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On a conjecture for trigonometric sums and starlike functions, II

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Abstract

We prove the case $\rho = \frac{1}{4}$ of the following conjecture of Koumandos and Ruscheweyh: let $s_n^\mu(z) := \sum_{k=0}^n \frac{(\mu)_k}{k!} z^k$, and for $\rho \in (0, 1]$ let $\mu^*(\rho)$ be the unique solution of

$$\int_0^{(\rho+1)\pi} \sin(t - \rho\pi) t^{\mu-1} \mathrm{d}t = 0$$

in (0, 1]. Then we have $|\arg[(1-z)^\rho s_n^\mu(z)]| \le \rho\pi/2$ for $0 < \mu \le \mu^*(\rho)$, $n \in \mathbb{N}$ and z in the unit disk of $\mathbb{C}$ and $\mu^*(\rho)$ is the largest number with this property. For the proof of this other new results are required that are of independent interest. For instance, we find the best possible lower bound μ_0 such that the derivative of $x - \frac{\Gamma(x+\mu)}{\Gamma(x+1)} x^{2-\mu}$ is completely monotonic on $(0, \infty)$ for $\mu_0 \le \mu < 1$.

Keywords: Positive trigonometric sums; Inequalities; Gegenbauer polynomials; Starlike functions; Subordination; Completely monotonic functions; Gamma and psi functions

1. Introduction

Let $\mathcal{A}$ be the class of functions that are analytic in the unit disk $\mathbb{D} := \{z : |z| < 1\}$ of the complex plane. For any function $f \in \mathcal{A}$ we will denote by $s_n(f, z)$ the nth partial sum

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of its power series expansion around the origin. If $f(z) = (1-z)^{-\mu}$, $\mu > 0$, we simply put $s_n^\mu(z) := s_n(f, z) = \sum_{k=0}^n \frac{(\mu)_k}{k!} z^k$. Here $(\mu)_k := \mu(\mu+1)\cdots(\mu+k-1)$ is the Pochhammer symbol. For two functions f and g in $\mathcal{A}$, we write $f \prec g$ and say that f is subordinate to g in $\mathbb{D}$ if there is a function w in $\mathcal{A}$ satisfying $|w(z)| \le |z|$, $z \in \mathbb{D}$, such that $f = g \circ w$. This implies in particular that $f(0) = g(0)$ and $f(\mathbb{D}) \subset g(\mathbb{D})$. On the other hand these two conditions are sufficient for $f \prec g$ if g is univalent in $\mathbb{D}$ (cf. [11, p. 35]).

For $\rho \in (0, 1]$ let $\mu^*(\rho)$ be the unique solution in (0,1] (cf. the proof of [9, Lemma 1]) of the equation

$$\int_0^{(\rho+1)\pi} \sin(t - \rho\pi) t^{\mu-1} \mathrm{d}t = 0. \tag{1}$$

In [9] Koumandos and Ruscheweyh proposed the following conjecture.

Conjecture 1. *For $\rho \in (0, 1]$ the number $\mu^*(\rho)$ is equal to the maximal number $\mu(\rho)$ such that for all $n \in \mathbb{N}$ and $0 < \mu \le \mu(\rho)$*

$$(1-z)^\rho s_n^\mu(z) \prec \left(\frac{1+z}{1-z}\right)^\rho. \tag{2}$$

As shown in [9], this conjecture contains the following weaker one.

Conjecture 2. *Let $\rho \in (0, 1]$. Inequality*

$$\mathrm{Re}\left[(1-z)^{2\rho-1} s_n^\mu(z)\right] > 0, \tag{3}$$

holds for all $n \in \mathbb{N}$ and $z \in \mathbb{D}$ when $0 < \mu \le \mu^*(\rho)$ and $\mu^*(\rho)$ is the largest number with this property.

These conjectures were motivated by the results found in [10], where Koumandos and Ruscheweyh proved the special case $\rho = \frac{1}{2}$ of (the then yet unknown) Conjecture 2. In [9] the case $\rho = \frac{1}{2}$ of Conjecture 1 and $\rho = \frac{3}{4}$ of Conjecture 2 were verified. There it is also shown that for $\rho \in (0, 1]$ we have $\mu(\rho) \le \mu^*(\rho)$. Here we will prove the case $\rho = \frac{1}{4}$ of Conjecture 1. The proof of Conjecture 1 for $\rho = \frac{1}{2}$, given in [9] relies on a sharp trigonometric inequality established in [5] that generalizes the celebrated Vietoris' Theorem [14], see also [3].

