A note on \( p/hp \) finite element methods for reaction–diffusion problems in polygonal domains

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SUMMARY

We present guidelines on how to choose the mesh and polynomial degree distribution for the approximation of reaction–diffusion problems in polygonal domains, in the context of the \( hp \) finite element method. These guidelines take into account the presence of both boundary layers and corner singularities in the solution. We present the results of numerical computations showing that robust, (near) exponential rates of convergence can be achieved using this technique. Copyright © 2000 John Wiley & Sons, Ltd.

1. INTRODUCTION

The numerical solution of singularly perturbed boundary value problems has received much attention recently, and there are numerous papers and even books written on this subject (cf. [1–3] and the reference therein). Problems of this type arise in many areas of engineering, such as fluid mechanics, heat transfer as well as problems in structural mechanics posed over thin domains. The solution of singularly perturbed problems will, in general, contain boundary layers along the boundary of the domain. These layers complicate the numerical approximation, and the method must be carefully tailored to account for their presence. If in addition the domain contains corners, then the solution will also include singularities in the neighbourhood of each vertex. The coupling of boundary layers and corner singularities is a complicated affair and the approximation of both such solution components is a challenging task.

In this note, we indicate how one can alleviate both difficulties associated with singularly perturbed problems in polynomial domains, through the use of the \( hp \) version of the finite element method (FEM). In particular, we will give precise guidelines on how the mesh and polynomial degree distribution should be designed for reaction–diffusion problems, in order for the solution to be uniformly approximated at a near exponential (spectral) rate. Our goal in this note is to illustrate, through numerical computations, that the guidelines for square domains, presented in References [4, 5], can be extended to general polygons. Even though the focus will be on the practical aspects of this technique, we provide references in which the mathematical details can...
be found. For a full exposition on the one-dimensional case we refer to Reference [6], and for the two-dimensional cases of smooth and square domains, we refer to References [5, 7, 8]. The guidelines presented here are not specific to reaction–diffusion problems. They are directly applicable to other singularly perturbed elliptic boundary value problems, and they have in fact been applied to the Reissner–Mindlin plate model [9]. We also refer to Reference [4] for an overview of the application of the hp version of the FEM to problems in mechanics with boundary layers, including the ‘prescription’ for mesh–degree combinations.

The organization of the present article is as follows. In Section 2 we describe the model problem and the properties of its solution and give the guidelines for choosing the mesh-degree combination for the approximation. In Section 3 we present the results of numerical computations over an L-shaped domain and finally, we present our conclusions in Section 4.

We will use $H^k(\Omega)$ to denote the Sobolev space of order $k \in \mathbb{N}_0$ on a domain $\Omega \subset \mathbb{R}^2$, with $H^0(\Omega) = L^2(\Omega)$, and $|.|_{k,\Omega}, |.|_{k,\Omega}$ denoting the norm and seminorm as usual. Also, $H^1_0(\Omega) = \{ u \in H^1(\Omega) : u = 0 \text{ on } \partial\Omega \}$. The letter $C$ will be used to denote generic positive constants, possibly not the same in each occurrence.

2. THE MODEL PROBLEM

We consider the two-dimensional reaction–diffusion problem

$$-\varepsilon^2 \Delta u + u = f \quad \text{in } \Omega \subset \mathbb{R}^2$$

$$u = 0 \quad \text{on } \partial\Omega$$

(1)

where $f$ is analytic and $\varepsilon \in (0,1]$ is a parameter that can approach 0. In References [10, 11] the asymptotic properties of the solution to (1) were studied and an expansion was given in the case of a rectangular domain $\Omega$. In Reference [12], a similar decomposition was presented for general polygonal domains. It was shown that

$$u = u_{2n} + R_{2n}$$

where $u_{2n}$ is the asymptotic expansion (with $2n$ terms) and $R_{2n}$ is a smooth remainder satisfying

$$|R_{2n}|_{m,\Omega} \leq C(n) \varepsilon^{2n+1-m}, \quad m \leq 2n + 1$$

The nature of the components in the asymptotic expansion $u_{2n}$ depends on whether $\Omega$ is a rectangle or a general polygon. In both cases, however, it includes the following:

1. A smooth part that depends on the right-hand side function $f$.
2. Boundary layers that are (essentially) sums of functions of the type $\tau(y) \exp(-x/\varepsilon)$, with $\tau(y)$ smooth.
3. Corner singularities that decay exponentially away from the corners.

