Research Article

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A Parameter Robust Finite Element Method for Fourth Order Singularly Perturbed Problems

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Abstract: We consider fourth order singularly perturbed problems in one-dimension and the approximation of their solution by the $h$ version of the finite element method. In particular, we use piecewise Hermite polynomials of degree $p \geq 3$ defined on an exponentially graded mesh. We show that the method converges uniformly, with respect to the singular perturbation parameter, at the optimal rate when the error is measured in both the energy norm and a stronger, ‘balanced’ norm. Finally, we illustrate our theoretical findings through numerical computations, including a comparison with another scheme from the literature.

Keywords: Fourth Order Singularly Perturbed Problem, Boundary Layers, Finite Element Method, Exponentially Graded Mesh, Uniform Convergence

MSC 2010: 65N30

1 Introduction

The numerical solution of singularly perturbed problems has been studied extensively over the last few decades (see, e.g., the books [12, 13, 18] and the references therein). As is well known, a main difficulty in these problems is the presence of boundary layers in the solution whose accurate approximation, independently of the singular perturbation parameter(s), is of great importance for the overall quality of the approximate solution to be considered reliable. In the context of the finite element method (FEM), the robust approximation of boundary layers requires either the use of the $h$ version on non-uniform, layer-adapted meshes (such as the Shishkin [20] or Bakhvalov [2] mesh), or the use of the high order $p$ and $hp$ versions on the so-called spectral boundary layer mesh [11, 19]. In this article we consider a different, layer-adapted, exponentially graded mesh, which first appeared in [19] and [22]. The finite element analysis on this mesh is carried out in [5] for one-dimensional reaction-diffusion and convection-diffusion problems, in [23] for a two-dimensional convection-diffusion problem posed in a square and in [24] for two-dimensional reaction-diffusion problems posed in smooth domains. All the aforementioned works concern second order singularly perturbed problems. Only recently fourth order singularly perturbed problems have truly attracted the attention of the research community (see, e.g., [4, 7, 14, 21] for recent results and [15, 17] for some earlier results). Such problems occur in a variety of applications, perhaps the most famous being the Orr–Sommerfeld equation from hydrodynamics (see, e.g., [6, Chapter 5]) which describes the shear flows of viscous, Newtonian, incompressible fluids. The problem reads: find $u(x)$ (the potential/stream-function) such that

\[
\left( \frac{1}{Re} \right) u^{(4)}(x) - i \alpha ((v(x) - y)u^{(2)}(x) - (a^2(v(x) - y) + v^{(2)}(x))u(x)) = 0,
\]

where $Re$ is the Reynolds number of the base flow, $v(x)$ is a given function (the profile of the velocity of the undisturbed flow), $y$ is a (spectral) parameter, $\alpha$ is the wavenumber and $i$ is the imaginary unit. Usually, $y$ is also an unknown, making (1.1) an eigenvalue problem. However, before we can tackle the eigen-

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value problem, we should study the boundary value problem as we do here. The solution of (1.1) gives the potential/stream-function in terms of the horizontal distance $x$. If $Re$ is large then $1/Re$ will be small, making the problem singularly perturbed. In this article we will study a simplified version of (1.1) and extend the results of [5] to one-dimensional fourth order singularly perturbed problems (see (2.1) ahead). The work presented here serves as a stepping stone for the treatment of multidimensional fourth order singularly perturbed boundary and eigenvalue problems.

The rest of the paper is organized as follows: In Section 2 we present the model problem and its regularity. The discretization using the exponentially graded mesh is presented in Section 3 and in Section 4 we present our main result of parameter robust, optimal convergence. We do so using the (natural) energy norm associated with our problem as well as a stronger, balanced norm. Balanced norm estimates have become popular in the literature, see [7, 9, 16], as they measure the error in each component of the solution in the ‘appropriate’ way (see Section 4.1). Finally, Section 5 shows the results of some numerical computations that illustrate the theoretical findings and Section 6 gives some closing remarks.