The main results of this paper are the following.

Theorem 1. *For $\rho = \frac{1}{4}$ the relation (2) holds for all $0 < \mu \le \mu^*\left(\frac{1}{4}\right) = 0.38556655\ldots$ and $\mu^*\left(\frac{1}{4}\right)$ is the largest number with this property.*

The proof of this Theorem relies on a new sharp trigonometric inequality.

Theorem 2. *For $0 < \mu \le \mu^*\left(\frac{1}{4}\right)$ we have*

$$U_n(\theta) := \sum_{k=0}^n \frac{(\mu)_k}{k!} \cos\left[\left(2k + \frac{1}{4}\right)\theta + \frac{\pi}{4}\right] > 0 \tag{4}$$

for $n \in \mathbb{N}$ and $0 \le \theta < \pi$ and $\mu^*\left(\frac{1}{4}\right)$ is the largest number with this property.

Theorem 1 has some interesting applications concerning starlike functions. Recall that the class $\mathcal{S}_\lambda$ of functions starlike of order λ, $\lambda < 1$, consists of those functions $f \in \mathcal{A}$ that satisfy $f(0) = f'(0) - 1 = 0$ and $\mathrm{Re}(zf'(z)/f(z)) > \lambda$ in $\mathbb{D}$. It is easy to check that $z/(1-z)^\mu \in \mathcal{S}_{1-\frac{\mu}{2}}$. Recall further that for $f(z) = \sum_{k=0}^\infty a_k z^k$ and $g(z) = \sum_{k=0}^\infty b_k z^k$ the Hadamard product or convolution $f * g$ is defined by $(f * g)(z) := \sum_{k=0}^\infty a_k b_k z^k$. As in [9] we denote by $\Phi_{\rho,\mu}$ the uniquely determined function f in $\mathcal{A}$ that satisfies

$$\frac{z}{(1-z)^\mu} * f(z) = \frac{z}{(1-z)^\rho}.$$

Our Theorem 1 implies the following two results.

Theorem 3. *If* $0 < \mu \le \mu^*\left(\frac{1}{4}\right)$*, then we have*

$$\frac{s_n(f,z)}{\Phi_{\frac{1}{4},\mu} * f} \prec \left(\frac{1+z}{1-z}\right)^{\frac{1}{4}}$$

for all $f \in \mathcal{S}_{1-\mu/2}$ *and* $n \in \mathbb{N}$.

Theorem 4. *We have*

$$\frac{1}{z} s_n(f,z) \prec \left(\frac{1+z}{1-z}\right)^{\frac{1}{2}} \tag{5}$$

for all $f \in \mathcal{S}_{1-\mu/2}$ *and* $n \in \mathbb{N}$ *if and only if* $0 < \mu \le \mu^*\left(\frac{1}{4}\right)$.

Note that (5) is equivalent to $\mathrm{Re}(s_n(f,z)/z)^2 > 0$ for $z \in \mathbb{D}$. Our proof of Theorem 1 will show that for the special case $f = z/(1-z)^\mu$ we even have

$$\mathrm{Re}\left(s_n^\mu(z)\right)^2 > 0 \quad \text{for } z \in \overline{\mathbb{D}}. \tag{6}$$

Let $C_k^\lambda(x)$ be the Gegenbauer polynomial of degree k and order $\lambda > 0$, defined by the generating function

$$G_\lambda(z,x) := (1 - 2xz + z^2)^{-\lambda} = \sum_{k=0}^\infty C_k^\lambda(x) z^k, \quad x \in [-1,1].$$

It is easily seen that $zG_\lambda(\cdot, x) \in \mathcal{S}_{1-\lambda}$ and so it is clear that Theorem 4 implies the following.

Corollary 1. *The inequality*

$$\left|\arg \sum_{k=0}^n C_k^\lambda(x) z^k\right| < \frac{\pi}{4}, \quad z \in \mathbb{D}, \tag{7}$$

holds for all $-1 < x < 1$ *precisely when* $0 < \lambda \le \frac{1}{2}\mu^*\left(\frac{1}{4}\right) = 0.1927\ldots$.

