HEIGHT-2 TODA SYSTEMS

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Joint work with Pantelis Damianou and Pol Vanhaecke
Let \((\mathbb{R}^{2n}, \{,\})\) be the usual Poisson manifold, and \((p_1, \ldots, p_n, q_1, \ldots, q_n)\) the corresponding canonical coordinates.

Define the Hamiltonian \(H\) by:

\[
H = \frac{1}{2} \sum_i p_i^2 + \sum_i e^{q_i - q_{i+1}}
\]

and Hamiltonian equations:

\[
\dot{q}_i = \{q_i, H\} = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = \{p_i, H\} = -\frac{\partial H}{\partial q_i}
\]

**Problem**: determine an Integrable System, relatively to \(H\), i.e. a set of \(n\) independent functions in involution which are constants of motion for Hamiltonian equations.
Using Flashka’s variables:

\[ \forall i, 1 \leq i \leq n - 1, a_i = \frac{1}{2} e^{\frac{1}{2}(q_i - q_{i+1})}, \forall i, 1 \leq i \leq n, b_i = -\frac{1}{2} p_i \]

Hamiltonian’s equations are then equivalent to the following Lax equation

\[ \dot{L} = [B, L] \]

where:

\[
L = \begin{pmatrix}
    b_1 & a_1 & 0 & \ldots & \ldots \\
    a_1 & b_2 & a_2 & 0 & \ldots \\
    0 & \ldots & \ldots & \ldots & \ldots \\
    0 & \ldots & \ldots & b_{n-1} & a_{n-1} \\
    0 & \ldots & \ldots & a_{n-1} & b_n
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
    0 & a_1 & 0 & \ldots & \ldots \\
    -a_1 & 0 & a_2 & 0 & \ldots \\
    0 & \ldots & \ldots & \ldots & \ldots \\
    0 & \ldots & \ldots & 0 & a_{n-1} \\
    0 & \ldots & \ldots & -a_{n-1} & 0
\end{pmatrix}
\]
The underlying Poisson structure on 
$L = L(a_1, \ldots, a_{n-1}, b_1, \ldots, b_n)$ is defined, up to a constant multiple, by:

$$\forall i, \{a_i, b_i\}_L = -a_i, \quad \{a_i, b_{i+1}\}_L = a_i$$  \hspace{1cm} (1)

The other brackets are zero.
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All the entries of $L$ vary over time, but the eigenvalues remain constant. It follows that the functions $H_i = \frac{1}{i} TrL^i$ are constants of motion and give us the right "Integrable system".
Given any semi-simple complex Lie algebra $\mathfrak{g}$, we introduce the following datas:

A Cartan subalgebra $\mathfrak{h}$ (i.e. a maximal abelian subalgebra composed of semi-simple elements) (ex: Diagonal matrices in $(\mathfrak{sl}_n)$).

A root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$.

A basis of simple roots $\Pi$ in $\Delta$, $\Pi = \{\alpha_1, ..., \alpha_l\}$, $l = \dim \mathfrak{h} = \text{Rk}\mathfrak{g}$.

$\forall \alpha \in \Delta, \alpha = \sum n_i \alpha_i$, $n_i \in \mathbb{Z}$, $n_i \geq 0$, $\forall i$, or $n_i \leq 0$, $\forall i$.

$\Delta = \Delta^+ \cup \Delta^-$.
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- $\forall \alpha \in \Delta$, $\alpha = \sum_j n_j \alpha_j$, $n_j \in \mathbb{Z}$, $n_j \geq 0$, $\forall i$, or $n_j \leq 0$, $\forall i$.
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If \( \alpha \in \Delta \), let \( g_\alpha = \{ X \in g \mid \forall H \in \mathfrak{h}, [H, X] = \alpha(H)X \} \) be the corresponding one dimensional root-space.

Then, \( g = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} g_\alpha \).
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• Then, $g = \mathfrak{h} \oplus \sum_{\alpha \in \Delta} g_\alpha$.

• Each positive root $\alpha$ leads to a triple $(X_\alpha, X_{-\alpha}, H_\alpha)$ of root-vectors which generate a subalgebra isomorphic to $sl_2$. The set $(X_\alpha, X_{-\alpha}, \alpha \in \Delta^+, H_\alpha, \alpha \in \Pi)$ is a basis of $g$. 
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$n^+ = \sum_{\alpha \in \Delta^+} \mathbb{C} X_\alpha$, $n^- = \sum_{\alpha \in \Delta^-} \mathbb{C} X_{-\alpha}$.