With $I \subset \mathbb{R}$ a bounded open interval with boundary $\partial I$ and measure $|I|$, we will denote by $C^k(I)$ the space of continuous functions on $I$ with continuous derivatives up to order $k$. We will use the usual Sobolev spaces $W^{k,m}(I)$ of functions on $I$ with $0, 1, 2, \ldots, k$ generalized derivatives in $L^m(I)$, equipped with the norm and seminorm $\| \cdot \|_{k,m,I}$ and $| \cdot |_{k,m,I}$, respectively. When $m = 2$, we will write $H^k(I)$ instead of $W^{k,2}(I)$, and for the norm and seminorm we will write $\| \cdot \|_{k,I}$ and $| \cdot |_{k,I}$, respectively. We will also use the space

$$H_0^k(I) = \{ u \in H^k(I) : u(0) = u'(0) = u'(1) = u(1) = 0, \ i = 0, \ldots, k - 1 \}.$$

The norm of the space $L^\infty(I)$ of essentially bounded functions is denoted by $\| \cdot \|_{\infty,I}$. Finally, the notation $a \leq b$ means $a \leq Cb$ with $C$ being a generic positive constant, independent of any discretization or singular perturbation parameters.

## 2 The Model Problem and Its Regularity

We consider the following model problem: Find $u$ such that

$$\varepsilon^2 u^{(0)}(x) - (a(x)u'(x))^2 + b(x)u(x) = f(x) \quad \text{in} \ I = (0, 1),$$

(2.1)

along with the boundary conditions

$$u(0) = u'(0) = u'(1) = u(1) = 0.$$  

(2.2)

The parameter $0 < \varepsilon \leq 1$ is given, as are the functions $a, b, f$, which are assumed to be sufficiently smooth on $\overline{I} = [0, 1]$. The variational formulation of (2.1)–(2.2) reads: Find $u \in H_0^2(I)$ such that

$$\mathcal{B}_\varepsilon(u, v) = \mathcal{F}(v) \quad \text{for all} \ v \in H_0^2(I),$$

(2.3)

where, with $\langle \cdot, \cdot \rangle_I$ the usual $L^2(I)$ inner product,

$$\mathcal{B}_\varepsilon(u, v) = \varepsilon^2 \langle u'', v'' \rangle_I + \langle au', v' \rangle_I + \langle bu, v \rangle_I,$$

$$\mathcal{F}(v) = \langle f, v \rangle_I.$$

(2.4)

It follows that the bilinear form $\mathcal{B}_\varepsilon(\cdot, \cdot)$ given by (2.4) is coercive with respect to the energy norm

$$\| u \|^2_{E,I} := \mathcal{B}_\varepsilon(u, u),$$

i.e.,

$$\mathcal{B}_\varepsilon(u, u) \geq \| u \|^2_{E,I} \quad \text{for all} \ u \in H_0^2(I).$$

In [14], the regularity of the solution to (2.3) was studied, under the assumption of analytic input data, i.e. $a, b, f$ are assumed to be analytic functions with known derivative growth estimates. It was shown that the solution $u$ is analytic in $I$ and its derivative features boundary layers at the endpoints. We quote the relevant result and refer to [14] for more details.
Theorem 2.1. Assume $a, b, f$ are analytic functions and let $u \in H_0^2(I)$ be the solution of (2.3).

(i) There exists a constant $K > 0$ independent of $\varepsilon$ and $n$ (but depending on the data $a, b, f$), such that

$$
\|u^{(n)}\|_{\text{loc}, I} \leq K^n \max[n^n, \varepsilon^{1-n}] \quad \text{for all } n = 0, 1, 2, \ldots
$$

(ii) $u$ can be decomposed as $u = u_S + u_{BL} + u_R$, where for some constants $C_S, C_{BL}, C_R, \beta > 0$ independent of $\varepsilon$, there holds

$$
\|u_S^{(n)}\|_{\text{loc}, I} \leq C_S \varepsilon^n, \quad |u_{BL}^{(n)}(x)| \leq C_{BL} \varepsilon^{1-n} e^{-\beta \text{dist}(x, \partial I)/\varepsilon} \quad \text{for all } n = 0, 1, 2, \ldots
$$

and

$$
\|u_R\|_{\text{loc}, I} + \|u_R^{I}\|_{\text{loc}, \partial I} + \|u_R\|_{E, I} \leq C_R \varepsilon^{-\beta/e}.
$$

Proof. This was shown in [14] in the case of constant coefficients $a(x) = b(x) = 1$ and in [3] in the case of variable coefficients.

For our purposes, we will need the following (simpler) version of the above result, which follows immediately.