For our proof of Theorem 2 it will be necessary to show that

$$0 < \xi(x) := x - \frac{\Gamma(x+\mu)}{\Gamma(x+1)} x^{2-\mu} < \frac{\mu(1-\mu)}{2}, \quad x > 0, \tag{8}$$

when $\mu = \mu^*\left(\frac{1}{4}\right) = 0.385\ldots$. Here $\Gamma(x)$ is Euler's gamma function. Inequality (8) was shown in [6, Thm. 1] for $\frac{1}{2} \le \mu < 1$ and then, in the special case $\mu = \mu^*\left(\frac{3}{4}\right) = 0.907\ldots$, applied in the proof of the case $\rho = \frac{3}{4}$ of Conjecture 2 in [9]. In fact, in [6, Thm. 1] a much stronger result is shown, namely that ξ' is completely monotonic on $(0, \infty)$ for $\frac{1}{2} \le \mu < 1$. Recall that a function $f : (0, \infty) \to \mathbb{R}$ is called completely monotonic if it has derivatives of all orders and satisfies

$$(-1)^n f^{(n)}(x) \ge 0 \quad \text{for all } x > 0 \text{ and } n \in \mathbb{N}. \tag{9}$$

It is known that if a non-constant function f is completely monotonic then strict inequality holds in (9) (cf. [4] or [13]). A characterization of completely monotonic functions is given by Bernstein's theorem, see [15, p. 161], which states that f is completely monotonic on $(0, \infty)$ if and only if

$$f(x) = \int_0^\infty \mathrm{e}^{-xt}\,\mathrm{d}\,m(t),$$

where m is a non-negative measure on $[0, \infty)$ such that the integral converges for all $x > 0$.

Here we will refine some techniques developed in [6] in order to obtain a best possible extension of [6, Thm. 1(i)] that will in particular imply (8) for $\mu = \mu^*\left(\frac{1}{4}\right)$.

Theorem 5. *The function $\xi'(x)$ is completely monotonic on $(0, \infty)$, when $\frac{1}{3} \le \mu < 1$. The lower bound 1/3 is best possible. In particular, the function $\xi(x)$ is strictly increasing and concave on $(0, \infty)$ and the inequality (8) holds for all $x > 0$, for this range of μ.*

For an extensive bibliography regarding completely monotonic functions and inequalities involving the gamma function we refer to the recent paper [6].

In the next section we will prove Theorem 2. In Section 3 we will show how this Theorem implies Theorems 1, 3 and 4. In Section 4 we will present the proof of Theorem 5.

2. Proof of Theorem 2

First, note that, using summation by parts, it is easy to see that it will be enough to show (4) for $\mu = \mu^*\left(\frac{1}{4}\right)$ and that an argument similar to the one given in the proof of [9, Lemma 1] shows that the upper bound $\mu^*\left(\frac{1}{4}\right)$ is sharp for the positivity of the trigonometric sums $U_n(\theta)$ in $[0, \pi)$.

Next, recall the well-known identity

$$\mathrm{e}^{\mathrm{i}c}\sum_{k=0}^{n}\mathrm{e}^{\mathrm{i}k\theta} = \mathrm{e}^{\mathrm{i}(c+n\theta/2)}\frac{\sin\frac{n+1}{2}\theta}{\sin\frac{\theta}{2}}, \quad n \in \mathbb{N}, \tag{10}$$

which holds for all $\theta \in \mathbb{R}$ for which $\sin\frac{\theta}{2}$ does not vanish and every $c \in \mathbb{R}$ (which might even depend on θ).

Further, observe that if we set

$$\Delta_n = \frac{1}{n^{1-\mu}}\left(\frac{1}{\Gamma(\mu)} - \frac{(\mu)_n}{n!\,n^{\mu-1}}\right)$$

for $n \in \mathbb{N}$ and $\frac{1}{3} \le \mu < 1$, then, because of Theorem 5, it follows as in the proof of [9, Prop. 4] that

$$n\Delta_n < \frac{1}{\Gamma(\mu)} \frac{\mu(1-\mu)}{2} \frac{1}{n^{1-\mu}}.$$

A small modification of the proof of this Proposition thus yields that

$$\left| \sum_{k=n+1}^{\infty} \Delta_k e^{2ik\theta} \right| \le \frac{\mu(1-\mu)}{2\sin a} \frac{1}{\Gamma(\mu)} \frac{1}{(n+1)^{2-\mu}} \tag{11}$$

for $0 < a < \theta < \frac{\pi}{2}$, $n \in \mathbb{N}$ and $\frac{1}{3} \le \mu < 1$. We therefore obtain the following inequality, which for $\mu = \mu^*\left(\frac{3}{4}\right)$ was the crucial result in the proof of [9, (3.3)] and will also play a crucial role in the proof of (4) that is presented here.