Our goal is to give guidelines on how to design the mesh and polynomial degree distribution, so that each component of $u$ is well approximated, uniformly in $\varepsilon$ and at a sufficiently fast rate. These guidelines will be in the context of high-order $p/hp$ versions of the FEM, in which the polynomial degree of each element changes to improve accuracy.

We first note that the smooth part and the remainder can be approximated at an arbitrarily high algebraic (spectral) rate by simply using the standard $p$ version [13], over an arbitrary mesh.
Hence the challenge lies in approximating the remaining components. For the corner singularities, it is well known [14] that the $hp$ version over geometrically graded meshes yields exponential rates of convergence. Boundary layers can also be approximated at an exponential rate, provided thin elements of width $pe$ are inserted along the boundary [5, 8]. With these observations in mind we choose the mesh–degree combination as follows:

1. Partition the domain $\Omega$ with (rectangular) elements of width $pe$ along the boundary. Note that as $p$ increases the size (and not the number) of the elements will change.
2. Insert a geometric refinement towards each vertex of the domain, that is concentrated within the boundary layer refinement. (See Reference [14] for the analysis of the $hp$ version with geometrically graded meshes.)
3. A uniform refinement in the interior of the domain may be included, depending on the right-hand side function $f$ in (1).

In the case of the $hp$ version, both $h$ and $p$ change; hence the number of geometric layers increases as $h \to 0$, and the size of the boundary layer elements also changes as $p \to \infty$. In the case of the $p$ version, the mesh is kept fixed and only $p$ changes. Thus, in the $p$ version the number of geometric layers does not change but is usually chosen to include a sufficient amount of refinement. The polynomial degree can then be taken to be uniform over all elements and since in practice $p$ only increases to a finite limit (typically as high as 8 in commercial codes), the boundary layer elements can be taken to have width $p_{\text{max}} \varepsilon$. Figure 1 shows how the above mesh design principles are applied to create the mesh in the vicinity of a vertex.

Using the above recommendations for the design of the mesh, problems like (1) can be approximated reliably. In fact, for the case of a square domain it was shown in Reference [5] that the following energy norm* estimate holds:

$$\|u - u_{\text{FEM}}\|_E \leq C(s) p^{1/2 - \delta} p^{-s} \forall \varepsilon$$

with $\delta > 0$ arbitrarily small and $C(s)$ a constant independent of $p$ and $\varepsilon$. This shows that the method is robust and converges at an arbitrarily high algebraic (spectral) rate. It also shows that as $\varepsilon \to 0$, the method not only does not deteriorate but actually performs better. The numerical results in the next section will suggest that an estimate like (2) holds for the case of polygonal domains as well, even though this has not been established mathematically. Moreover, since the right-hand side function $f$ will be taken as constant the numerical results will show that an exponential rate of convergence is observed, which is consistent with the one-dimensional analysis of Reference [6].

3. NUMERICAL RESULTS

In this section we present the results of numerical computations for problem (1) with $f = 1$ and $\Omega = (0, 2)^2 \setminus (1, 2)^2$ an $L$-shaped domain. The computations were performed on a SUN Ultra 1 workstation using the finite element package StressCheck. The polynomial degree $p$ varied from 1 to 8 uniformly over all elements and the meshes were kept fixed. This will illustrate that even without $hp$ mesh refinement, choosing the mesh properly is of great importance. These results

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*The energy norm is defined as $\|u\|^2_E := \varepsilon^2 \|u\|^2_{H^1,\Omega} + \|u\|^2_{0,\Omega}$. 
Figure 1. Mesh design in the vicinity of a vertex, with two geometric layers of ratio \( s = \frac{1}{2} \): (a) Vicinity of vertex \( v \); (b) boundary layer mesh; (c) boundary layer and geometrically graded mesh.

will also show how the \( p \) version performs for various choices of (fixed) meshes. We will be comparing four schemes as described below:

