psi(x)}\, dx\Big) \\ &\lesssim e^{\frac{x_j}{(p+1)\varepsilon}} \int_{(j-1)/N}^{j/N} (-\psi'(x))\, dx,\end{aligned}$$

we see that

$$e^{\frac{2px_j}{(p+1)\varepsilon}} (e^{-2x_{j-1}/\varepsilon} - e^{-2x_j/\varepsilon}) \lesssim e^{\frac{2px_j}{(p+1)\varepsilon}} e^{-2x_{j-1/2}/\varepsilon} e^{\frac{x_j}{(p+1)\varepsilon}} \int_{(j-1)/N}^{j/N} (-\psi'(x))\, dx \lesssim 1.$$

Hence

$$|u_{BL} - u_{BL}^I|_{1,[0,x_{N/2-1}]} \lesssim \varepsilon^{-1/2} N^{-p}.$$

In $[x_{N/2}, 1]$ we have

$$|u_{BL} - u_{BL}^I|_{1,[x_{N/2-1},1]}^2 \leq |u_{BL}|_{1,[x_{N/2-1},1]}^2 + |u_{BL}^I|_{1,[x_{N/2-1},1]}^2 \lesssim \int_{x_{N/2}-1}^{1} |\varepsilon^{-1} e^{-x/\varepsilon}|^2\, dx \lesssim \varepsilon^{-1} e^{-2x_{N/2-1}/\varepsilon} \lesssim \varepsilon^{-1} N^{-2(p+1)},$$

where (8) was used once more. This completes the proof. □

4 A Reaction-Diffusion Problem

In this section, we apply the previous results to the following one-dimensional reaction-diffusion boundary value problem: Find $u(x)$ such that

$$-\varepsilon^2 u''(x) + c(x)u(x) = f(x) \quad \text{in } I = (0,1),$$
$$u(0) = u(1) = 0,$$

with $0 < \varepsilon \le 1$, $c(x) \ge 0$, $f(x) \in L^2(I)$ given. Its variational formulation reads: Find $u \in H_0^1(I)$ such that

$$B(u,v) = F(v) \quad \text{for all } v \in H_0^1(I), \tag{14}$$

where

$$B(u,v) = \int_I \{\varepsilon^2 u'(x)v'(x) + c(x)u(x)v(x)\}\,dx, \quad F(v) = \int_I f(x)v(x)\,dx.$$

Associated with the above problem is the ε-weighted H^1 norm, the so-called *energy norm*, defined by

$$\|u\|_\varepsilon^2 = \varepsilon^2\|u'\|_{0,I}^2 + \|u\|_{0,I}^2.$$

The solution will exhibit boundary layer behavior at both endpoints of I, hence the mesh will be given by

$$x_j = \begin{cases} (p+1)\varepsilon\phi(\frac{j}{N}) & \text{if } j = 0,1,\dots,N/4-1,\\ x_{N/4-1} + (\frac{x_{3N/4+1}-x_{N/4-1}}{N/2+2})(j - N/4 + 1) & \text{if } j = N/4,\dots,3N/4,\\ 1-(p+1)\varepsilon\phi(\frac{N-j}{N}) & \text{if } j = 3N/4+1,\dots,N, \end{cases} \tag{15}$$

with N now a multiple of 4. It is well known (see, e.g., [8]) that the solution u to (14) can be decomposed as

$$u = u_S + u_{BL}^{\pm}, \tag{16}$$

with

$$\begin{aligned} |(u_S)^{(k)}(x)| &\lesssim 1, & k &= 0,1,\dots,q,\\ |(u_{BL}^{\pm})^{(k)}(x)| &\lesssim \varepsilon^{-k}e^{-\mathrm{dist}(x,\partial I)/\varepsilon}, & k &= 0,1,\dots,q, \end{aligned} \tag{17}$$

for some q which depends only on the data – the smoother the data, the higher the value of q.

The finite element approximation $u_h \in V_h$ satisfies (14) for all $v \in V_h$ and we have

$$\|u-u_h\|_\varepsilon \le C\|u-v\|_\varepsilon \quad \text{for all } v \in V_h, \tag{18}$$

where the space V_h is given by (2).

