On recursion operators for elliptic integrable models

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1based on joint work with V V Sokolov, Landau Institute for Theoretical Physics.
Some basics about recursion operators

One of the main algebraic structures related to 1 + 1-dimensional integrable PDE of the form

\[ u_t = F(u, u_x, \ldots, u_n), \quad n \geq 2, \quad u_i = \partial_x^i u(x, t) \quad (1.1) \]

is an infinite hierarchy of commuting flows or, the same, (generalized) symmetries

\[ u_{t_i} = G_i(u, \ldots, u_{m_i}). \quad (1.2) \]

We identify the symmetry (1.2) with its right hand side \( G_i \).
Some basics about recursion operators

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is an infinite hierarchy of commuting flows or, the same, (generalized) symmetries
\[ u_{t_i} = G_i(u, \ldots, u_{m_i}) . \quad (1.2) \]
We identify the symmetry (1.2) with its right hand side \( G_i \). The symmetry \( G \) satisfies the equation
\[ D_t(G) - F_*(G) = 0, \quad (1.3) \]
where \( D_t \) stands for the evolution derivative in virtue of (1.1) and \( F_* \) denotes the Fréchet derivative of \( F \):
\[ F_* = \sum_{i=0}^{n} \frac{\partial^i F}{\partial u_i} D_x^i . \]
Some basics about recursion operators.

**Definition**

Linear (pseudo-)differential operator $\mathcal{R}$ is said to be a recursion operator for (1.1) if it satisfies the operator equation

$$[D_t - F_*, \mathcal{R}] = \mathcal{R}_t - [F_*, \mathcal{R}] = 0.$$  \hspace{1cm} (1.4)
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The usual way to get higher commuting flows of (1.1) is to act to $u_x$ by a recursion operator.
The structure of Recursion operators

Most of known recursion operators have the following special form

\[ \mathcal{R} = R + \sum_{i=1}^{k} G_i D_x^{-1} g_i, \]  

(1.5)

where \( R \) is a differential operator, \( G_i \) and \( g_i \) are some fixed symmetries and cosymmetries.
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where \( R \) is a differential operator, \( G_i \) and \( g_i \) are some fixed symmetries and cosymmetries. For all known examples the cosymmetries \( g_i \) are variational derivatives of conserved densities.
Hamiltonian and recursion operators

Most of known integrable evolution equations can be written in a Hamiltonian form

$$u_t = \mathcal{H}\left(\frac{\delta \rho}{\delta u}\right),$$

where $\rho$ is a conserved density and $\mathcal{H}$ is a Hamiltonian operator.
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where \( \rho \) is a conserved density and \( \mathcal{H} \) is a Hamiltonian operator. It is known that this operator satisfies the equation

\[ (D_t - F_*) \mathcal{H} = \mathcal{H}(D_t + F^*_t), \quad (1.6) \]

which means that \( \mathcal{H} \) takes cosymmetries to symmetries.
Besides (1.6) the Hamiltonian operator should satisfy certain identities equivalent to the skew-symmetricity and the Jacobi identity for the corresponding Poisson bracket. It is easy to see that the ratio $\mathcal{H}_2\mathcal{H}_1^{-1}$ of any two Hamiltonian operators is a recursion operator.
Besides (1.6) the Hamiltonian operator should satisfy certain identities equivalent to the skew-symmetricity and the Jacobi identity for the corresponding Poisson bracket. It is easy to see that the ratio $\mathcal{H}_2 \mathcal{H}_1^{-1}$ of any two Hamiltonian operators is a recursion operator. As the rule, the Hamiltonian operators are local or quasilocal operators:

$$\mathcal{H} = H + \sum_{i=1}^{m} G_i D_x^{-1} \bar{G}_i,$$

(1.7)

where $H$ is a differential operator and $G_i$, $\bar{G}_i$ are fixed symmetries.
The KdV equation

For the Korteweg-de Vries equation

\[ u_t = u_{xxx} + 6 u u_x \]

the simplest recursion operator

\[ \mathcal{R} = D_x^2 + 4u + 2u_x D_x^{-1} \] \hspace{1cm} (1.8)

is quasilocal with \( k = 1 \), \( G_1 = 2u_x \), and \( g_1 = 1 \).
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is quasilocal with \( k = 1, \; G_1 = 2u_x, \; \text{and} \; g_1 = 1. \) This operator is the ratio of two local Hamiltonian operators

\[ \mathcal{H}_1 = D_x, \quad \mathcal{H}_2 = D_x^3 + 4uD_x + 2u_x. \]
For the KdV equation the associative algebra of all quasilocal recursion operators is generated by operator (1.8). In other words, this algebra is isomorphic to the algebra of all polynomials in one variable.
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**Observation**