Theorem 2.2. Assume $a, b, f$ are sufficiently smooth functions and let $u \in H_0^2(I)$ be the solution of (2.3). Then

$$
u = w + u_{BL},
$$

where

$$
\|w^{(n)}\|_{\text{loc}, I} \leq 1, \quad |w_{BL}^{(n)}(x)| \leq \varepsilon^{1-n} e^{-\beta \text{dist}(x, \partial I)/\varepsilon} \quad \text{for all } n = 0, 1, 2, \ldots.
$$

3 Discretization by an Exponentially Graded $h$-FEM

The discrete version of (2.3) reads: find $u_{\text{FEM}} \in V_h \subset H_0^2(I)$ such that

$$
B_{\varepsilon}(u_{\text{FEM}}, v) = F(v) \quad \text{for all } v \in V_h \subset H_0^2(I),
$$

with the finite-dimensional subspace $V_h$ defined below. Note that

$$
\|u - u_{\text{FEM}}\|_{E, I} \leq \|u - v\|_{E, I} \quad \text{for all } v \in V_h. \tag{3.1}
$$

Let

$$
\Delta = \{0 = x_0 < x_1 < \cdots < x_N = 1\}
$$

be an arbitrary partition of $I = (0, 1)$ and set

$$
I_j = (x_{j-1}, x_j), \quad h_j = |I_j| = x_j - x_{j-1}, \quad j = 1, \ldots, N.
$$

With $P_p(\alpha, \beta)$ the space of polynomials of degree less than or equal to $p \geq 2N + 1$ on the interval $(\alpha, \beta)$, we define the subspace $V_h \subset H_0^2(I)$ as

$$
V_h = \{u \in H_0^2(I) : u|_{I_j} \in P_p(I_j), \; j = 1, \ldots, N\}. \tag{3.2}
$$

We note that the space $V_h$ consists of the classical (piecewise) Hermite polynomials (see, e.g., [1]), hence we quote the following relevant results.

Definition 3.1 ([1]). Let $\{x_i\}_{i=0}^N$ be an arbitrary partition of the interval $[a, b]$ and suppose that for a sufficiently smooth function $f(x), \; x \in [a, b]$, the values

$$
f(x_i) = y_i \in \mathbb{R}, \quad f^i(x_i) = y_i^i \in \mathbb{R}, \quad i = 0, 1, \ldots, N
$$

are given. Then there exists a unique polynomial $f^i \in P_{2N+1}(a, b)$, called the Hermite interpolant of $f$, given by

$$
f^i(x) = \sum_{i=0}^{N} (y_i H_{0,i}(x) + y_i^i H_{1,i}(x)),
$$

where $H_{0,i}$ and $H_{1,i}$ are

...
The meshwidth where, with node \( x_i \),
\[
H_{0,j}(x) = \left[ 1 - 2(x - x_i) \frac{dL_j}{dx}(x_i) \right] L_j^2(x), \quad H_{1,j}(x) = (x - x_i)L_j^2(x).
\]

**Theorem 3.2** ([1, Theorem 1.12]). Let \( v \in C^{2n+2}([a, b]) \) and let \( \Delta = \{ x_i \}_{i=0}^{N} \) be a mesh on \([a, b]\) with maximum meshsize \( h \) and with \( N \) a multiple of \( n \). If \( v' \) is the piecewise Hermite interpolant of \( v \) from Definition 3.1, having degree at most \( 2n+1 \) on each subinterval \([x_{i-1}, x_i]\), \( i = 1, \ldots, N \), then
\[
\| v^{(k)} - (v')^{(k)} \|_{\infty, I} \leq \| v^{(2n+2)} \|_{\infty, I} h^{2n+2-k}, \quad k = 0, 1, \ldots, 2n+1.
\]

In view of Theorem 2.2, the ‘challenge’ lies in approximating the one-dimensional boundary layer function
\[
e^{-\beta x/h}, \quad \beta \in \mathbb{R}^+, \quad x \in [0, 1], \quad \varepsilon \in (0, 1].
\]

