The Chaplygin ball on a rotating table: conservation laws, integrable case

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Consider the problem of an inhomogeneous balanced ball rolling without slipping on a horizontal plane rotating with constant angular velocity Ω. We use the following notation: $\boldsymbol{v}$ is the velocity of the center of the ball; $\boldsymbol{\omega}$ is the angular velocity of the ball; $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ — are the unit vectors of the fixed axes $Oxyz$.

In this case, the constraint equations (which express the no-slip constraint) are represented in vector form:

$$\boldsymbol{f} = \boldsymbol{v} - a\boldsymbol{\omega} \times \boldsymbol{\gamma} - \Omega\boldsymbol{\gamma} \times \boldsymbol{R} = 0, \quad \boldsymbol{R} = x\boldsymbol{\alpha} + y\boldsymbol{\beta}, \tag{1}$$

here, $\boldsymbol{R}$ is the radius vector of the center of the ball in the moving axes, and a is the radius of the ball. Complete system governing the dynamics in the form

$$\begin{gathered}\mathbf{I}\dot{\boldsymbol{\omega}} + ma^2\boldsymbol{\gamma} \times (\dot{\boldsymbol{\omega}} \times \boldsymbol{\gamma}) = \mathbf{I}\boldsymbol{\omega} \times \boldsymbol{\omega} - ma\Omega(\boldsymbol{\omega}, \boldsymbol{\gamma})\boldsymbol{R} \times \boldsymbol{\gamma} - ma\Omega(\dot{\boldsymbol{R}} \times \boldsymbol{\gamma}) \times \boldsymbol{\gamma}, \\ \dot{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\beta}} = \boldsymbol{\beta} \times \boldsymbol{\omega}, \quad \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma} \times \boldsymbol{\omega}, \\ \dot{x} = -\Omega y + a(\boldsymbol{\omega}, \boldsymbol{\beta}), \quad \dot{y} = \Omega x - a(\boldsymbol{\omega}, \boldsymbol{\alpha}).\end{gathered} \tag{2}$$

We define the momentum vector of the system

$$\boldsymbol{M} = \mathbf{I}\boldsymbol{\omega} + ma^2\boldsymbol{\gamma} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma}) - ma\Omega\boldsymbol{R}. \tag{3}$$

From equations (2) we find that its evolution is governed by the equation

$$\dot{\boldsymbol{M}} = \boldsymbol{M} \times \boldsymbol{\omega}.$$

This equation implies that the vector M remains constant in the fixed coordinate system $Oxyz$, and, as a consequence, we find that the system (2) admits three linear first integrals

$$F_1 = (\boldsymbol{M}, \boldsymbol{\alpha}), \quad F_2 = (\boldsymbol{M}, \boldsymbol{\beta}), \quad F_3 = (\boldsymbol{M}, \gamma). \tag{4}$$

In addition, it was shown in [1] that in the system under consideration one can construct an integral similar to the Jacobi integral in mechanics

$$E = \frac{1}{2}\big(\boldsymbol{\omega}, \mathbf{I}\boldsymbol{\omega} + ma^2\boldsymbol{\gamma} \times (\boldsymbol{\omega} \times \boldsymbol{\gamma})\big) - \frac{m}{2}\Omega^2\big(x^2 + y^2\big). \tag{5}$$

This integral was found in [2].

Another general invariant of the system (2) is the invariant measure $\mu = \rho dxdyd^3\boldsymbol{\alpha}d^3\boldsymbol{\beta}d^3\boldsymbol{\gamma}d^3\boldsymbol{\omega}$, where density is given by the expression

$$\rho = \frac{\det(\mathbf{I} + ma^2\mathbf{E} - ma^2\boldsymbol{\gamma} \otimes \boldsymbol{\gamma})}{\sqrt{1 - ma^2\big(\boldsymbol{\gamma}, (\mathbf{I} + ma^2\mathbf{E})^{-1}\boldsymbol{\gamma}\big)}}.$$

Equations (2) admit an obvious symmetry field corresponding to invariance under rotations of the plane of support:

$$\hat{\boldsymbol{u}}_s = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - \beta_1\frac{\partial}{\partial \alpha_1} + \alpha_1\frac{\partial}{\partial \beta_1} - \beta_2\frac{\partial}{\partial \alpha_2} + \alpha_2\frac{\partial}{\partial \beta_2} - \beta_3\frac{\partial}{\partial \alpha_3} + \alpha_3\frac{\partial}{\partial \beta_3}. \tag{6}$$