The above claim is not true for such integrable models as the Krichever-Novikov and the Landau-Lifshitz equations. It turns out that for these models, known to be elliptic, the algebra of recursion operators is isomorphic to the coordinate ring of the elliptic curve.
Recursion operators and multiplicators

The Landau-Lifshitz equation

\[ \mathbf{U}_t = \mathbf{U} \times \mathbf{U}_{xx} + \mathbf{U} \times J\mathbf{U}, \quad (2.9) \]

where \( \mathbf{U} = (u_1, u_2, u_3) \), \( |\mathbf{U}| = 1 \), symbol \( \times \) stands for the vector product, and \( J = \text{diag}(p, q, r) \) is an arbitrary constant diagonal matrix.
Underlying algebraic structure of L-L equation

Let

\[
    e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
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be the standard basis in the Lie algebra \( so(3) \).
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be the standard basis in the Lie algebra so(3). The Lax operator \( L \) for (2.9) is given by

\[ L = D_x - \sum_{j=1}^{3} u_i E_i, \tag{2.10} \]

where

\[ E_1 = \frac{1}{\lambda} e_1 \sqrt{1 - p\lambda^2}, \quad E_2 = \frac{1}{\lambda} e_2 \sqrt{1 - q\lambda^2}, \quad E_3 = \frac{1}{\lambda} e_3 \sqrt{1 - r\lambda^2}. \]
The operators $A_i$ defining the Lax representations

$$L_{t_i} = [A_i, L]$$

(2.11)

for the commuting flows

$$U_{t_i} = H^{(i)}(U, U_x, \ldots)$$

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from the Landau-Lifshitz hierarchy belong to the Lie algebra $\mathcal{G}$
generated by $E_1, E_2, E_3$. One can check that the algebra $\mathcal{G}$ is
goinged by

$$\frac{1}{\lambda^2} E_1, \quad \frac{1}{\lambda^2} E_2, \quad \frac{1}{\lambda^2} E_3, \quad \frac{1}{\lambda^2} \bar{E}_1, \quad \frac{1}{\lambda^2} \bar{E}_2, \quad \frac{1}{\lambda^2} \bar{E}_3,$$

(2.12)

where

$$\bar{E}_1 = \frac{1}{\lambda^2} e_1 \sqrt{1 - q\lambda^2} \sqrt{1 - r\lambda^2}, \quad \bar{E}_2 = \frac{1}{\lambda^2} e_2 \sqrt{1 - p\lambda^2} \sqrt{1 - r\lambda^2},$$

$$\bar{E}_3 = \frac{1}{\lambda^2} e_3 \sqrt{1 - p\lambda^2} \sqrt{1 - q\lambda^2}.$$
**Multiplicators**

**Definition.**

A (scalar) function $\mu(\lambda)$ is called the *multiplicator* for the algebra $G$ if $\mu(\lambda) G \subset G$. The order of pole of $\mu(\lambda)$ at $\lambda = 0$ is called the order of the multiplicator.
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**Lemma**
The set of all multiplicators for $\mathcal{G}$ coincides with the polynomial ring generated by $1$, 

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\mu_1(\lambda) = \frac{1}{\lambda^2}, \quad \mu_2(\lambda) = \frac{\sqrt{1 - p\lambda^2} \sqrt{1 - q\lambda^2} \sqrt{1 - r\lambda^2}}{\lambda^3}.
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It is clear that

\[
\mu_2^2 = (\mu_1 - p)(\mu_1 - q)(\mu_1 - r).
\]

Thus the ring of multiplicators is isomorphic to the coordinate ring of an elliptic curve.
How it works

Let $\mu$ be a multiplicator of order $k > 0$. To find a relation between operators $A_n$ and $A_{n+k}$ we use the following anzats

$$A_{n+k} = \mu A_n + R_n, \quad R_n \in \mathcal{G}, \quad \text{ord } R_n < k.$$
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Substituting this into Lax equation, we get

$$\sum_{j=1}^{3} H_{j}^{(n+k)} E_{j} = \mu \sum_{j=1}^{3} H_{j}^{(n)} E_{j} + \left[ \sum_{j=1}^{3} u_{j} E_{j}, R_{n} \right] - \frac{dR_{n}}{dx}. \quad (2.13)$$
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Both sides of this relation belong to $\mathcal{G}$. Equating in (2.13) the coefficients of basis elements (2.12), we find step by step unknown coefficients of $R_n$ and eventually an expression for $H_{j}^{(n+k)}$ in terms of $H_{j}^{(n)}$ i.e. a recursion operator of order $k$. 
The first recursion operator of the L-L equation