Theorem 4.1. *Let u be the solution of* (14) *and u_h its finite element approximation based on the space V_h given by* (2) *using the nodes* (15). *Then*

$$\|u-u_h\|_\varepsilon \lesssim N^{-p}.$$

Proof. Let $u^I \in \mathbb{P}_p(0,1)$ be the pth-degree piecewise interpolant of u, given by Definition 3.1. We have from (16) that $u = u_S + u_{BL}^{\pm}$, hence we split the interpolant $u^I = u_S^I + (u_{BL}^+)^I + (u_{BL}^-)^I$ and we have, using (18),

$$\|u-u_h\|_\varepsilon \lesssim \|u-u^I\|_\varepsilon \lesssim \|u_S-u_S^I\|_\varepsilon + \|u_{BL}-(u_{BL}^+)^I\|_\varepsilon + \|u_{BL}-(u_{BL}^-)^I\|_\varepsilon.$$

For the smooth part, standard interpolation estimates and (17) give

$$\|u_S-u_S^I\|_{0,I} + \varepsilon^{1/2}|u_S-u_S^I|_{1,I} \lesssim N^{-p},$$

while for the boundary layer, Lemma 3.2 gives

$$\|u_{BL}-(u_{BL}^{\pm})^I\|_{0,I} + \varepsilon^{1/2}|u_{BL}-(u_{BL}^{\pm})^I|_{1,I} \lesssim N^{-p}.$$

Combining the above gives the desired result. □

5 A Convection-Diffusion Problem

In this section, we consider the following convection-diffusion boundary value problem: Find $u(x)$ such that

$$-\varepsilon u''(x) - b(x)u'(x) + c(x)u(x) = f(x), \quad x \in I = (0,1), \tag{19}$$

$$u(0) = u(1) = 0, \tag{20}$$

where $0 < \varepsilon \le 1$ and $b(x), c(x) \in L^\infty(I)$, $f(x) \in L^2(I)$ are given with $b(x) > 1$ for $x \in \overline{I}$. The solution to the above problem will exhibit boundary layer behavior near $x = 0$, and the following decomposition may be established (see, e.g., [6]):

$$u = u_S + u_{BL},$$

with

$$\left|(u_S)^{(k)}(x)\right| \lesssim 1, \qquad k = 0,1,2\ldots, \tag{21}$$

$$\left|(u_{BL})^{(k)}(x)\right| \lesssim \varepsilon^{-k} e^{-x/\varepsilon}, \quad k = 0,1,2,\ldots.$$

The variational formulation of (19), (20) reads: Find $u \in H_0^1(I)$ such that

$$B(u,v) = F(v) \quad \text{for all } v \in H_0^1(I), \tag{22}$$

where

$$B(u,v) = \varepsilon \int_I u'v'\,dx - \int_I (bu' - cu)v\,dx \quad \text{for all } u,v \in H_0^1(I),$$

$$F(u,v) = \int_I fv\,dx.$$

The finite element approximation $u_h \in V_h$ satisfies

$$B(u_h, v) = F(v) \quad \text{for all } v \in V_h,$$

and since $b > 1$, we can always ensure that $c + \frac{1}{2}b' > 0$, hence we have coercivity of the bilinear form in the energy norm:

$$B(v,v) \ge \alpha \|v\|_\varepsilon^2, \quad \alpha \in \mathbb{R}^+, \tag{23}$$

where

$$\|u\|_\varepsilon^2 = \varepsilon \|u'\|_{0,I}^2 + \|u\|_{0,I}^2.$$

We also have the orthogonality property

$$B(u - u_h, v) = 0 \quad \text{for all } v \in V_h.$$

The main result of this section is the following.

Theorem 5.1. *Let u be the solution of* (22) *and u_h its finite element approximation based on the space V_h given by* (2) *using the nodes* (3). *Then*

$$\|u - u_h\|_\varepsilon \sim \|u - u_h\|_{0,I} + \varepsilon^{1/2} |u - u_h|_{1.I} \lesssim N^{-p}.$$

Proof. The proof begins with the triangle inequality:

$$\|u - u_h\|_\varepsilon \le \|u - u^I\|_\varepsilon + \|u^I - u_h\|_\varepsilon, \tag{24}$$

where u^I is the interpolant of Definition 3.1. For the first term in (24), we have, using the splittings $u = u_S + u_{BL}$ and $u^I = u_S^I + u_{BL}^I$,

$$\|u - u^I\|_\varepsilon \le \|u_S - u_S^I\|_\varepsilon + \|u_{BL} - u_{BL}^I\|_\varepsilon.$$

Standard interpolation estimates give the result for the smooth part, thanks to (21), while Lemma 3.2 provides the desired bound for the boundary layer part.