1. Scheme 1: (Figure 2) The mesh includes boundary layer elements of width \( p_{\text{max}} \varepsilon \) along the boundary and geometric refinement with 6 layers \( \text{within} \) the region \( (1 - p_{\text{max}} \varepsilon, 1)^2 \) near the re-entrant corner (i.e. the geometric mesh is \( \varepsilon \)-dependent).
2. Scheme 2: (Figure 3) The mesh includes boundary layer elements of width $p_{\text{max}}\varepsilon$ along the boundary and geometric refinement with 6 layers that is now independent of $\varepsilon$ and extends outside of the region $(1 - p_{\text{max}}\varepsilon, 1)^2$ near the re-entrant corner.

3. Scheme 3: (Figure 4) The mesh includes only boundary layer elements of width $p_{\text{max}}\varepsilon$ along the boundary.

4. Scheme 4: (Figure 5) The mesh includes only geometric refinement with 6 layers that is independent of $\varepsilon$. This is the typical mesh one would use in the case of $\varepsilon = 1$, i.e. when...
no boundary layers are present and when the corner singularities do not decay exponentially away from the corner.

Figures 6 and 7 show the (estimated) percentage relative error in the energy norm

$$\text{Error} = 100 \times \frac{\|u - u_{\text{FEM}}\|_E}{\|u\|_E}$$

versus the number of degrees of freedom, in a log-log scale. We see that Schemes 1–3 perform very well and do not deteriorate as $\varepsilon \to 0$. This is due to the fact that the elements of width $p_{\text{max,}}$
that are present along the boundary are able to capture the boundary layer effect extremely well. Hence, Scheme 4 with no such elements gives poor results for small values of $\varepsilon$ and will no longer be considered in our study. We also see that when the error in the energy norm is of interest, Scheme 3 with only boundary layer refinement gives the 'best' results.
When quantities such as the derivative of the solution are measured near the boundary, the picture is very different. Figures 8 and 9 show the flux \( \frac{\partial u}{\partial x} \) as predicted by Schemes 1–3 for \( \varepsilon = 0.01 \) and 0.001, respectively, along a 45° line of length \( \varepsilon \) emanating from the point \((1 - \varepsilon, 1 - \varepsilon)\) near the re-entrant corner. Schemes 2 and 3 do not yield satisfactory results as \( \varepsilon \to 0 \), even though
their performance in the energy norm was quite good. For Scheme 3 this is to be expected, since the lack of proper refinement near the point of singularity causes the deterioration in the performance. It is interesting to see that the geometric refinement must be \( \epsilon \)-dependent, as in Scheme 1, in order for quantities such as the flux to be measured reliably near the boundary. Failure to do so will result in incorrectly predicting \( \partial u / \partial x \) or other similar quantities, as seen by the results shown in Figure 9.

Finally, Figure 10 shows the (estimated) percentage relative error in the energy norm for Scheme 1, with \( \epsilon = 10^{-j} \), \( j = 2, 3, 4 \). Note that the errors decrease as \( \epsilon \) decreases, which is in agreement with the estimate (2) established for square domains.

4. CONCLUSIONS

Based on our numerical study dealing with the approximation of reaction–diffusion problems, we conclude the following:

1. The mesh must be designed to account for the presence of both boundary layers and corner singularities. This can be achieved by using elements of width \( p \epsilon \) along the boundary and geometric refinement towards each corner of the domain.
2. The geometric refinement must be \( \epsilon \)-dependent and concentrated within the boundary layer region near each vertex. This will allow for quantities such as the flux to be predicted with confidence, even when measured very near the point of singularity.
3. The above mesh–degree recommendations lead to an (observed) exponential rate of convergence, when the given data (i.e. the function \( f \)) is analytic.
As a final comment, we would like to mention that these mesh–degree guidelines are applicable to other elliptic singularly perturbed problems, such as plate and shell models. It is important to note, however, that (at least) in the case of the Reissner–Mindlin plate, considered in Reference [9], the geometric refinement must extend outside the boundary layer region in order for quantities of engineering interest to be predicted reliably.

REFERENCES

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