For the simplest multiplicator $\mu_1 = \lambda^{-2}$, we have

$$R_n = \sum_{j=1}^{3} M_j E_j + \sum_{j=1}^{3} F_j \bar{E}_j.$$
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$$R_n = \sum_{j=1}^{3} M_j E_j + \sum_{j=1}^{3} F_j \bar{E}_j.$$ 

It is easy to verify that (2.13) is equivalent to the identities:

1. $H^{(n)} - F \times U = 0,$
2. $F_x + U \times M = 0,$
3. $H^{(n+2)} = M_x + F \times JU,$

where $M = (M_1, M_2, M_3), F = (F_1, F_2, F_3).$
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where \( M = (M_1, M_2, M_3) \), \( F = (F_1, F_2, F_3) \). From the first identity and the condition \( |U| = 1 \) we get

\[
F = U \times H^{(n)} + f U.
\]
The first recursion operator of the L-L equation

To find $f$ we note that the second identity implies $(U, F_x) = 0$. Substituting the expression (2.15) to this relation, we get

$$f = D_x^{-1}(U, H^{(n)} \times U_x).$$
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The second identity can be rewritten as

$$M = U \times F_x + mU = U(U, H_x^{(n)}) - H_x^{(n)} + fU \times U_x + mU$$

for some $m$. 
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To find $f$ we note that the second identity implies $(U, F_x) = 0$. Substituting the expression (2.15) to this relation, we get

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for some $m$. To find $m$, we substitute into the third identity the latter expression for $M$ and take the scalar product by $U$ of both sides of the relation thus obtained.
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for some $m$. To find $m$, we substitute into the third identity the latter expression for $M$ and take the scalar product by $U$ of both sides of the relation thus obtained. Taking into account that

$$(H^{(n+2)}, U) = 0,$$

we get

$$m = D_x^{-1}\left((JU, H^{(n)}) - (U_x, H_x^{(n)})\right).$$
The first recursion operator of the L-L equation

Finally, the third identity produces

\[ H^{(n+2)} = (U, H_x^{(n)})U - H_x^{(n)} - (U, U_x \times H^{(n)})U \times U_x \]

\[ + (U, JU)H^{(n)} - (U \times U_{xx} + U \times JU)D_x^{-1}(U, U_x \times H^{(n)}) \]

\[ - U_x D_x^{-1} \left( (U_x, H_x^{(n)}) - (JU, H^{(n)}) \right) + (U, H_x^{(n)})U_x. \]
The second recursion operator of the L-L equation

For the second multiplicator $\mu_2$ we set

\[
R_n = \sum_{j=1}^{3} K_j E_j + \sum_{j=1}^{3} M_j \tilde{E}_j + \frac{1}{\lambda^2} \sum_{j=1}^{3} F_j E_j. \tag{2.16}
\]
The second recursion operator of the L-L equation

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Upon the substitution of (2.16) to (2.13) we get the following identities

1. $H^{(n)} = F \times U$,
2. $F_x = M \times U$,
3. $M_x - JH^{(n)} = K \times U$,
4. $H^{(n+3)} = K_x + M \times JU$.  \hspace{1cm} (2.17)
The second recursion operator of the L-L equation

The first two relations in (2.17) are analogous to (2.14), and therefore

\[ F = U \times H^{(n)} + fU, \quad f = D_x^{-1}(U, H^{(n)} \times U_x), \]

\[ M = (U, H_x^{(n)})U - H_x^{(n)} + fU \times U_x + mU. \]
The second recursion operator of the L-L equation

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\[ M = (U, H_x^{(n)})U - H_x^{(n)} + f U \times U_x + mU. \]

It follows from (2.17)₃ that

\[ K = U \times (M_x - JH^{(n)}) + \kappa U = (U, H_x^{(n)})U \times U_x \]

\[ -U \times JH^{(n)} - U \times H_x^{(n)} - U_x(U, H^{(n)} \times U_x) \]

\[ -((U_x, U_x)U + U_{xx})f + mU \times U_x + \kappa U. \]
The second recursion operator of the L-L equation