The system reduced by the symmetry field (6) can be written in terms of the variables $\boldsymbol{\omega}$, $\boldsymbol{\gamma}$, $\boldsymbol{R}$:

$$\begin{gathered}\tilde{\mathbf{I}}\dot{\boldsymbol{\omega}} = \mathbf{I}\boldsymbol{\omega}\times\boldsymbol{\omega} - ma\Omega(\boldsymbol{\omega},\boldsymbol{\gamma})\boldsymbol{R}\times\boldsymbol{\gamma} - ma\Omega(\dot{\boldsymbol{R}}\times\boldsymbol{\gamma})\times\boldsymbol{\gamma}\\ \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\times\boldsymbol{\omega}, \quad \dot{\boldsymbol{R}} = \boldsymbol{R}\times(\boldsymbol{\omega}-\Omega\boldsymbol{\gamma}) - a\boldsymbol{\gamma}\times\boldsymbol{\omega},\end{gathered} \tag{7}$$

where $\tilde{\mathbf{I}} = \mathbf{I} + ma^2(\boldsymbol{\gamma}^2\mathbf{E} - \boldsymbol{\gamma}\otimes\boldsymbol{\gamma})$ — is the tensor of inertia relative to the point of contact.

For the system (7) to be integrable, we also need (with suitable parameter values of the system) pairs of tensor invariants, for example, two integrals, or an integral and a symmetry field. In this case, the map exhibits chaotic trajectories and hence, in the general case, there is no additional integral.

Let the angular momentum $\boldsymbol{M}$ be parallel to the normal vector $\boldsymbol{\gamma}$:

$$\boldsymbol{M} = \lambda\boldsymbol{\gamma}, \quad \lambda = F_3 = \mathrm{const}. \tag{8}$$

This case requires a separate analysis.

In view of (8) the relation for the angular velocity has the form

$$\begin{gathered}\boldsymbol{\omega} = \mathbf{A}\boldsymbol{K} + \frac{(\boldsymbol{\gamma},\mathbf{A}\boldsymbol{K})}{d^{-1} - (\boldsymbol{\gamma},\mathbf{A}\boldsymbol{\gamma})}\mathbf{A}\boldsymbol{\gamma},\\ \mathbf{A} = \mathrm{diag}(a_1,a_2,a_3), \quad \boldsymbol{K} = \lambda\boldsymbol{\gamma} + ma\Omega\boldsymbol{R},\\ a_i = (I_i + d)^{-1}, \quad d = ma^2.\end{gathered} \tag{9}$$

As a result, we obtain a closed system of equations governing the evolution of $\boldsymbol{K}$ and $\boldsymbol{\gamma}$ in the form

$$\begin{gathered}\dot{\boldsymbol{K}} = \Omega\boldsymbol{\gamma}\times\boldsymbol{K} + (\boldsymbol{K} - d\Omega\boldsymbol{\gamma})\times\boldsymbol{\omega},\\ \dot{\boldsymbol{\gamma}} = \boldsymbol{\gamma}\times\boldsymbol{\omega}.\end{gathered} \tag{10}$$

The first integrals of this system can be represented as

$$\begin{gathered}\boldsymbol{\gamma}^2 = 1, \quad (\boldsymbol{K},\boldsymbol{\gamma}) = \lambda,\\ \tilde{E} = \frac{1}{2}(\boldsymbol{K},\mathbf{A}\boldsymbol{K}) - \frac{\boldsymbol{K}^2}{2d} + \frac{d}{2(1 - d(\boldsymbol{\gamma},\mathbf{A}\boldsymbol{\gamma}))}(\mathbf{A}\boldsymbol{K},\boldsymbol{\gamma})^2.\end{gathered}$$

If the case is dynamically symmetric ($a_1 = a_2$), there exists an additional integral F_3 (linear in $\boldsymbol{K}$):

$$F_3 = \rho\gamma_3(K_1\gamma_1 + K_2\gamma_2) - \rho\left(1 - \gamma_3^2 - \frac{1}{a_1 d}\right)K_3 + \Omega\Psi(\gamma_3),$$

where the function $\Psi(\gamma_3)$ is defined, depending on the moments of inertia. Thus, in the case of a dynamically symmetric ball the system (10) is integrable by quadratures.

References

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