$b^+ = \mathfrak{h} \oplus n^+$, $b^- = \mathfrak{h} \oplus n^-$ are Borel subalgebras of $g$. 
Lie-Poisson Structure

Let $\mathfrak{g}$ be any complex Lie algebra. The Lie-Poisson structure on its dual $\mathfrak{g}^*$ is defined as follows:

$$\forall F, G \in \mathcal{F}(\mathfrak{g}^*), \forall \mu \in \mathfrak{g}^*, \{ F, G \}(\mu) = \mu([d_\mu F, d_\mu G])$$
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- The Poisson-Rank of $\mathfrak{g}^*$, $Prk(\mathfrak{g}^*)$, is then the maximal dimension of co-adjoint orbits.
- Define the *Index* of $\mathfrak{g}$ to be the lowest dimension of stabilizers in $\mathfrak{g}$ of elements of $\mathfrak{g}^*$, denoted by $Ind(\mathfrak{g})$. Then,

$$Prk(\mathfrak{g}^*) = \dim \mathfrak{g} - Ind(\mathfrak{g})$$
Let \( g \) be any complex Lie algebra. The Lie-Poisson structure on its dual \( g^* \) is defined as follows:

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- The Poisson-Rank of \( g^* \), \( Prk(g^*) \), is then the maximal dimension of co-adjoint orbits.
- Define the Index of \( g \) to be the lowest dimension of stabilizers in \( g \) of elements of \( g^* \), denoted by \( Ind(g) \). Then,

\[
Prk(g^*) = \dim g - Ind(g)
\]

Suppose that \( g \) is semi-simple. We can identify \( g \) with its dual, via the Killing form \( \langle X, Y \rangle = Tr(adX.adY) \). The Lie-Poisson structure on \( g \) is then given by:

\[
\forall F, G \in \mathcal{F}(g), \forall X \in g, \{ F, G \}(X) = \langle X, [d_\mu F, d_\mu G] \rangle
\]
The Lax pair \((L, B)\) in \(\mathfrak{g}\) can be described as follows, using the root-system \(\Delta\):

\[
L = \sum_{i=1}^{i=l} b_i H_{\alpha_i} + \sum_{i=1}^{i=l} a_i (X_{\alpha_i} + X_{-\alpha_i})
\]

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B = \sum_{i=1}^{i=l} a_i (X_{\alpha_i} - X_{-\alpha_i})
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\]

Generalizing this procedure, we can introduce the pair \((L_\phi, B_\phi)\):

\[
L_\phi = \sum_{\alpha \in \phi} b_\alpha H_\alpha + \sum_{\alpha \in \phi} a_\alpha (X_\alpha + X_{-\alpha})
\]

\[
B_\phi = \sum_{\alpha \in \phi} a_\alpha (X_\alpha - X_{-\alpha})
\]

Where \(\phi\) is any subset of \(\Delta^+\), containing \(\Pi\).
In order to get similar Lax equation, the bracket \([L_\phi, B_\phi]\) should give an element of the same form than \(L_\phi\). In that case, we say that \(\phi\) is *adapted*. 
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Let \(\mathcal{I}\) be the set of nilpotent ideals of \(\mathfrak{b}^+\). We know that:

- If \(I \in \mathcal{I}\), \(I \subset \mathfrak{n}^+\).
- \(I\) is generated by \(<X_\alpha, \alpha \in \Delta_I>\), where \(\Delta_I\) is some subset of \(\Delta^+\).
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**Proposition**: \(\phi\) is adapted if and only if the subspace \(I_\phi\) generated by \(\langle X_\alpha, \alpha \in \Delta^+ \setminus \phi \rangle\) is a nilpotent ideal of \(\mathfrak{b}^+\).
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**Proposition**: $\phi$ is adapted if and only if the subspace $I_\phi$ generated by $\langle X_\alpha, \alpha \in \Delta^+ \setminus \phi \rangle$ is a nilpotent ideal of $\mathfrak{b}^+$.