Lemma 1. *Let $c(\theta)$ be a real integrable function depending on $\theta \in \mathbb{R}$, $\frac{1}{3} \le \mu < 1$, $0 < a < b \le \frac{\pi}{2}$ and $d_k = \frac{(\mu)_k}{k!}$, $k \in \mathbb{N}_0$. Then for $f(\theta) = \sin\theta$ or $f(\theta) = \cos\theta$ we have for all $a \le \theta \le b$ and $n \in \mathbb{N}$*

$$2^{\mu}\theta^{\mu-1}\Gamma(\mu) \sum_{k=0}^{n} d_k f(2k\theta + c(\theta))$$
$$> \kappa_n(\theta) - a_n - b_n - c_n + \Gamma(\mu)\left(2q(\theta)\frac{\sin\frac{\mu\theta}{2}}{\sin\theta} - r(\theta)s(\theta)\right), \tag{12}$$

where

$$a_n := \frac{b}{\sin b} \frac{1-\mu}{(2an)^{1-\mu}} \frac{1}{4n}, \qquad b_n := \frac{b^2}{\sin^2 b} \frac{1-\mu}{(2an)^{1-\mu}} \frac{1}{3n}, \qquad c_n := \frac{\pi\mu(1-\mu)}{(2a(n+1))^{2-\mu}},$$
$$q(\theta) := f\left(\frac{\mu}{2}(\pi - \theta) + c(\theta) - \frac{\pi}{2}\right), \qquad r(\theta) := f\left(\frac{\mu}{2}(\pi - 2\theta) + c(\theta)\right),$$
$$s(\theta) := \frac{1}{\sin\theta}\left[1 - \left(\frac{\sin\theta}{\theta}\right)^{1-\mu}\right]$$

and

$$\kappa_n(\theta) := \frac{1}{\sin\theta} \int_0^{(2n+1)\theta} \frac{f(t + c(\theta))}{t^{1-\mu}} dt.$$

The function $s(\theta)$ is positive and increasing on $(0, \pi)$.

Proof. By [9, (3.8)] we have

$$\sum_{k=0}^{n} d_k e^{2ik\theta} = F(\theta) + \frac{1}{\Gamma(\mu)} \frac{\theta}{\sin\theta} \frac{1}{(2\theta)^{\mu}} \int_0^{(2n+1)\theta} \frac{e^{it}}{t^{1-\mu}} dt$$
$$- \frac{1}{\Gamma(\mu)} \frac{\theta}{\sin\theta} \left\{ \sum_{k=n+1}^{\infty} A_k(\theta) + \sum_{k=n+1}^{\infty} B_k(\theta) \right\} + \sum_{k=n+1}^{\infty} \Delta_k e^{2ik\theta}, \tag{13}$$

with

$$F(\theta) := \sum_{k=0}^{\infty} d_k e^{2ik\theta} - \frac{\theta}{\sin\theta}\frac{e^{i\mu\frac{\pi}{2}}}{(2\theta)^{\mu}}$$

and where

$$\left|\sum_{k=n+1}^{\infty} A_k(\theta)\right| < \frac{1-\mu}{8}\frac{1}{n^{2-\mu}},$$

$$\left|\sum_{k=n+1}^{\infty} B_k(\theta)\right| < \frac{\theta}{\sin\theta}\frac{1-\mu}{6}\frac{1}{n^{2-\mu}}$$

for $\theta \in \mathbb{R}$ by [9, Prop. 1]. As in the proof of [9, Prop. 2] it follows that

$$F(\theta) = \frac{\theta^{1-\mu}}{2^{\mu}}\frac{e^{i\mu\frac{\pi}{2}}}{\sin\theta}\left\{\left(e^{-i\mu\theta}-1\right) - \left[1-\left(\frac{\sin\theta}{\theta}\right)^{1-\mu}\right]e^{-i\mu\theta}\right\}.$$