For the second term in (24) we have from (23), with $\langle\cdot,\cdot\rangle_I$ denoting the usual $L^2(I)$ inner product,

$$\begin{aligned}\alpha\|u^I-u_h\|_\varepsilon^2 &\le B(u^I-u_h,u^I-u_h)\le B(u^I-u,u^I-u_h)\\ &\lesssim \varepsilon\langle(u^I-u)',(u^I-u_h)'\rangle_I+\langle c(u^I-u),u^I-u_h\rangle_I-\langle b(u^I-u)',u^I-u_h\rangle_I.\end{aligned}$$

It suffices to only consider the last term above. In fact, using the splittings $u=u_S+u_{BL}$ and $u^I=u_S^I+u_{BL}^I$, we get

$$|\langle b(u^I-u)',u^I-u_h\rangle_I|\le\langle b(u_S^I-u_S)',u^I-u_h\rangle_I+\langle b'(u_{BL}^I-u_{BL}),u^I-u_h\rangle_I+\langle b(u_{BL}^I-u_{BL}),(u^I-u_h)'\rangle_I \tag{25}$$

and, again, only the last term in (25) needs to be considered. Using the Cauchy–Schwarz inequality and (10), we have

$$\begin{aligned}|\langle b(u_{BL}^I-u_{BL}),(u^I-u_h)'\rangle_{[0,x_{N/2-1}]}| &\lesssim \|u_{BL}^I-u_{BL}\|_{0,[0,x_{N/2-1}]}\|(u^I-u_h)'\|_{0,[0,x_{N/2-1}]}\\ &\lesssim \varepsilon^{1/2}N^{-(p+1)}\|(u^I-u_h)'\|_{0,[0,x_{N/2-1}]}\\ &\lesssim N^{-(p+1)}\|u^I-u_h\|_\varepsilon.\end{aligned}$$

Next, using (8) and an inverse inequality [8, Theorem 3.91], we get

$$\begin{aligned}|\langle b(u_{BL}^I-u_{BL}),(u^I-u_h)'\rangle_{[x_{N/2-1},1]}| &\lesssim \|u_{BL}^I-u_{BL}\|_{\infty,[x_{N/2-1},1]}\|(u^I-u_h)'\|_{0,[x_{N/2-1},1]}\\ &\lesssim \|u_{BL}^I-u_{BL}\|_{\infty,[x_{N/2-1},1]}\sum_{j=N/2-1}^{N-1}\|(u^I-u_h)'\|_{0,[x_j,x_{j+1}]}\\ &\lesssim N^{-(p+1)}\sum_{j=N/2-1}^{N-1}\frac{p}{h_j}\|u^I-u_h\|_{0,[x_j,x_{j+1}]}\\ &\lesssim N^{-p}\|u^I-u_h\|_\varepsilon.\end{aligned}$$

Combining the above, we get the desired result. □

6 Numerical Results

Since for reaction-diffusion problems there exist several numerical evidence that corroborate the claims made in this article (see [11–13]), in this section we present the results of numerical computations for the following model convection-diffusion problem:

$$\begin{aligned}-\varepsilon u''(x)-b(x)u'(x)+c(x)u(x)&=f(x),\quad x\in(0,1),\\ u(0)=u(1)&=0.\end{aligned}$$

We will consider two examples: one with constant data and one with variable data. In both cases we will be comparing the performance of the h version FEM with polynomials of degree $p=1,2$, using the following meshes:

- Bakhvalov–Shishkin (BS) mesh, for which $\phi(t)=-\ln[1-2(1-N^{-1})t]$.
- Exponential (eXp) mesh, for which $\phi(t)=-\ln[1-2C_{p,\varepsilon}t]$, with $C_{p,\varepsilon}$ given by (5).

Let us comment on the similarities and differences between the two meshes. They both have a "transition" point that separates the domain into a layer region and a non-layer one. The mesh points outside the layer region are equidistant for both meshes, while within the layer region the mesh points are chosen via the mesh generating function ϕ. The meshes differ in the choice of the transition point as well in the definition of ϕ. But the major difference between them is that for the BS mesh, the error estimates hold under the assumption $\varepsilon\lesssim N^{-1}$, while for the eXp mesh this assumption is not needed. In fact (see, e.g., [2]), the interpolation error for the BS mesh with piecewise linear basis functions is of the order $O(\varepsilon+N^{-1})$ which shows that for relatively large ε the BS mesh might not yield the optimal convergence rate.