Thus, for such $\phi$, we obtain an Hamiltonian system; the problem is to study this new system and determine whether it is integrable.
\[ \phi = \Pi, \quad I_\phi = \langle X_\alpha, \alpha \in \Delta^+ \setminus \Pi \rangle, \text{ Classical Toda lattice.} \]

\[ \phi = \Delta^+, \quad I_\phi = \{0\}, \text{ Full "symmetric" Toda lattice} \]

"Intermediate" Toda lattices.
Kostant form

By conjugation,

\[
L = \begin{pmatrix}
  b_1 & a_1 & 0 & \ldots & \ldots \\
  a_1 & b_2 & a_2 & 0 & \ldots \\
  0 & \ldots & \ldots & \ldots & \ldots \\
  0 & \ldots & \ldots & b_{n-1} & a_{n-1} \\
  0 & \ldots & \ldots & a_{n-1} & b_n
\end{pmatrix}
\rightarrow
\begin{pmatrix}
  b_1 & 1 & 0 & \ldots & \ldots \\
  a_1 & b_2 & 1 & 0 & \ldots \\
  0 & \ldots & \ldots & \ldots & \ldots \\
  0 & \ldots & \ldots & b_{n-1} & 1 \\
  0 & \ldots & \ldots & a_{n-1} & b_n
\end{pmatrix}
\]

The latter matrix is the "Kostant form" of the phase space of the classical Toda.
Kostant form

By conjugation,

\[ L = \begin{pmatrix} b_1 & a_1 & 0 & \ldots & \ldots \\ a_1 & b_2 & a_2 & 0 & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & b_{n-1} & a_{n-1} \\ 0 & \ldots & a_{n-1} & b_n \end{pmatrix} \rightarrow \begin{pmatrix} b_1 & 1 & 0 & \ldots & \ldots \\ a_1 & b_2 & 1 & 0 & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & b_{n-1} & 1 \\ 0 & \ldots & a_{n-1} & b_n \end{pmatrix} \]

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- Lax equation : \( \dot{X} = [p_-(X), X] \), where \( p_- \) is the projection on the strictly lower triangular part of \( X \).
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- Lax equation : \( \dot{X} = [p_-(X), X] \), where \( p_- \) is the projection on the strictly lower triangular part of \( X \).

- Hamiltonian : \( H = \frac{1}{2} \text{Tr}(X^2) \).
\[ \varepsilon = \sum_{i=1}^{l} X_{\alpha_i}, \]

\[ M = \varepsilon + b^- \simeq (b^+)^* \]

\[ (\varepsilon + X \rightarrow \phi_X = \langle X, \cdot \rangle) \]

\[ \forall I \in \mathbb{I}, M_I = \{ X \in M | \langle X, I \rangle \geq 0 \} = \varepsilon + I^\perp \simeq (b^+/I)^*. \]
The Poisson structure on $M$ is defined by using the theory of $R$-matrices.
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$\mathfrak{g} = \mathfrak{b}^+ \oplus \mathfrak{n}^-, \quad P^+: \mathfrak{g} \longrightarrow \mathfrak{b}^+$ canonical projection.

$\forall F, G \in \mathcal{F}(M), \forall X \in M,$

$$\{F, G\}(X) = \langle X, [P^+(d_X \tilde{F}), P^+(d_X \tilde{G})] \rangle$$

where $\tilde{F}, \tilde{G}$ are arbitrary extensions of $F, G$ to $\mathfrak{g}$. 
Poisson structure on $M$ and $M_I$.

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- $\forall I \in \mathbb{I}, M_I$ is a Poisson submanifold of $M$. 

Poisson rank on $M$ and $M_I$

$$Prk(M_I) = \dim \mathfrak{b}^+ / I - \text{Ind}(\mathfrak{b}^+ / I)$$
Poisson rank on $M$ and $M_I$ 

$$Prk(M_I) = \dim \mathfrak{b}^+/I - Ind(\mathfrak{b}^+/I)$$

We compute the index, using \textit{stable forms}.

\textbf{Definition :} \textit{Let $\mathfrak{a}$ be a complex algebraic Lie algebra, $A$ its corresponding adjoint Lie group. A linear form $f \in \mathfrak{a}^*$ is said to be stable if there exists a neighborhood $V$ of $f$ in $\mathfrak{a}^*$ such that, $\forall \varphi \in V$, the stabilizers of $f$ and $\varphi$ in $\mathfrak{a}$ are $A$-conjugate.}
Poisson rank on $M$ and $M_I$

$$Prk(M_I) = \dim b^+ / I - \text{Ind}(b^+ / I)$$

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$f$ stable $\implies$ $f$ regular
Poisson rank on $M$ and $M_I$

\[ \Prk(M_I) = \dim \mathfrak{b}^+ / I - \Ind(\mathfrak{b}^+ / I) \]