In [5] a layer adapted, exponentially graded mesh for the approximation of functions of the type (3.3) was analyzed for second order singularly perturbed problems. Our goal is to utilize this mesh and extend the results to fourth order singularly perturbed problems. To this end, let the mesh points be chosen as follows: with \( N > 4 \) a multiple of \( 4 \), we split our interval into
\[
[0, x_{N/4-1}], \quad [x_{N/4-1}, x_{3N/4+1}], \quad [x_{3N/4+1}, 1]
\]
and on \([x_{N/4-1}, x_{3N/4+1}]\) we choose an equidistant mesh. For the other two subintervals the mesh will be **exponentially graded**. In particular, the mesh is given by a continuous, monotonically increasing, piecewise continuously differentiable, generating function \( \phi \) with \( \phi(0) = 0 \). The nodal points in our mesh are given by
\[
x_j = \begin{cases} 
\frac{E}{\beta}(p + 1)\phi \left( \frac{j}{N} \right) & \text{if } j = 0, 1, \ldots, N/4 - 1, \\
x_{N/4-1} + \left( \frac{x_{3N/4} - x_{N/4-1}}{N/2 + 2} \right) \left( i - \frac{N}{4} + 1 \right) & \text{if } j = N/4, 3N/4, \\
1 - \frac{E}{\beta}(p + 1)\phi \left( \frac{N - j}{N} \right) & \text{if } j = 3N/4 + 1, \ldots, N,
\end{cases}
\]  
(3.4)

with
\[
\phi(t) = -\ln[1 - 4C_{p,\varepsilon} t], \quad t \in [0, 1/4 - 1/N],
\]  
(3.5)

where
\[
C_{p,\varepsilon} = 1 - \exp\left( -\frac{\beta}{E(p + 1)} \right) \in \mathbb{R}^+.
\]

An example of this mesh is shown in Figure 1. Recently, it has been shown that the exponential mesh may be thought of as a generalized S-type mesh [8].

![Figure 1](image.png)

**Figure 1.** Example of the exponential mesh.

We also define the function \( \psi \) by \( \phi = -\ln \psi \), which gives \( \psi(t) = 1 - 2C_{p,\varepsilon} t \) as well as \( \psi'(t) = -2C_{p,\varepsilon} \in \mathbb{R}^- \). The meshwidth \( h_j \) in the intervals \([0, x_{N/4-1}], [x_{3N/4+1}, 1]\) satisfies
\[
h_j \leq \frac{E}{\beta}(p + 1)N^{-1} \max_{I_j} \phi' \leq \frac{E}{\beta}(p + 1) \psi_{x_{N/4-1}}(p+1), \quad j = 1, \ldots, N/4 - 1, 3N/4 + 1, \ldots, N;
\]  
(3.6)

see [5]. Moreover, under the assumption \( \frac{E}{\beta}(p + 1) \ln(N - 4) < 1 \), it was shown in [5] that
\[
e^{-\beta x_{N/4-1}/h} + e^{-\beta x_{N/4+1}/h} \leq N^{-(p+1)}.
\]  
(3.7)

Instead, we will make the (stronger, and very common) assumption
\[
\varepsilon < N^{-1},
\]  
(3.8)

in order to be able to approximate the smooth part of the solution at the correct rate. Practically, this means we are in the singularly perturbed regime. Note that under this assumption, one has \( h_j \leq N^{-1} \) for all \( I_j \subseteq I \).
4 Error Estimates

We begin by noting that in our setting, Theorem 3.2 gives

$$\|v^{(k)} - (v_j^{(k)})\|_{\text{co}, I_j} \leq \|v^{(p+1)}\|_{\text{co}, I_j} h_j^{p+1-k}, \quad k = 0, 1, \ldots, p, \quad j = 1, \ldots, N. \quad (4.1)$$

Using the above and the definition of the exponential mesh, we establish the following lemma which will be the main tool in the analysis.