Functions $m$ and $\kappa$ can be found from the conditions

$$(\mathbf{U}, \mathbf{M}_x - J\mathbf{H}^{(n)}) = 0, \quad (\mathbf{U}, \mathbf{H}^{(n+3)}) = 0,$$

which yield

$$m = D_x^{-1}((\mathbf{U}, J\mathbf{H}^{(n)}) - (\mathbf{U}_x, \mathbf{H}_x^{(n)})),
\kappa = D_x^{-1}((\mathbf{U}, \mathbf{U}_x \times J\mathbf{H}^{(n)}) - (\mathbf{U}, \mathbf{H}_x^{(n)} \times \mathbf{U}_x)
- \frac{1}{2}(\mathbf{U}, \mathbf{H}^{(n)} \times \mathbf{U}_x)((\mathbf{U}, J\mathbf{U}) + (\mathbf{U}_x, \mathbf{U}_x))
+ (\mathbf{U}, \mathbf{H}_x^{(n)} \times J\mathbf{U})) - \frac{1}{2}((\mathbf{U}_x, \mathbf{U}_x) - (\mathbf{U}, J\mathbf{U}))D_x^{-1}(\mathbf{U}, \mathbf{H}^{(n)} \times \mathbf{U}_x).$$
The second recursion operator of the L-L equation

\[ H^{(n+3)} = H^{(n)}_{xxx} \times U + H^{(n)}_{xx} \times U_x + (H^{(n)}_{xx}, U)U \times U_x \\
- (U, H^{(n)}_{xx} \times U_x)U + (H^{(n)}_{x}, U)U \times JU + (H^{(n)}_{x}, U)U \times U_{xx} \\
- H^{(n)}_{x} \times JU + (U, H^{(n)}_{x} \times JU)U - (U, H^{(n)}_{x} \times U_x)U_x \\
- 2(U, H^{(n)} \times U_x)(U_x, U_x)U - U_x \times JH^{(n)} - U \times JH^{(n)}_x \\
+ (U, U_x \times JH^{(n)})U + (JH^{(n)}, U)U \times U_x \\
- (U, H^{(n)} \times U_{xx})U_x - 2(U, H^{(n)} \times U_x)U_{xx} \\
- (U \times U_{xx} + U \times JU)D_x^{-1} \left( (H^{(n)}_x, U_x) - (JH^{(n)}, U) \right) \\
- \left( (U_{xx} + \frac{3}{2}U(U_x, U_x))_x - \frac{3}{2}U_x(U, JU) \right)D_x^{-1}(U, H^{(n)} \times U_x) \\
+ U_xD_x^{-1} \left( (U, U_x \times JH^{(n)}) - (U, H^{(n)}_{xx} \times U_x) \right) \\
- \frac{1}{2}(U, H^{(n)} \times U_x)(U, JU) \\
- \frac{1}{2}(U, H^{(n)} \times U_x)(U_x, U_x) + (H^{(n)}_x \times JU, U) \right). \]
L-L equation as a two-component system and its recursion operators

In this section we consider the Landau-Lifshitz equation written in the form

\[ u_t = -u_{xx} + 2\psi \left( u_x^2 - P(u) \right) + \frac{1}{2} P'(u) \]
\[ v_t = v_{xx} + 2\psi \left( v_x^2 - P(v) \right) - \frac{1}{2} P'(v), \]

where

\[ \psi = (u - v)^{-1}, \]

and \( P \) is an arbitrary fourth degree polynomial. The usual vectorial form of Landau-Lifshitz equation gives rise to a system of the form (3.18) after the stereographic projection.

The third order symmetry of (3.18) is given by

\[ u_{t3} = u_{xxx} - 6u_x u_{xx} \psi + 6u_x^3 \psi^2 - \frac{1}{2} u_x P''(u) - 3u_x \psi P'(u) - 6u_x \psi^2 P(u), \]
\[ v_{t3} = v_{xxx} + 6v_x v_{xx} \psi + 6v_x^3 \psi^2 - \frac{1}{2} v_x P''(v) - 3v_x \psi P'(v) - 6v_x \psi^2 P(v). \]
In case of the system of evolution equation (3.18) the symmetries and cosymmetries can be treated as two-dimensional vectors. We introduce the following notation for symmetries:

\[ G_1 = (u_x, v_x)^t, \quad G_2 = (u_{t_2}, v_{t_2})^t, \quad G_3 = (u_{t_3}, v_{t_3})^t. \]