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\[ f \text{ stable} \implies f \text{ regular} \]

\[ \Ind(\mathfrak{a}) = \dim \mathfrak{a} - \dim A.f, f \text{ stable} \]
Using Kostant’s "Cascade Method", **P. Tauvel** and **R. Yu** (2005) get an explicit stable form on $\mathfrak{b}^+$, which leads to the formula:

$$\text{Ind}(\mathfrak{b}^+) = rk(\mathfrak{g}) - \kappa_g$$

**Question**: Stable forms on $\mathfrak{b}^+/I$, $I \in \mathfrak{I}$?
From now, assume that each positive root of height 2 is of the form $\alpha_k + \alpha_k + 1$, $1 \leq k \leq l - 1$.

Let $S = \{\Pi_1, \Pi_1', \Pi_2, \Pi_2', \ldots, \Pi_k, \Pi_k'\}$ be an "ordered" partition of $\Pi$, composed of subsets of consecutive simple roots.

$\Pi_j = \{\alpha_{i_j}, \ldots, \alpha_{i_j + l_j - 1}\}$

$E_S = \{\alpha = \alpha_{i} + \alpha_{i + 1} \in \Pi_j, 1 \leq j \leq k\}$

Let $IS$ be the nilpotent ideal defined by:

$\langle X_\alpha \mid \text{ht}(\alpha) \geq 3 \text{ or } \text{ht}(\alpha) = 2 \text{ and } \alpha \not\in E_S \rangle$

$MS = MI_S$ is the corresponding Height-2-Toda manifold.
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- Let $S = \{\Pi_1, \Pi'_1, \Pi_2, \Pi'_2, \ldots, \Pi_k, \Pi'_k\}$ be an "ordered" partition of $\Pi$, composed of subsets of consecutive simple roots.
- $\Pi_j = \{\alpha_{i_j}, \ldots, \alpha_{i_j+l_j-1}\}$
- $E_S = \{\alpha = \alpha_{i_j} + \alpha_{i_j+1}, \alpha_i, \alpha_{i+1} \in \Pi_j, 1 \leq j \leq k\}$
- Let $I_S$ be the nilpotent ideal defined by:
  $$I_S = \langle X_\alpha \mid ht(\alpha) \geq 3 \text{ or } ht(\alpha) = 2 \text{ and } \alpha \notin E_S \rangle$$
- $M_S = M_{I_S}$ is the corresponding Height-2-Toda manifold.
The Poisson rank of a Height-2-Toda manifold is obtained by computing an explicit stable form:
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Let $X_j, X'_j \in b^-, f_j, f'_j \in (b^+)^*$ defined by

$$
X_j = \varepsilon_j X_{-\alpha_i} + \sum_{i=i_j}^{i_j+l_j-2} X_{-\alpha_i-\alpha_{i+1}}, \text{ where } \varepsilon_j = \frac{1 - (-1)^l_j}{2}
$$

\forall Z \in b^+, f_j(Z) = \langle X_j, Z \rangle

$$
X_j' = \sum_{\alpha \in \Pi'_j} X_{-\alpha}
$$

\forall Z \in b^+, f'_j(Z) = \langle X'_j, Z \rangle
Let $f_S = \sum_{j=1}^{k} (f_j + f'_j)$.

**Proposition:**
- $f_S$ is a stable form on $b^+/I_S$.
- $\text{Ind}(b^+/I_S) = \sum_{j=1}^{k} \frac{1 + (-1)^{l_j}}{2}$
- $\text{Prk}(M_S) = 2l + \sum_{j=1}^{k} l_j - 1 - \frac{1 + (-1)^{l_j}}{2}$. 

Integrability for the Full-Toda ($A_n$-case)

$g = \mathfrak{sl}_n, \ G = SL_n, \ b^+ = n \times n$ upper triangular matrices

$$\dim M = \frac{n(n+1)}{2}, \ Ind(b) = \left[ \frac{n-1}{2} \right], \ Prk(M) = \frac{n(n+1)}{2} - \left[ \frac{n-1}{2} \right]$$
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$$\dim M = \frac{n(n+1)}{2}, \text{Ind}(b) = \left\lfloor \frac{n-1}{2} \right\rfloor, \text{Prk}(M) = \frac{n(n+1)}{2} - \left\lfloor \frac{n-1}{2} \right\rfloor$$

Number of functions required : $\dim M - 1/2\text{Prk}(M) = \left\lfloor \left(\frac{n+1}{2}\right)^2 \right\rfloor$. 
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Number of functions required : \( \dim M - \frac{1}{2} Prk(M) = \left( \frac{n+1}{2} \right)^2 \).

