On integrable Weingarten surfaces

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Immersed surfaces and ZCRs

The Gauss–Mainardi–Codazzi equations describing surfaces immersed into three-dimensional Euclidean space are always of the form

\[ A_y - B_x + [A, B] = 0, \]  

(1)

being the compatibility condition of an auxiliary linear system

\[ \Psi_x = A\Psi, \quad \Psi_y = B\Psi, \]  

(2)

identifiable with the Gauss–Weingarten equations.
Gauge invariance

Equation $A_y - B_x + [A, B] = 0$ is called a zero curvature representation (ZCR) and is invariant under a group of gauge transformations

$$A' = S_x S^{-1} + SAS^{-1},$$
$$B' = S_y S^{-1} + SBS^{-1},$$

resulting from the transformation $\Psi' = S \Psi$ of the linear system $\Psi_x = A \Psi, \; \Psi_y = B \Psi$ (2). Here $S$ is an invertible matrix.
Coordinates and fundamental forms

Our coordinate system of choice will be the curvature coordinates, denoted by $x, y$. They are unique up to arbitrary changes $x = X(x), y = Y(y)$.

The first and second fundamental forms are respectively given by

\[ I = u^2 \, dx^2 + v^2 \, dy^2, \]
\[ II = u^2 p \, dx^2 + v^2 q \, dy^2. \]

Consequently, $p, q$ are the principal curvatures.
Gauss–Mainardi–Codazzi equations

The corresponding Gauss–Mainardi–Codazzi equations are

\[ uu_{yy} + vv_{xx} - \frac{v}{u} u_x v_x - \frac{u}{v} u_y v_y + u^2 v^2 pq = 0, \quad (4a) \]

\[ (p - q) u_y + up_y = 0, \quad (4b) \]

\[ (q - p) v_x + vq_x = 0. \quad (4c) \]

Such equations are always equivalent to a ZCR

\[ A_y - B_x + [A, B] = 0 \]

with suitable matrices \( A, B \).
The GMC equations possess an $\mathfrak{sl}(2, \mathbb{C})$-valued parameter-free ZCR

$$A = \begin{pmatrix} \frac{iu_y}{2v} & -\frac{1}{2} up \\ \frac{1}{2} up & -\frac{iuy}{2v} \end{pmatrix},$$

$$B = \begin{pmatrix} \frac{iux}{2u} & -\frac{1}{2} iqv \\ -\frac{1}{2} iqv & \frac{iux}{2u} \end{pmatrix}.$$
Geometric constraints, Weingarten surfaces

The functional relation

\[ f(p, q) = 0 \iff q = F(p), \]  

(6)

between principal curvatures is a general algebraic constraint invariant w. r. to coordinate changes restricting us to the class of *Weingarten surfaces*. The Mainardi–Codazzi equations (4b), (4c) becomes

\[
\begin{align*}
(\ln u)_y &= \frac{u_y}{u} = \frac{p_y}{q - p}, & (\ln v)_x &= \frac{v_x}{v} = \frac{q_x}{p - q}.
\end{align*}
\]
Weingarten surfaces II – reduction

Those equations have the solution

\[ u = u_0 \int \frac{dp}{q - p}, \quad v = v_0 \int \frac{dq}{p - q} \]

Substituting \( q = F = 1/(\ln G)_p + p \), where \( G = G(p) \) is an arbitrary function we obtain

\[ u = u_0 G, \quad v = v_0 \frac{G_p}{G^2}. \]

Without loss of generality, we can put \( u_0 = 1, \ v_0 = 1 \) (this can be always achieved by a change of curvature coordinates).
The GMC equations (4a), (4b), (4c) then reduce to the single equation

\[ p_{yy} = - \frac{G_{pp}^2}{G^3} - \frac{p}{G^2} - 2 \frac{G_{pp}^2}{G} + \frac{G_{pp}^2}{G^2} \frac{-2G_p^2 + G_{pp}G}{G^6} \frac{p_{xx}}{G^7} - \frac{G_{pp}G_p G}{G^6} \frac{8G_p^3 - 7G_{pp}G_p G + G_{ppp}G^2}{G^7} \frac{p_x^2}{G^7} \] (7)
The ZCR (5) becomes