**Lemma 4.1.** Let $u_{BL}$ be given by (3.3) and let $u_{BL}^I \in V_h$ be its interpolant as in Theorem 3.2 based on the mesh $\Delta = \{x_j\}_{j=1}^N$ with nodes (3.4) obtained with the mesh generating function $\phi$ given by (3.5). Then

$$\|(u_{BL} - u_{BL}^I)^{(k)}\|_{\text{co}, I} \leq \varepsilon^{1-k} N^{-p+1-k}, \quad k = 0, 1, \ldots, p, \quad (4.2)$$

and

$$|u_{BL} - u_{BL}^I|_{2, I} \leq \varepsilon^{-1/2} N^{-p+1}. \quad (4.3)$$

**Proof.** Throughout the proof let $k \in \{0, 1, \ldots, p\}$. We first show (4.2): for $I_j \subset [0, x_{N/4-1}] \cup [x_{3N/4+1}, 1]$ we have from (4.1), (3.6) and Theorem 2.2,

$$\|(u_{BL} - u_{BL}^I)^{(k)}\|_{\text{co}, I_j} \leq h_{j}^{p+1-k} \|u_{BL}^{(p+1)}\|_{\text{co}, I_j}$$

$$\leq \left[ \frac{\varepsilon}{\beta}(p + 1)N e^{x_j/(\varepsilon(p+1))} \right]^{p+1-k} e^{-(p+1)\varepsilon \text{dist}(x, \partial I_j)/\varepsilon} \|\phi\|_{\text{co}, I_j}$$

$$\leq \varepsilon^{1-k} N^{-p+1-k} e^{x_j/\varepsilon} e^{-x_j/\varepsilon} \leq \varepsilon^{1-k} N^{-p+1-k} e^{p+1} \text{max}_j \phi' \leq \varepsilon^{1-k} N^{-p+1-k},$$

where (3.6) was used along with the fact that for this choice of $\phi$ there holds $\text{max}_j \phi' \leq N$ (cf. the proof of [5, Lemma 3.2]). For $I_j \subset \{x_{N/4-1}, x_{3N/4+1}\}$ we have, using (3.7),

$$\|(u_{BL} - u_{BL}^I)^{(k)}\|_{\text{co}, I_j} \leq \|u_{BL}^{(k)}\|_{\text{co}, I_j} \leq \varepsilon^{1-k} \left( e^{-6\sinh(1/\varepsilon)} + e^{-6\sinh(1/\varepsilon)} \right) \leq \varepsilon^{1-k} N^{-p+1}.$$

Combining the above estimates establishes (4.2). We next show (4.3): for $I_j \subset [0, x_{N/4-1}] \cup [x_{3N/4+1}, 1]$ we have (cf. again the proof of [5, Lemma 3.2])

$$|u_{BL} - u_{BL}^I|_{2, I_j} \leq h_{j}^{2p+2} \left( \right)^{1/2}$$

$$\leq h_{j}^{2p+2} \left( e^{-6 \text{dist}(x, \partial I_j)/\varepsilon} \right)^2$$

$$\leq e^{-2p+1} h_{j}^{2p+2} \varepsilon^{-2x_{j-1}/\varepsilon} - e^{-2x_{j}/\varepsilon}$$

$$\leq e^{-2p+1} \left( \frac{\varepsilon}{\beta}(p + 1)N e^{x_j/(\varepsilon(p+1))} \right)^{2p+2} \varepsilon^{-2x_{j-1}/\varepsilon} - e^{-2x_{j}/\varepsilon}$$

$$\leq e^{-1} N^{-2p+2} e^{(2p-2)x_j/(\varepsilon(p+1))} \varepsilon^{-2x_{j-1}/\varepsilon} - e^{-2x_{j}/\varepsilon}.$$
we see that
\[ e^{(2p-2)x_j/(\epsilon(p+1))}(e^{2x_j/\epsilon} - e^{2x_j/\epsilon}) \leq e^{x_j/\epsilon} e^{-2x_j/(\epsilon(p+1))} \int_{(j-1)/N}^{j/N} (-\psi'(x)) dx \leq 1. \]

Thus,
\[ |u_{BL} - u^I_{BL}|_{2, I_j} \leq e^{-1/2} N^{-p+1}. \]

Finally, for \( I_j \subset [x_{N/4-1}, x_{3N/4+1}] \), we have
\[
|u_{BL} - u^I_{BL}|_{2, I_j} \leq |u_{BL}|_{2, I_j} + |u^I_{BL}|_{2, I_j} \\
\leq \left[ e^{-1/2} e^{-\beta \text{dist}(\partial I_j)/\epsilon} \right]^{1/2} \\
\leq e^{-1/2} \left( e^{-\beta x_{N/4-1}/\epsilon} + e^{-\beta x_{3N/4+1}/\epsilon} \right) \\
\leq e^{-1/2} N^{-p+1},
\]

where (3.7) was used once more. Combining the above estimates gives (4.3).