Thus \( G_{ij} \) is \( j \)-th component of the symmetry \( G_i \).
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Thus \( G_{ij} \) is \( j \)-th component of the symmetry \( G_i \). The simplest cosymmetries are given by

\[ g_1 = \psi^2(v_x, -u_x)^t, \quad g_2 = \psi^2(v_{t_2}, -u_{t_2})^t, \quad g_3 = \psi^2(v_{t_3}, -u_{t_3})^t. \] (3.20)
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These cosymmetries are variational derivatives \( (\frac{\delta \rho}{\delta u}, \frac{\delta \rho}{\delta v})^t \) of the following conserved densities

\[ \rho_1 = \frac{1}{2} \psi (u_x + v_x), \quad \rho_2 = \psi^2 (u_x v_x - P(v)) - \frac{1}{2} \psi P'(v) - \frac{1}{12} P''(v), \]
\[ \rho_3 = \frac{1}{2} \psi^2 (u_x v_{xx} - u_{xx} v_x) + \psi^3 u_x v_x (u_x + v_x) - \frac{1}{2} u_x \psi (4 \psi^2 P(u) + P''(u)). \]
L-L equation as a two-component system and its recursion operators

Theorem

The L-L equation possesses the following quasilocal recursion operators:

\[ R_1 = \begin{pmatrix} R_{11} & 0 \\ 0 & R_{22} \end{pmatrix} - 2 \begin{pmatrix} G_{11} & G_{21} \\ G_{12} & G_{22} \end{pmatrix} D^{-1}_x \begin{pmatrix} g_{21} & g_{22} \\ g_{11} & g_{12} \end{pmatrix} \]

and

\[ R_2 = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} + 2 \begin{pmatrix} G_{11} & G_{21} & G_{31} \\ G_{12} & G_{22} & G_{32} \end{pmatrix} D^{-1}_x \begin{pmatrix} g_{31} & g_{32} \\ g_{21} & g_{22} \\ g_{11} & g_{12} \end{pmatrix}, \]
L-L equation as a two-component system and its recursion operators

Theorem

Operators $\mathcal{R}_1$ and $\mathcal{R}_2$ are related by the following elliptic curve equation

$$\mathcal{R}_2^2 - \mathcal{R}_1^3 - \varphi \mathcal{R}_1 - \vartheta E = 0,$$

(3.21)

where $E$ stands for the unity matrix, and $\varphi$, $\vartheta$ are some functions of coefficients of polynomial $P$. 


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Theorem

Operators $\mathcal{R}_1$ and $\mathcal{R}_2$ are related by the following elliptic curve equation

$$\mathcal{R}_2^2 - \mathcal{R}_1^3 - \varphi \mathcal{R}_1 - \vartheta E = 0,$$

where $E$ stands for the unity matrix, and $\varphi$, $\vartheta$ are some functions of coefficients of polynomial $P$.

It is easy to verify that $\varphi$ and $\vartheta$ are constants for any polynomial $P$, if $\deg P \leq 4$. It turns out that the above recursion operators are ratios

$$\mathcal{R}_1 = \mathcal{H}_1 \mathcal{H}_0^{-1}, \quad \mathcal{R}_2 = \mathcal{H}_2 \mathcal{H}_0^{-1}$$

of some quasilocal Hamiltonian operators.
Recursion operators of the K-N equation

The Krichever-Novikov equation

\[ u_{t_1} = u_{xxx} - \frac{3}{2} \frac{u_x^2}{u_x} + \frac{P(u)}{u_x}, \quad P^{(V)} = 0. \] (4.22)
The Krichever-Novikov equation

\[ u_{t_1} = u_{xxx} - \frac{3}{2} \frac{u_{xx}^2}{u_x} + \frac{P(u)}{u_x}, \quad P(V) = 0. \]  

(4.22)