Hence

$$2^{\mu}\theta^{\mu-1}F(\theta)e^{ic(\theta)} = 2e^{i(\frac{\mu}{2}(\pi-\theta)+c(\theta)-\frac{\pi}{2})}\frac{\sin\frac{\mu\theta}{2}}{\sin\theta} - s(\theta)e^{i(\frac{\mu}{2}(\pi-2\theta)+c(\theta))}$$

and thus (12) follows from (11) and the well-known inequality $\sin x > \frac{2}{\pi}x$ for $0 < x < \frac{\pi}{2}$. It is straightforward to check that $s(\theta)$ is positive and increasing on $(0, \pi)$. □

2.1. The cases $n = 1, 2$ and $\theta \in [0, \frac{\pi}{4n+1}] \cup [\pi - \frac{\pi}{n+1}, \pi)$ of (4)

For the rest of Section 2 set $\mu := \mu^*\left(\frac{1}{4}\right)$ and $d_k := \frac{(\mu)_k}{k!}$, $k \in \mathbb{N}_0$. Observe that

$$W_n(\theta) := U_n(\pi-\theta) = \sum_{k=0}^{n} d_k \sin\left(2k+\frac{1}{4}\right)\theta. \tag{14}$$

We obviously have $U_n(0) > 0$ for all $n \in \mathbb{N}$ and a summation by parts, together with (10), shows that $U_n(\theta) > 0$ and $W_n(\theta) > 0$ for $0 < \theta \le \frac{\pi}{4n+1}$ and $0 < \theta \le \frac{\pi}{n+1}$, respectively. Because of (14), this shows (4) for $\theta \in [0, \frac{\pi}{4n+1}] \cup [\pi - \frac{\pi}{n+1}, \pi)$, $n \in \mathbb{N}$.

Since $\mu^*(\frac{1}{4}) < \frac{2}{5}$, it follows from (14) and a summation by parts that it will be sufficient to show that

$$w_n(\theta) := \sum_{k=0}^{n} \frac{\left(\frac{2}{5}\right)_k}{k!} \sin\left(2k+\frac{1}{4}\right)\theta > 0 \quad \text{for } 0 \le \theta < \pi \text{ and } n = 1, 2,$$

in order to prove (4) for $n = 1, 2$. A straightforward calculation gives $w_n(\theta) = \sin\frac{\theta}{4}\, p_n(\cos\frac{\theta}{2})$, $n = 1, 2$, where

$$p_1(x) = \frac{1}{5}\left(7 - 12x - 4x^2 + 28x^3 + 4x^4\right)$$

$$p_2(x) = \frac{2}{25}\left(21 - 72x - 10x^2 + 539x^3 - 445x^4 - 574x^5 + 448x^6 + 203x^7 + 7x^8\right).$$

By the method of Sturm sequences we see that $p_n(x)$ does not vanish in $(0, 1)$ when $n = 1, 2$. Clearly $p_n(0) > 0$ and thus the cases $n = 1, 2$ of (4) are proven.

2.2. The case $\frac{5\pi}{8} \le \theta < \pi - \frac{\pi}{n+1}$ of (4)

Because of (14), the proof of this case will be completed if we can show that $W_n(\theta) > 0$ for $\frac{\pi}{n+1} < \theta \le \frac{3\pi}{8}$. In order to do this we will apply Lemma 1 with the parameters $f(\theta) = \sin\theta$, $c(\theta) = \frac{\theta}{4}$, $a = \frac{\pi}{n+1}$ and $b = \frac{3\pi}{8}$ $(n \ge 3)$.

In this case

$$\kappa_n(\theta) = \frac{\cos\frac{\theta}{4}}{\sin\theta}\int_0^{(2n+1)\theta}\frac{\sin t}{t^{1-\mu}}\mathrm{d}t + \frac{\sin\frac{\theta}{4}}{\sin\theta}\int_0^{(2n+1)\theta}\frac{\cos t}{t^{1-\mu}}\mathrm{d}t.$$