\[ A_0 = \begin{pmatrix} \frac{1}{2} i G^2 p_y & - \frac{1}{2} G p \\ \frac{1}{2} G p & - \frac{1}{2} i G^2 p_y \end{pmatrix}, \]

\[ B_0 = \begin{pmatrix} -i(GG_{pp} - 2G^2_\rho)p_x & -i(G_{pp} + G) \\ \frac{2G^4}{2G^2} i(GG_{pp} - 2G^2_\rho)p_x & \frac{-2G^2}{2G^4} i(G_{pp} + G) \end{pmatrix}. \]
Jets language

In the context of reduced GMC equation (7) it is more appropriate to rewrite (1) as

\[ D_y A_0 - D_x B_0 + [A_0, B_0] = 0, \]  

(9)

where \( D_x, D_y \) denote the total derivatives

\[ D_x = \frac{\partial}{\partial x} + \sum_I p_{xI} \frac{\partial}{\partial p_I}, \quad D_y = \frac{\partial}{\partial y} + \sum_I p_{yI} \frac{\partial}{\partial p_I}. \]
More on inserting the spectral parameter: restriction procedure

Consider the formal Taylor expansions

\[ A(\lambda) = \sum_{i=0}^{\infty} A_i \lambda^i, \quad B(\lambda) = \sum_{i=0}^{\infty} B_i \lambda^i. \]  

(10)

Plugging (10) into \( A_y - B_x + [A, B] = 0 \) yields

\[ D_y A_k - D_x B_k + \sum_{i+j=k} [A_i, B_j] = 0, \]  

(11)

for all \( k \geq 0 \). For \( k = 0 \) we obtain an identity once \( A_0, B_0 \) is a ZCR.
For $k = 1$, eq.(11) is linear in $A_1, B_1$ and reads

$$D_y A_1 - D_x B_1 + [A_1, B_0] + [A_0, B_1] = 0.$$  (12)

Solutions $A_1, B_1$ are called cocycles. This equation always has solutions of the form

$$A_1 = D_x C + [A_0, C], \quad B_1 = D_y C + [B_0, C].$$  (13)

where $C$ is an arbitrary matrix. These are called coboundaries. Cocycles that differ by a coboundary are called cohomological.
Cocycles modulo coboundaries are in one-to-one correspondence with ZCRs of the form

$$A^{[1]} = \begin{pmatrix} A_0 & 0 \\ A_1 & A_0 \end{pmatrix}, \quad B^{[1]} = \begin{pmatrix} B_0 & 0 \\ B_1 & B_0 \end{pmatrix}, \quad (14)$$

modulo gauge equivalence with respect to gauge matrices of the form

$$S^{[1]} = \begin{pmatrix} E & 0 \\ S & E \end{pmatrix},$$

where $E$ is the unit matrix.
It is easily verified that the subsystem of equations (11)

\[ D_yA_k - D_xB_k + \sum_{i+j=k} [A_i, B_j] = 0, \quad k = 0, \ldots, m \]

is equivalent to the same zero curvature condition

\[ D_yA^m - D_xB^m + [A^m, B^m] = 0, \]

where
\[ D_y A^m - D_x B^m + [A^m, B^m] = 0, \quad (15) \]

where

\[
A^m = \begin{pmatrix}
A_0 & 0 & \cdots & 0 \\
A_1 & A_0 & \cdots & : \\
: & \cdots & \cdots & 0 \\
A_m & \cdots & A_1 & A_0
\end{pmatrix}, \quad B^m = \begin{pmatrix}
B_0 & 0 & \cdots & 0 \\
B_1 & B_0 & \cdots & : \\
: & \cdots & \cdots & 0 \\
B_m & \cdots & B_1 & B_0
\end{pmatrix}.
\]

If \( A_k, B_k \) are already known for all \( k < m \), then (15) says whether expansions (10) can be extended one step further.
If, for some $m$, eq. (15) has no solution, then there is no possibility to extend the expansions (10) beyond the first $m$ terms.