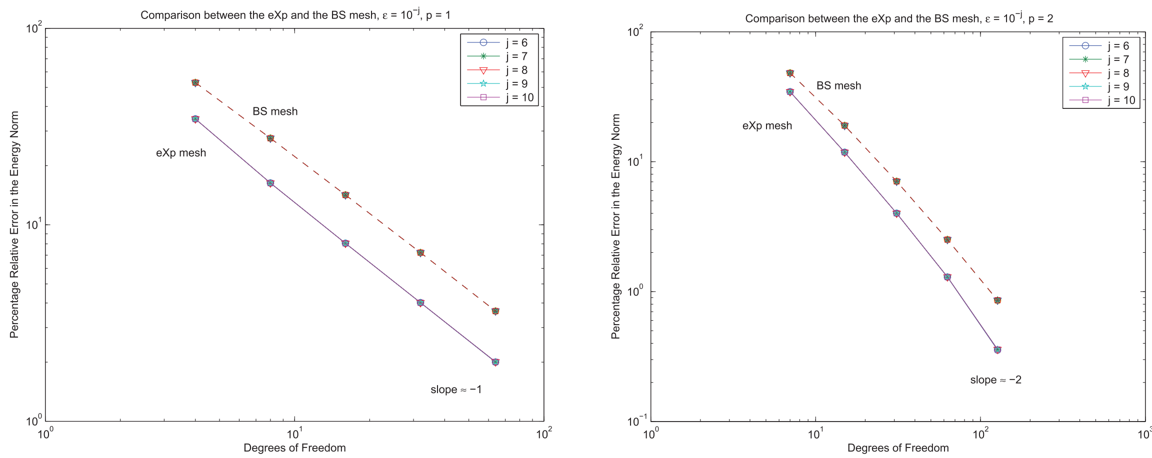

Figure 2. Energy norm convergence for Example 6.1. Left: $p = 1$. Right: $p = 2$.

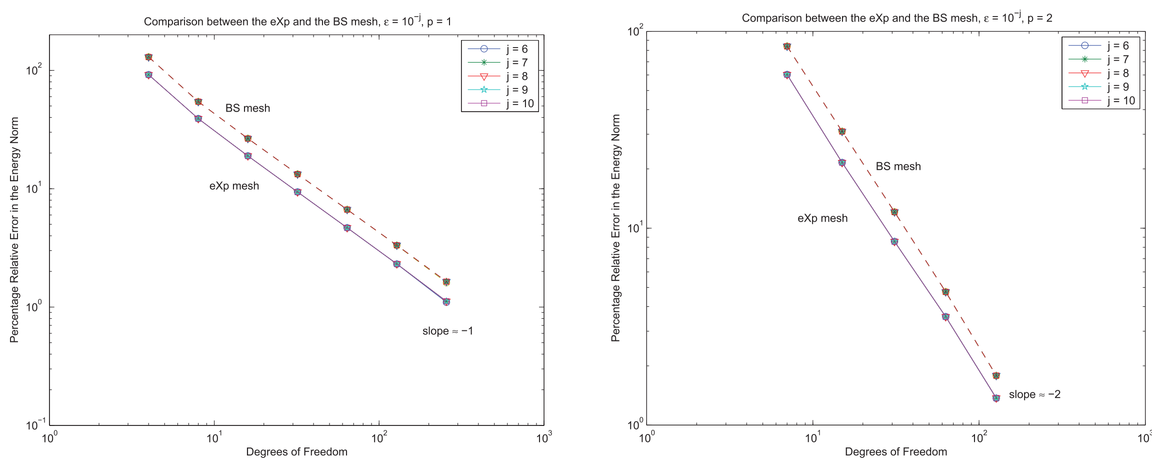

Figure 3. Energy norm convergence for Example 6.2. Left: $p = 1$. Right: $p = 2$.

Example 6.1. $b(x) = c(x) = f(x) = 2$ and an exact solution is available making our reported results reliable. In Figure 2 we show the convergence of the method for $p = 1, 2$, for both meshes – the eXp mesh in solid and the BS mesh in dotted lines. The robustness and optimal rate of the methods are clearly visible; even though the optimal rate of convergence is achieved by both methods, the eXp mesh yields an error with a smaller constant, as the solid curve lies below the dotted one. Other values of p (not shown here) gave the same results.

Example 6.2. $b(x) = \exp(x)$, $c(x) = 1/(x^2 + 1)$, $f(x) = \sin(x)$, and an exact solution is not available. For the computations we use a reference solution obtained with the *hp* version of the FEM with high enough p as to guarantee adequate accuracy (which is achieved at an exponential rate). The observations made for the previous example remain valid for this one as well, as seen in Figure 3.

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