The functions \( H_i = 1/i \text{Tr}X^i \) are still in involution, but not enough to ensure integrability. We need more functions.
let $k$ be an integer, $0 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Let $X \in \mathfrak{g}$. Define $X_k$ to be the matrix obtained by removing the first $k$ rows and the last $k$ columns from $X$. Let $G_k$ be the (Parabolic) subgroup of $G$ consisting of matrices of the form:

$$
\begin{pmatrix}
\Delta & A & B \\
0 & D & C \\
0 & 0 & \Delta'
\end{pmatrix}
$$

where $\Delta, \Delta'$ are upper triangular matrices of size $k \times k$, and $A, B, C, D$ arbitrary.
let $k$ be an integer, $0 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. Let $X \in g$. Define $X_k$ to be the matrix obtained by removing the first $k$ rows and the last $k$ columns from $X$.

Let $G_k$ be the (Parabolic) subgroup of $G$ consisting of matrices of the form:

$$
\begin{pmatrix}
\Delta & A & B \\
0 & D & C \\
0 & 0 & \Delta'
\end{pmatrix}
$$

where $\Delta, \Delta'$ are upper triangular matrices of size $k \times k$, and $A, B, C, D$ arbitrary.

**Proposition:** $\forall X \in g, \forall g \in G_k,$

$$
\det(gXg^{-1})_k = \frac{\det \Delta}{\det \Delta'} \det X_k
$$
Define, $\forall X \in \mathfrak{g}, \forall \lambda \in \mathbb{C},$

$$Q_k(X, \lambda) = \det(X - \lambda I d_n)_k = \sum_{i=0}^{i=n-2k} E_{i,k} \lambda^{n-2k-i}$$
Define, $\forall X \in \mathfrak{g}, \forall \lambda \in \mathbb{C}$,

$$Q_k(X, \lambda) = \det(X - \lambda \text{Id}_n)_k = \sum_{i=0}^{i=n-2k} E_{i,k} \lambda^{n-2k-i}$$

All $E_{i,k}$’s are $G_k$-semi-invariant, with the same character.
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All $E_{i,k}$’s are $G_k$-semi-invariant, with the same character.

Set 

$$\forall (i, k), 0 \leq i \leq n - 2k, 0 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, H_{i,k} = \frac{E_{i,k}}{E_{0,k}}$$

The $(H_{i,k})$’s are $G_k$-invariant functions := $k$-chop integrals.

$$H = 1/2(H_{1,0}^2 - 2H_{2,0})$$
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$$\forall (i, k), 0 \leq i \leq n - 2k, 0 \leq k \leq \left[\frac{n-1}{2}\right], H_{i,k} = \frac{E_{i,k}}{E_{0,k}}$$

The $(H_{i,k})$’s are $G_k$-invariant functions := $k$-chop integrals.

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Number of $k$-chop integrals : $\left[\left(\frac{n+1}{2}\right)^2\right]$. 
Define, \( \forall X \in \mathfrak{g}, \forall \lambda \in \mathbb{C}, \)

\[
Q_k(X, \lambda) = \det(X - \lambda \text{id}_n)_k = \sum_{i=0}^{i=n-2k} E_{i,k} \lambda^{n-2k-i}
\]

All \( E_{i,k} \)'s are \( G_k \)-semi-invariant, with the same character.

Set

\[
\forall (i, k), 0 \leq i \leq n - 2k, 0 \leq k \leq \left\lfloor \frac{n-1}{2} \right\rfloor, H_{i,k} = \frac{E_{i,k}}{E_{0,k}}
\]

The \( (H_{i,k})'s \) are \( G_k \)-invariant functions := \( k \)-chop integrals.

\[
H = 1/2(H_{1,0}^2 - 2H_{2,0})
\]

Number of \( k \)-chop integrals : \( \left\lfloor (\frac{n+1}{2})^2 \right\rfloor \).