The simplest symmetries and conservation laws of (4.22) are given by

\[ G_1 = u_x, \quad G_2 = u_5 - 5 \frac{u_4 u_2}{u_1} - \frac{5}{2} \frac{u_3^2}{u_1} + \frac{25}{2} \frac{u_3 u_2^2}{u_1^2} - \frac{45}{8} \frac{u_2^4}{u_1^3} \]

\[ -\frac{5}{3} P \frac{u_3}{u_1^2} + \frac{25}{6} P \frac{u_2^2}{u_1^3} - \frac{5}{3} P' \frac{u_2}{u_1} - \frac{5}{18} \frac{P^2}{u_1^3} + \frac{5}{9} u_1 P'' \].
The simplest three conserved densities of (4.22) are

\[ \rho_1 = -\frac{1}{2} \frac{u_2^2}{u_1^2} - \frac{1}{3} \frac{P}{u_1^2}, \quad \rho_2 = \frac{1}{2} \frac{u_3^2}{u_1^2} - \frac{3}{8} \frac{u_2^4}{u_1^4} + \frac{5}{6} \frac{P u_2^2}{u_1^4} + \frac{1}{18} \frac{P^2}{u_1^4} - \frac{5}{9} P''', \]

\[ \rho_3 = \frac{u_4^2}{u_1^2} + 3 \frac{u_3^3}{u_1^3} - \frac{19}{2} \frac{u_3 u_2^2}{u_1^4} + \frac{7}{3} P \frac{u_3^2}{u_1^4} + \frac{35}{9} P' \frac{u_2^3}{u_1^4} \]
\[ + \frac{45}{8} \frac{u_2^6}{u_1^6} - \frac{259}{36} \frac{u_2^4 P}{u_1^6} + \frac{35}{18} P^2 \frac{u_2^2}{u_1^6} - \frac{14}{9} P''' \frac{u_2^2}{u_1^6} + \frac{1}{27} \frac{P^3}{u_1^6} - \frac{14}{27} \frac{P''' P}{u_1^2} - \frac{7}{27} \frac{P''^2}{u_1^2} - \frac{14}{9} P^{(IV)} u_1^2. \]
Recursion operators of the K-N equation

The forth order quasilocal recursion operator

\[ R_1 = D_x^4 + a_1 D_x^3 + a_2 D_x^2 + a_3 D_x + a_4 + G_1 D_x^{-1} \frac{\delta \rho_1}{\delta u} + u_x D_x^{-1} \frac{\delta \rho_2}{\delta u}, \]

where the coefficients \( a_i \) are given by

\[ a_1 = -4 \frac{u_2}{u_1}, \quad a_2 = 6 \frac{u_2^2}{u_1^2} - 2 \frac{u_3}{u_1} - \frac{4}{3} \frac{P}{u_1^2}, \]

\[ a_3 = -2 \frac{u_4}{u_1} + 8 \frac{u_3 u_2}{u_1^2} - 6 \frac{u_2^3}{u_1^3} + 4P \frac{u_2}{u_1^3} - 2 \frac{P'}{3 u_1}, \]

\[ a_4 = \frac{u_5}{u_1} - 2 \frac{u_3^2}{u_1^2} + 8 \frac{u_3 u_2^2}{u_1^3} - 4 \frac{u_4 u_2}{u_1^2} - \frac{3}{3} \frac{u_2^4}{u_1^4} + \frac{4}{9} \frac{P^2}{u_1^4} \]

\[ + \frac{4}{3} \frac{P u_2^2}{u_1^4} + \frac{10}{9} \frac{P''}{P'} - \frac{8}{3} \frac{P'}{u_1^2}. \]
Theorem

There exists one more quasilocal recursion operator for (4.22) of the form

\[ R_2 = D_x^6 + b_1 D_x^5 + b_2 D_x^4 + b_3 D_x^3 + b_4 D_x^2 + b_5 D_x + b_6 \]

\[ - \frac{1}{2} u_x D_x^{-1} \frac{\delta \rho_3}{\delta u} + G_1 D_x^{-1} \frac{\delta \rho_2}{\delta u} + G_2 D_x^{-1} \frac{\delta \rho_1}{\delta u}, \]

where

\[ b_1 = -6 \frac{u_2}{u_1}, \quad b_2 = -9 \frac{u_3}{u_1} - 2 \frac{P}{u_1^2} + 21 \frac{u_2}{u_1^2}, \quad b_3 = -11 \frac{u_4}{u_1} + 60 \frac{u_3 u_2}{u_1^2} \]

\[ + 14 P \frac{u_2}{u_1^3} - 57 \frac{u_3}{u_1^3} - 3 \frac{P'}{u_1}, \]

\[ b_4 = -4 \frac{u_5}{u_1} + 38 \frac{u_4 u_2}{u_1^2} + 22 \frac{u_3^2}{u_1^2} + 99 \frac{u_2^4}{u_1^4} - 155 \frac{u_3 u_2^2}{u_1^3} + \frac{34}{3} P \frac{u_3}{u_1} \]