Above considerations reduce the spectral parameter problem to that of computation of ZCRs modulo a gauge group. The substantial benefit is that the problem is linear in all unknowns $A_k, B_k$, $k \geq 1$ (except $A_0, B_0$, which are not unknowns).
To solve the system (15)

\[ D_y A^{[m]} - D_x B^{[m]} + [A^{[m]}, B^{[m]}] = 0, \]

Application to Weingarten surfaces I

We start with the already known ZCR (8)

\[ A_0 = \begin{pmatrix} \frac{1}{2} iG^2 p_y - \frac{1}{2} Gp \\ \frac{1}{2} Gp - \frac{1}{2} iG^2 p_y \end{pmatrix}, \]

\[ B_0 = \begin{pmatrix} -i(GG_{pp} - 2G^2_p)p_x \\ \frac{2G^4}{i(G_{pp} + G)} \\ -i(G_{pp} + G) \\ \frac{2G^2}{2G^2} \\ \frac{i(GG_{pp} - 2G^2_p)p_x}{2G^4} \end{pmatrix}. \]
A routine computation shows that for every function $G$ satisfying

$$G_{pp} + 2\frac{G_p}{p} + \frac{G}{p^2} - \frac{G_p^2}{G} = 0,$$

solutions of equation (12)

$$D_y A_1 - D_x B_1 + [A_1, B_0] + [A_0, B_1] = 0.$$ 

are gauge equivalent to a particular solution of the following form:
Application to Weingarten surfaces III

The particular solution in question reads

\[
A_1 = \begin{pmatrix}
-a_1 \frac{(G_{pp} + G)p_x}{G^2p^2} & a_1 \\
\frac{a_1((G_{pp} + G)p_x)}{G^2p^2} & a_1 \\
\end{pmatrix},
\]

\[
B_1 = \begin{pmatrix}
-a_1 \frac{(G_{pp} + G)p_y}{G^2p^2} & 0 \\
\frac{a_1((G_{pp} + G)p_y)}{G^2p^2} & 0 \\
\end{pmatrix}.
\]
Application to Weingarten surfaces IV

Here $a_1 = a_1(p)$ a function subject to the differential equation

$$\frac{d}{dp} a_1 = \frac{a_1 G_p}{G} + \frac{a_1}{p}.$$
Application to Weingarten surfaces V

The second step reveals that the matrices $A_2, B_2$ exist and are of the form

$$A_2 = \begin{pmatrix} -\frac{a_2(G_{pp} + G)p_x}{G^2p^2} - \frac{ia_1^2p_y}{p^2} & \cdots \\ a_2 & \cdots \end{pmatrix},$$

$$B_2 = \begin{pmatrix} -\frac{ia_1^2(G_{pp}^2 + 2G_{p}G_{pp} + G^2)p_x}{p^4G^6} - \frac{a_2(G_{pp} + G)p_y}{G^2p^2} & \cdots \\ \frac{ia_1^2(G_{pp} + G)}{G^4p^2} & \cdots \end{pmatrix}.$$
Application to Weingarten surfaces VI

Here the functions $a_2 = a_2(p)$ are subject to the differential equation

$$\frac{d}{dp} a_2 = \frac{a_2 G_p}{G} + \frac{a_2}{p}.$$
Application to Weingarten surfaces VII

An analogous third step gives

\[ A_3 = \begin{pmatrix} -\frac{a_3(Gpp + G)p_x}{G^2p^2} & -\frac{2ia_1a_2p_y}{p^2} & \cdots \\ \\ a_3 & \cdots \\ \end{pmatrix}, \]

\[ B_3 = \begin{pmatrix} -\frac{2ia_1a_2(pG_p + G)^2p_x}{p^4G^6} & -\frac{a_3(pG_p + G)p_y}{p^2G^2} & \cdots \\ \\ 2ia_1a_2(Gpp + G) & \frac{2ia_1a_2G^4p^2}{G^4p^2} & \cdots \\ \end{pmatrix} \]

under the same condition on \( G \).
The condition (16) has the general solution

\[ G = e^{1+c/p} \]

with the numbers \( b, c \) as parameters; after substituting \( G \) back into \( F \) we obtain

\[ F = \frac{pc}{p + c} \]
Thus, the class of surfaces we arrived at is characterized by the nonlinear condition

\[ pq = c(p - q). \]