Using the best approximation result (3.1) and Lemma 4.1, we establish the following.

**Theorem 4.2.** Let \( u \) be the solution of (2.3) and \( u_{\text{FEM}} \) its finite element approximation based on the space \( V_h \) given by (3.2) using the nodes (3.4). Then
\[ \|u - u_{\text{FEM}}\|_{E, I} \sim \|u - u_{\text{FEM}}\|_{1, I} + \epsilon \|u - u_{\text{FEM}}\|_{2, I} \leq N^{-p+1}. \]

**Proof.** With \( u = w + u_{BL} \) and \( u^I = w^I + u^I_{BL} \) its interpolant with the obvious notation, we have from (3.1)
\[ \|u - u_{\text{FEM}}\|_{E, I} \leq \|u - u^I\|_{E, I} \leq \|w - w^I\|_{E, I} + \|u_{BL} - u^I_{BL}\|_{E, I}. \]

Lemma 4.1 gives the desired result for second term above as follows:
\[ |u_{BL} - u^I_{BL}|_{E, I} \approx \epsilon |u_{BL} - u^I_{BL}|_{2, I} + \|u_{BL} - u^I_{BL}\|_{1, I} \leq e^{1/2} N^{-p+1} + N^{-p}. \]

For the first term, we use the interpolation estimate (3.6) and Theorem 2.2, to get
\[ \|w - w^I\|_{E, I} \leq \begin{cases} \epsilon h_I^{-p-1} + h_I^{-p} & \text{for } I_j \subset [x_0, x_{N/4-1}] \cup [x_{3N/4+1}, x_1], \\
\epsilon h_I^{-p-1} & \text{for } I_j \subset [x_{N/4-1}, x_{3N/4+1}], \\
\end{cases} \]

since we have \( h_I \leq N^{-1} \) for all \( I_j \subset I \) (cf. (3.8)).

**Remark 4.3.** The result of the above theorem is not considered *balanced* since
\[ \|u_{BL}\|_{E, I} = O(\epsilon^{1/2}) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0. \]

This means that, as \( \epsilon \rightarrow 0 \), the energy norm 'does not see the layer'; as a result, the method appears to perform better, with respect to this norm, as \( \epsilon \) gets smaller; see the numerical results in Section 5. A balanced norm is the following:
\[ \|u - u_{\text{FEM}}\|^2_B := \|u - u_{\text{FEM}}\|^2_{1, I} + \epsilon \|u - u_{\text{FEM}}\|^2_{2, I}, \quad (4.4) \]

since \( \|u_{BL}\|_B = O(1) \). The results of Lemma 4.1 allow us to infer robust, optimal convergence in this norm as well, as is described next.
4.1 Estimates in a Balanced Norm

We will follow the idea from [16] (see also [9] for another approach): we begin by defining the bilinear form \( B_0 : H^1_0(I) \times H^1_0(I) \to \mathbb{R} \),
\[
B_0(u, v) = \langle au', v' \rangle_I + \langle bu, v \rangle_I,
\]
corresponding to the reduced/limit problem. We also define the operator \( P_0 : H^1_0(I) \to V_h \) by the orthogonality condition
\[
B_0(u - P_0 u, v) = 0 \quad \text{for all } v \in V_h. \tag{4.5}
\]
Then, with \( u_{\text{FEM}} \in V_h \) the finite element approximation of \( u \), we have
\[
u - u_{\text{FEM}} = u - P_0 u + P_0 u - u_{\text{FEM}}
\]
and we focus on the term \( \xi := P_0 u - u_{\text{FEM}}:\)
\[
\|\xi\|_{E, I}^2 = B_0(\xi, \xi) = B_0(P_0 u - u, \xi) = \varepsilon^2 ((P_0 u - u)^{''}, (P_0 u - u_{\text{FEM}})^{''}),
\]
where Galerkin orthogonality (satisfied by \( u - u_{\text{FEM}} \)) with respect to the bilinear form \( B_0 \) and (4.5) were used. Hence
\[
|P_0 u - u_{\text{FEM}}|_{2, I} \leq |P_0 u - u|_{2, I} \tag{4.6}
\]
and the triangle inequality will allow us to infer the desired bound provided we can show
\[
|P_0 u - u|_{2, I} \leq \varepsilon^{-1/2} N^{-p+1}.
\]
This is achieved in the following.