\[ - 44 P \frac{u_2}{u_1^4} + 4 \frac{P^2}{u_1^4} + 12 P' \frac{u_2}{u_1^2} - P''', \]
\[ b_5 = -2 \frac{u_6}{u_1} + 29 \frac{u_4 u_3}{u_1^2} + 80 P \frac{u_2^3}{u_1^3} + \frac{23}{3} P' \frac{u_3}{u_1^2} - 104 \frac{u_2 u_3^2}{u_1^4} - 70 \frac{u_4 u_2^2}{u_1^3} \]
\[ + 241 \frac{u_2^2 u_3}{u_1^4} + 14 \frac{u_5 u_2}{u_1^2} + \frac{20}{3} P \frac{u_4}{u_1^3} - \frac{170}{3} P \frac{u_2 u_3}{u_1^4} + \frac{4}{3} P' P \]
\[ - 22 P' \frac{u_2}{u_1^3} + 2 P'' \frac{u_2}{u_1} - \frac{16}{3} P^2 \frac{u_2}{u_1^5} - 108 \frac{u_5^2}{u_1^3}, \]
\[ b_6 = \frac{u_7}{u_1} - 6 \frac{u_2 u_6}{u_1^2} + \frac{8}{9} P^2 \frac{u_2^2}{u_1^6} - 195 \frac{u_3^2 u_2}{u_1^4} + 6P \frac{u_2^3}{u_1^6} + \frac{142}{3} P \frac{u_3^2}{u_1^5} + \frac{28}{9} P' P \frac{u_2}{u_1^4} \]
\[ + 101 \frac{u_4 u_3 u_2}{u_1^3} + \frac{34}{3} P \frac{u_4 u_2}{u_1^4} - 72 \frac{u_6^2}{u_1^6} - \frac{28}{9} P''' u_2 + \frac{38}{3} P'' \frac{u_2^2}{u_1^5} - \frac{19}{3} P' \frac{u_4}{u_1^2} \]
\[ - \frac{122}{3} P' \frac{u_3}{u_1^2} - 10 \frac{u_4^2}{u_1^4} + 22 \frac{u_3^3}{u_1^6} - \frac{178}{3} P \frac{u_3 u_2}{u_1^5} + \frac{14}{9} P (IV) u_1^2 - 13 \frac{u_5 u_3}{u_1^2} \]
\[ - \frac{2}{3} P \frac{u_5}{u_1^3} - \frac{17}{3} P'' \frac{u_3}{u_1} - \frac{4}{3} P^2 \frac{u_3^2}{u_1^5} - 89 \frac{u_4 u_2^3}{u_1^4} + 236 \frac{u_3 u_2^4}{u_1^5} + \frac{113}{3} P' \frac{u_3 u_2}{u_1^3} \]
\[ + 25 \frac{u_5^2 u_2^2}{u_1^3} - \frac{7}{9} \frac{P' P}{u_1^2} - \frac{8}{27} \frac{P^3}{u_1^6} - \frac{4}{9} \frac{P'' P}{u_1^2}. \]
The relation between the operators $\mathcal{R}_1$ and $\mathcal{R}_2$ is as follows

$$\mathcal{R}_2^2 = \mathcal{R}_1^3 - \phi \mathcal{R}_1 - \theta,$$

(4.23)

where

$$\phi = \frac{16}{27} \left( (P'')^2 - 2P'''P' + 2P^{(IV)}P \right),$$

$$\theta = \frac{128}{243} \left( P'P''P''' - \frac{1}{3}(P'')^3 - \frac{3}{2}(P')^2P^{(IV)} ight. + 2P^{(IV)}P''P - P(P''')^2).$$
**Remark** The relations between recursion operators for the K-N and L-L equations are understood as identities in the non-commutative field of pseudo-differential series of the form

\[ A = a_mD_x^m + a_{m-1}D_x^{m-1} + \cdots + a_0 + a_{-1}D_x^{-1} + a_{-2}D_x^{-2} + \cdots \]

where \( a_i \) are local functions and the multiplication is defined by

\[ aD_x^k bD_x^m = a(bD_x^{m+k} + C_k^1D_x(b)D_x^{k+m-1} + C_k^2D_x^2(b)D_x^{k+m-2} + \cdots) , \]

where \( k, m \in \mathbb{Z} \) and

\[ C_n^j = \frac{n(n-1)(n-2) \cdots (n-j+1)}{j!} . \]
Hamiltonian structures of the K-N equation