The reduced GMC equations (7) now can be rewritten in the following simple form:

\[
pyy = \frac{c^2p_{xx}}{e^{4(p+c)/p}} - 2\frac{c^2(p - c)p_x^2}{e^{4(p+c)/p}p^2}
\]

\[
+ 2\frac{(p + c)p_y^2}{p^2} + \frac{cp^2}{e^{2(p+c)/p}}.
\]
Application to Weingarten surfaces X

Using the already known first four terms of the Taylor expansion, the general form of
the rest is rather obvious. The ZCR of reduced GMC (17) with spectral parameter
\( \lambda \) is then easily found to be

\[
A = \begin{pmatrix}
\frac{c(\lambda + \frac{1}{2})p_x + \sqrt{\lambda^2 + \lambda e^{2+2c/p}p_y}}{p^2} & \lambda e^{1+c/p} \\
(\lambda + 1)e^{1+c/p} & -\frac{c(\lambda + \frac{1}{2})p_x + \sqrt{\lambda^2 + \lambda e^{2+2c/p}p_y}}{p^2}
\end{pmatrix}
\]

\[
B = \begin{pmatrix}
\frac{c^2\sqrt{\lambda^2 + \lambda p_x}}{e^{2+2c/p}p^2} + \frac{c(\lambda + \frac{1}{2})p_y}{p^2} & \frac{c\sqrt{\lambda^2 + \lambda}}{e^{1+c/p}} \\
\frac{c\sqrt{\lambda^2 + \lambda}}{e^{1+c/p}} & -\frac{c^2\sqrt{\lambda^2 + \lambda p_x}}{e^{2+2c/p}p^2} - \frac{c(\lambda + \frac{1}{2})p_y}{p^2}
\end{pmatrix}
\]
Application to Weingarten surfaces XI

Alternatively, the above ZCR can be written in the gauge equivalent form

\[
A = \begin{pmatrix}
\frac{\sqrt{\lambda^2 + \lambda e^{2(p+c)/p} p_y}}{p^2} & \frac{\lambda e^{(p+2c\lambda+2c)/p}}{p^2} \\
(\lambda + 1)e^{(p-2c\lambda)/p} & -\frac{\sqrt{\lambda^2 + \lambda e^{2(p+c)/p} p_y}}{p^2}
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
\frac{c^2 \sqrt{\lambda^2 + \lambda} p_x}{e^{2(p+c)/p} p^2} & \frac{c \sqrt{\lambda^2 + \lambda}}{e^{(p-2c\lambda)/p}} \\
\frac{c \sqrt{\lambda^2 + \lambda}}{e^{(p+2c\lambda+2c)/p}} & -\frac{c^2 \sqrt{\lambda^2 + \lambda} p_x}{e^{2(p+c)/p} p^2}
\end{pmatrix}.
\]
Link to an evolution system

Consider an evolution system

\[ u_y = v_x, \quad v_y = -c \frac{\sqrt{c \cdot x^2 + 1}}{u_x}, \tag{18} \]

and set

\[ p = \frac{2c}{\ln(c^2 u_x) - 2}. \]

Then so defined \( p \) satisfies the reduced GMC(17), i.e., (17) is a differential consequence of (18).
Rescaling

Moreover, upon introducing rescaled independent variables \( t = \sqrt{c} t \), \( z = \sqrt{c} x \) we can transform (18) into

\[
\begin{align*}
    u_t &= v_z, \\
    v_t &= -\frac{1}{u_z} - z^2.
\end{align*}
\] (19)
Symmetries of the evolution system