**Lemma 4.4.** Let \( u \) be the solution to (2.4) and let \( P_0 u \in V_h \) be its projection defined by (4.5). Then
\[
|P_0 u - u|_{2, I} \leq \varepsilon^{-1/2} N^{-p+1}.
\]

**Proof.** Since \( P_0 \) is a projection, there holds \( u - P_0 u = u - u^I + P_0(u^I - u) \), where \( u^I \) is the interpolant of \( u \). Thus
\[
|u - P_0 u|_{2, I} \leq |u - u^I|_{2, I} + |u^I - P_0 u|_{2, I}
\]
and the first term above is handled by (4.3). For the second term, we first use an inverse inequality to get
\[
|P_0 u - u^I|_{2, I} \leq N|P_0 u - u^I|_{1, I}.
\]
Next, using the stability estimate
\[
\|P_0 u\|_{1, I} \leq \|u\|_{1, I} \quad \text{for all } u \in H^1_0(I),
\]
which follows by taking \( v = P_0 u \) in (4.5), we obtain
\[
|P_0 u - u^I|_{2, I} \leq N|P_0 u - u^I|_{1, I} \leq N\|u - u^I\|_{1, I} \leq N\|u - u^I\|_{\infty, I} \leq N^{-p+1},
\]
where we used Lemma 4.1. The result follows by combining the above with (4.3).

We are now in a position to state our main result.

**Theorem 4.5.** Let \( u \) be the solution of (2.3) and \( u_{\text{FEM}} \) its finite element approximation based on the space \( V_h \) given by (3.2) using the nodes (3.4). Then
\[
\|u - u_{\text{FEM}}\|_B \sim \|u - u_{\text{FEM}}\|_{1, I} + \varepsilon^{1/2}|u - u_{\text{FEM}}|_{2, I} \leq N^{-p+1}.
\]

**Proof.** By the triangle inequality
\[
\|u - u_{\text{FEM}}\|_B \leq \|u - P_0 u\|_B + \|P_0 u - u_{\text{FEM}}\|_B.
\]
By (4.6) and Lemma 4.4 we get the desired result.
5 Numerical Results

In this section we present the results of numerical computations for two model problems, one with constant data and one with variable data. We will be using the \( h \) version FEM with (piecewise) polynomials of degree \( p = 3 \) and \( 5 \), defined on the exponentially graded mesh. We will measure the percentage relative error in the energy norm, versus the number of degrees of freedom and provide log-log plots that show straight lines with slope \((-p + 1)\), thus verifying the result of Theorem 4.2. Similarly, we will plot the error in the balanced norm (4.4) versus the number of degrees of freedom, in order to illustrate the result of Theorem 4.5.

**Example 5.1.** \( b(x) = c(x) = f(x) = 1 \) and an exact solution is available making our reported results reliable. In Figure 2 we show the convergence of the method for \( p = 3, 5 \), for various values of \( \varepsilon \). The robustness and optimal rate of the method are clearly visible, as is the fact that the energy norm is not balanced (hence the method performs better as \( \varepsilon \to 0 \)). Tables 1 and 2 list the errors in these computations.

In Figure 3 we show the same results but with respect to the balanced norm (4.4). The lines, which coincide, have slope \((-p + 1)\) illustrating the result of Theorem 4.5. In Tables 3 and 4 we list the errors in the balanced norm.

We also show, in Figure 4, a comparison between the exponential mesh (indicated by eXp) and the Bakhvalov–Shishkin mesh (indicated by B-S) from the literature (see, e.g., [10]), for \( \varepsilon = 10^{-j}, j = 3, 5, 7 \) and \( p = 3 \). (Other values of these parameters gave similar results.) Both methods are parameter robust and converge at the optimal rate, as can be seen from Figure 4, with the proposed mesh performing equally well as the B-S mesh.

**Example 5.2.** \( b(x) = \exp(-x), c(x) = 0, f(x) = e^{-x^2} + 1 \), and an exact solution is not available. For the computations we use a reference solution obtained with the \( hp \) version of the FEM with high enough \( p \) as to guarantee adequate accuracy (which is achieved at an exponential rate [4]). The observations made for the previous example remain valid for this one as well, as seen in Figure 5. For \( p = 5 \), the linear system obtained for \( \varepsilon \leq 10^{-7} \) was too ill-conditioned and roundoff error made the computed solution unreliable. The errors for these computations are shown in Tables 5 and 6.