The recursion operators presented above appear to be the ratios

\[ R_1 = \mathcal{H}_1 \mathcal{H}_0^{-1}, \quad R_2 = \mathcal{H}_2 \mathcal{H}_0^{-1} \]

of the following quasilocal Hamiltonian operators

\[
\mathcal{H}_1 = \frac{1}{2}(u_x^2 D_x^3 + D_x^3 u_x^2) + (2u_{xxx} u_x - \frac{9}{2} u_{xx}^2 - \frac{2}{3} P)D_x \\
+ D_x(2u_{xxx} u_x - \frac{9}{2} u_{xx}^2 - \frac{2}{3} P) \\
+ G_1 D_x^{-1} G_1 + u_x D_x^{-1} G_2 + G_2 D_x^{-1} u_x,
\]

\[
\mathcal{H}_2 = \frac{1}{2}(u_x^2 D_x^5 + D_x^5 u_x^2) + (3u_{xxx} u_x - \frac{19}{2} u_{xx}^2 - P)D_x^3 \\
+ D_x(3u_{xxx} u_x - \frac{19}{2} u_{xx}^2 - P) + hD_x + D_x h + G_1 D_x^{-1} G_2 \\
+ G_2 D_x^{-1} G_1 + u_x D_x^{-1} G_3 + G_3 D_x^{-1} u_x,
\]

where

\[
h = u_{xxxxxx} u_x - 9u_{xxxx} u_{xx} + \frac{19}{2} u_{xxx}^2 - \frac{2}{3} \frac{u_{xxx}}{u_x} (5P - 39u_{xx}^2) \\
+ \frac{u_{xx}^2}{u_x^2} (5P - 9u_{xx}^2) + \frac{2}{3} \frac{P^2}{u_x^2} + u_x^2 P''',
\]
and $G_3 = \mathcal{R}_1(G_1) = \mathcal{R}_2(u_x)$ is the seventh order symmetry of (4.22):

$$G_3 = u_7 - 7 \frac{u_2 u_6}{u_1} - \frac{7}{6} \frac{u_5}{u_1^2} (2P + 12u_3 u_1 - 27u_2^2) - \frac{21}{2} \frac{u_4^2}{u_1}$$

$$+ \frac{21}{2} \frac{u_4^3}{u_1^3} u_2 (2P - 11u_2^2) - \frac{7}{3} \frac{u_4^2}{u_1^2} (2P' u_1 - 51u_2 u_3) + \frac{49}{2} \frac{u_3^2}{u_1^2}$$

$$+ \frac{7}{12} \frac{u_3^2}{u_1^3} (22P - 417u_2^2) + \frac{2499}{8} \frac{u_4^3}{u_1^3} u_3 + \frac{91}{3} P' \frac{u_2}{u_1} u_3 - \frac{595}{6} P \frac{u_2^2}{u_1^3} u_3$$

$$- \frac{35}{18} \frac{u_3^4}{u_1^4} (2P'' u_1^4 - P^2) - \frac{1575}{16} \frac{u_6}{u_1^5} + \frac{1813}{24} \frac{u_2^4}{u_1^5} P$$

$$- \frac{203}{6} \frac{u_3^3}{u_1^4} P' + \frac{49}{36} \frac{u_2^2}{u_1^3} (6P'' u_1^4 - 5P^2) - \frac{7}{9} \frac{u_2^3}{u_1^2} (2P''' u_1^4 - 5PP') + \frac{7}{54} \frac{P^3}{u_1^5}$$

$$- \frac{7}{9} P'' \frac{P}{u_1} + \frac{7}{9} P'''' u_1^3 - \frac{7}{18} \frac{P'^2}{u_1}.$$
The Krichever-Novikov and the Landau-Lifshitz equations play a role of the universal models for the classes of KdV and NLS-type equations.

The associative algebra of quasilocal recursion operators for these models is generated by a couple of operators related by an elliptic curve equation.

A theoretical explanation of the above fact for the Landau-Lifshitz equation is given in terms of multiplicators of the corresponding Lax structure.

New quasilocal Hamiltonian operators for these models are found as well.
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Conclusions

- The Krichever-Novikov and the Landau-Lifshitz equations play a role of the universal models for the classes of KdV and NLS-type equations.
- The associative algebra of quasilocal recursion operators for these models is generated by a couple of operators related by an elliptic curve equation.
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References