**Theorem**: The chop integrals \( (H_{i,k}) \) form an integrable system.
Integrability of Height-2-Toda: $A_n$ case.

- $S = \{\Pi_1, \Pi'_1, \Pi_2, \Pi'_2, \ldots, \Pi_k, \Pi'_k\}$
- $\#\Pi_j = l_j$, $\#\Pi'_j = l'_j$.

$$X_S = \begin{pmatrix} b_1 & 1 & 0 & \ldots & \ldots \\ a_1 & b_2 & 1 & 0 & \ldots \\ c_1 & a_2 & b_3 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & 1 \\ 0 & \ldots & c_{n-2} & a_{n-1} & b_n \end{pmatrix}$$
There are only 0-chops and 1-chops.
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- 0-chop integrals: \( H_i = \frac{1}{i} \text{Tr}(X^i) \), \( 2 \leq i \leq n \).
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- 0-chop integrals: \( H_i = \frac{1}{i} \text{Tr}(X^i), \ 2 \leq i \leq n \).

- 1-chop integrals.

Define \( Q_S(X, \lambda) = \det(X_S - \lambda I_n)_1 \).

\[
(X_S - \lambda I)_1 = \\
\begin{pmatrix}
M_1(\lambda) & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & M'_1(\lambda) & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & M_2(\lambda) & \cdots & \cdots & \cdots \\
& & & & & \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & \cdots & \cdots & M'_k(\lambda)
\end{pmatrix}
\]

\( D_i(\lambda) = \det M_i(\lambda), \ D'_i = \det M'_i(\lambda) \)
- \( M_i(\lambda) \) (resp. \( M'_i(\lambda) \)) is a square matrix of size \( l_i \) (resp. \( l'_i \)).

- \( D_i(\lambda) \) is a polynomial in \( \lambda \), \( D'_i(\lambda) \) is independent of \( \lambda \).

- \( Q_S(X, \lambda) = \prod_{i=1}^{k} D_i(\lambda)D'_i(\lambda) \).
• $M_i(\lambda)$ (resp. $M'_i(\lambda)$) is a square matrix of size $l_i$ (resp. $l'_i$).

• $D_i(\lambda)$ is a polynomial in $\lambda$, $D'_i(\lambda)$ is independent of $\lambda$.

• $Q_S(X, \lambda) = \prod_{i=1}^{k} D_i(\lambda) D'_i(\lambda)$.

Proposition :

(i) $\forall i$, $D_i(\lambda)$ is a polynomial of degree $d_i = \left[\frac{l_i}{2}\right]$.

(ii) Let $D_i(\lambda) = \sum_{j=1}^{d_i} I_{i,j} \lambda^j$, $H_{i,j} = \frac{I_{i,j}}{I_{i,d_i}}$, $0 \leq j \leq d_i - 1$.

Then, the functions $(H_{i,j}, 1 \leq i \leq k, 0 \leq j \leq d_i - 1, H_i, 1 \leq i \leq n)$ are constants of motion in involution.

(iii) If $l_i$ is odd, $(H_{i,d_i-1})$ is a Casimir.
• $M_i(\lambda)$ (resp. $M'_i(\lambda)$) is a square matrix of size $l_i$ (resp. $l'_i$).

• $D_i(\lambda)$ is a polynomial in $\lambda$, $D'_i(\lambda)$ is independent of $\lambda$.

• $Q_S(X, \lambda) = \prod_{i=1}^{k} D_i(\lambda)D'_i(\lambda)$.

Proposition :
(i) $\forall i, D_i(\lambda)$ is a polynomial of degree $d_i = \left\lfloor \frac{l_i}{2} \right\rfloor$.

(ii) Let $D_i(\lambda) = \sum_{j=1}^{d_i} l_{i,j} \lambda^j$, $H_{i,j} = \frac{l_{i,j}}{l_{i,d_i}}, 0 \leq j \leq d_{i-1}$

Then, the functions $(H_{i,j}, 1 \leq i \leq k, 0 \leq j \leq d_i - 1, H_i, 1 \leq i \leq n)$ are constants of motion in involution.

(iii) If $l_i$ is odd, $(H_{i,d_i-1})$ is a Casimir.

This gives the right number of functions in involution and the exact number of Casimirs.
Next step (work in progress) : Generalize this procedure to other classical Lie algebras ($B_n, C_n, D_n$).

Last step : All the functions ($H_{i,j}, H_i$) are independent.
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Last step : All the functions ($H_{i,j}$, $H_i$) are independent.

Conjecture : *For any classical Lie algebra, the corresponding set of functions ($H_{i,j}$, $H_i$) form an integrable system.*