In Figure 6 and Tables 7, 8 we show the corresponding balanced norm convergence results for Example 5.2. Finally, in Figure 7, we show the comparison of the eXp mesh and the B-S mesh; the conclusions are the same as in the previous example.

6 Closing Remarks

We have shown that the finite element approximation to the solution of fourth order singularly perturbed problems, based on piecewise Hermite polynomials defined on an exponentially graded mesh, yields uniform, optimal convergence in both the energy norm and a stronger, balanced norm.

The results presented here are immediately applicable to two-dimensional fourth order singularly perturbed problems, since the boundary layer effect is one-dimensional. In [3] this is established, among other things, and for two-dimensional analogs of (2.1) posed in smooth domains \( \Omega \subset \mathbb{R}^2 \), the boundary layer is of the form

\[
u_{BL}(\rho, \theta) = S(\theta) n(\rho) e^{-b(\rho)/\varepsilon},
\]

where \((\rho, \theta)\) are boundary fitted coordinates (distance from the boundary and arclength, respectively), \(S(\theta)\) is a smooth function, \(n(\rho)\) is a polynomial and \(b \in \mathbb{R}^+\). The exponentially graded mesh may easily be constructed in the direction normal to the boundary (see Figure 8) and via tensor product arguments, similar results may be obtained. This is the focus of our current research efforts.
Figure 2. Energy norm convergence for Example 5.1. Left: $p = 3$. Right: $p = 5$.

Figure 3. Balanced norm convergence for Example 5.1. Left: $p = 3$. Right: $p = 5$.

Figure 4. Energy norm convergence for Example 5.1: Comparison between the eXp and the B-S mesh, $p = 3$. 

<table>
<thead>
<tr>
<th>$\varepsilon$ \ DOF</th>
<th>18</th>
<th>42</th>
<th>90</th>
<th>186</th>
</tr>
</thead>
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Table 1. Percentage relative error in the energy norm for Example 5.1 with $p = 3$, $\varepsilon = 10^{-j}$, $j = 3, \ldots, 8$.

<table>
<thead>
<tr>
<th>$\varepsilon$ \ DOF</th>
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<th>130</th>
<th>274</th>
<th>562</th>
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</table>

Table 2. Percentage relative error in the energy norm for Example 5.1 with $p = 5$, $\varepsilon = 10^{-j}$, $j = 3, \ldots, 8$.

<table>
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<th>18</th>
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Table 3. Estimated error in the balanced norm for Example 5.1 with $p = 3$, $\varepsilon = 10^{-j}$, $j = 3, \ldots, 8$.

<table>
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<th>$\varepsilon$ \ DOF</th>
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Table 4. Estimated error in the balanced norm for Example 5.1 with $p = 5$, $\varepsilon = 10^{-j}$, $j = 3, \ldots, 8$. 
Figure 5. Estimated energy norm convergence for Example 5.2. Left: $p = 3$. Right: $p = 5$.

Figure 6. Balanced norm convergence for Example 5.2. Left: $p = 3$. Right: $p = 5$.

Figure 7. Energy norm convergence for Example 5.2: Comparison between the eXp and the B-S mesh, $p = 3$. 
Table 5. Estimated percentage relative error in the energy norm for Example 5.2 with \( p = 3, \varepsilon = 10^{-j}, j = 3, \ldots, 8 \).

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Table 6. Estimated percentage relative error in the energy norm for Example 5.2 with \( p = 5, \varepsilon = 10^{-j}, j = 3, \ldots, 8 \).

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<tr>
<th>( \varepsilon ) \ DOF</th>
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Table 7. Estimated error in the balanced norm for Example 5.2 with \( p = 3, \varepsilon = 10^{-j}, j = 3, \ldots, 8 \).

<table>
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Table 8. Estimated error in the balanced norm for Example 5.2 with \( p = 5, \varepsilon = 10^{-j}, j = 3, \ldots, 8 \).

<table>
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<th>( \varepsilon ) \ DOF</th>
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Figure 8. Example of the exponential mesh in two dimensions.
References