Define a nonlocal variable $w$ by the formulas $w_z = u, w_t = v$. Then we have the following symmetries of (19):

$$(U_1) = \begin{pmatrix} t \\ z \end{pmatrix}, \quad (U_2) = \begin{pmatrix} u_z \\ v_z \end{pmatrix}, \quad (U_3) = \begin{pmatrix} v_z, -1/u_z - z^2 \end{pmatrix}^T,$$

$$(U_4) = \begin{pmatrix} t^2 \\ 2zt \end{pmatrix}, \quad (U_5) = \begin{pmatrix} zu_z - tv_z + u \\ zv_z + tu_z^{-1} - v + z^2t \end{pmatrix},$$

$$(U_6) = \begin{pmatrix} v_z^2 + 2z^2u_z - 2\ln(u_z) + 4zu - 4w \\ 2z^2v_z - 2u_z^{-1}v_z - 4zv \end{pmatrix},$$

$$(U_7) = \begin{pmatrix} 2u_zv_z + ztu_z - 2t^2v_z + tu \\ v_z^2 + 4ztv_z + 2t^2u_z^{-1} + 2\ln(u_z) - 4tv + 4w + 2z^2t^2 \end{pmatrix} \cdots$$
Integrability of the evolution system – ZCR

System (19) has a ZCR

\[ A = \begin{pmatrix} \frac{(\lambda + \frac{1}{2})u_{zz}}{2u_z} + \frac{1}{2} \sqrt{\lambda^2 + \lambda v_{zz}} & (\lambda + 1)\sqrt{u_z} \\ \lambda \sqrt{u_z} & -\frac{(\lambda + \frac{1}{2})u_{zz}}{2u_z} - \frac{1}{2} \sqrt{\lambda^2 + \lambda v_{zz}} \end{pmatrix}, \]

\[ B = \begin{pmatrix} \frac{\sqrt{\lambda^2 + \lambda u_{zz}}}{2u_z^2} + \frac{(\lambda + \frac{1}{2})v_{zz}}{2u_z} & \frac{\sqrt{\lambda^2 + \lambda}}{u_z} \\ \frac{\sqrt{\lambda^2 + \lambda}}{\sqrt{u_z}} & -\frac{\sqrt{\lambda^2 + \lambda u_{zz}}}{2u_z^2} - \frac{(\lambda + \frac{1}{2})v_{zz}}{2u_z} \end{pmatrix}, \]

with the spectral parameter \( \lambda \).
Recursion operator – preliminaries

Given a symmetry \((U, V)^T\) of (19), introduce nonlocal variables \(W, R, S\) given by the formulas

\[
W_z = U,
\]
\[
W_t = V,
\]
\[
R_z = \frac{u_z v_z V_z + z^2 u_z U_z - 2W u_z + 2z U u_z - U_z}{2 u_z}
\]
\[
R_t = \frac{z^2 u_z^2 V_z - 2z V u_z^2 - u_z V_z + v_z U_z}{2 u_z^2}
\]
\[
S_z = \frac{1}{2} u_z V_z + \frac{1}{2} v_z U_z + z t U_z - \frac{1}{2} t^2 V_z + t U
\]
\[
S_t = \frac{u_z^2 v_z V_z + 2z t u_z^2 V_z - 2t V u_z^2 + 2W u_z^2 + u_z U_z - t^2 U_z}{2 u_z^2}.
\]
Recursion operator

**Proposition 1** *Under the above assumptions*

\[
\mathcal{R} \begin{pmatrix} U \\ V \end{pmatrix} = \begin{pmatrix} (v_z + 2zt)U - (u_z + t^2)V - 2S \\ -(u_z^{-1} + z^2)U - v_zV + 2R \end{pmatrix}
\]

is a new symmetry for (19), i.e., \( \mathcal{R} \) is a recursion operator for (19).

Thus, the system (19) has infinitely many symmetries. Indeed, applying \( \mathcal{R} \) to the above symmetries \((U_i, V_i)^T\) yields infinite hierarchies of (nonlocal) symmetries.
Conclusions

• We disproved the conjecture that only linear Weingarten surfaces are integrable.

• We found a link from an evolution system to the hyperbolic equation describing the non-linear integrable Weingarten surfaces and found a recursion operator for the evolution system in question.