It is immediately clear that $S(x) := \int_0^x t^{\mu-1}\sin t\,\mathrm{d}t$ is positive for all $x \in \mathbb{R}$ and it is shown in [10, Section 2] that the same holds for the integral $C(x) := \int_0^x t^{\mu-1}\cos t\,\mathrm{d}t$. Therefore, since it is readily verified that $\cos\frac{\theta}{4}/\sin\theta$ and $-\sin\frac{\theta}{4}/\sin\theta$ are decreasing on $(0, \frac{\pi}{2}]$, we obtain that for $\frac{\pi}{n+1} < \theta \le \frac{3\pi}{8}$

$$\begin{aligned}\kappa_n(\theta) &\ge \frac{\cos\frac{3\pi}{32}}{\sin\frac{3\pi}{8}}\int_0^{(2n+1)\theta}\frac{\sin t}{t^{1-\mu}}\mathrm{d}t + \frac{1}{4}\int_0^{(2n+1)\theta}\frac{\cos t}{t^{1-\mu}}\mathrm{d}t \\ &\ge \frac{\cos\frac{3\pi}{32}}{\sin\frac{3\pi}{8}}\int_0^{2\pi}\frac{\sin t}{t^{1-\mu}}\mathrm{d}t + \frac{1}{4}\int_0^{\frac{7\pi}{4}}\frac{\cos t}{t^{1-\mu}}\mathrm{d}t, \qquad (15)\end{aligned}$$

where the latter inequality is obtained by minimizing $S(x)$ and $C(x)$ over $x \ge \frac{7\pi}{4}$.

Furthermore, it is easy to see that, for $f(\theta)$ and $c(\theta)$ as defined above, the functions $-q(\theta)$ and $r(\theta)$ are positive and decreasing on $(0, \frac{3\pi}{8}]$. Since $\sin\frac{\mu\theta}{2}/\sin\theta$ is increasing on this interval, we obtain that for $0 < \theta \le \frac{3\pi}{8}$

$$\Gamma(\mu)\left(2q(\theta)\frac{\sin\frac{\mu\theta}{2}}{\sin\theta} - r(\theta)s(\theta)\right) \ge \Gamma(\mu)\left(2q(0)\frac{\sin\frac{3\mu\pi}{16}}{\sin\frac{3\pi}{8}} - r(0)s\left(\frac{3\pi}{8}\right)\right). \qquad (16)$$

Finally, for a and b as defined above, it follows that for $n \ge 3$ the coefficients a_n, b_n and c_n are smaller than

$$\frac{3\pi}{8\sin\frac{3\pi}{8}}\frac{1-\mu}{12\left(\frac{3\pi}{2}\right)^{1-\mu}}, \quad \left(\frac{3\pi}{8\sin\frac{3\pi}{8}}\right)^2\frac{1-\mu}{9\left(\frac{3\pi}{2}\right)^{1-\mu}} \quad \text{and} \quad \frac{\pi\mu(1-\mu)}{(2\pi)^{2-\mu}}, \qquad (17)$$

respectively.

Lemma 1, together with (15)–(17), now yields that for $\frac{\pi}{n+1} < \theta \le \frac{3\pi}{8}$ and $n \ge 2$

$$2^\mu\theta^{\mu-1}\Gamma(\mu)W_n(\theta) > 0.2109\ldots.$$

2.3. The case $\frac{\pi}{4n+1} < \theta \le \frac{\pi}{3}$ of (4)

In order to prove this case of (4) we will apply Lemma 1 on the three intervals $I_1 := (\frac{\pi}{4n+1}, \frac{\pi}{2n+1}]$, $I_2 := (\frac{\pi}{2n+1}, \frac{\pi}{n+2}]$ and $I_3 := (\frac{\pi}{n+2}, \frac{\pi}{3}]$ $(n \ge 3)$.

To this end, observe that with $f(\theta) = \cos\theta$ and $c(\theta) = \frac{\pi+\theta}{4}$ we have

$$\kappa_n(\theta) = \frac{\sin\frac{\theta}{4}}{\sin\theta}\int_0^{(2n+1)\theta}\frac{\sin\left(t - \frac{3\pi}{4}\right)}{t^{1-\mu}}\mathrm{d}t - \frac{\cos\frac{\theta}{4}}{\sin\theta}\int_0^{(2n+1)\theta}\frac{\sin\left(t - \frac{\pi}{4}\right)}{t^{1-\mu}}\mathrm{d}t.$$

As described in the proof of [9, Lemma 1] it follows from the definition of $\mu^*\left(\frac{1}{4}\right)$ and $\mu^*\left(\frac{3}{4}\right)$ that $\int_0^x t^{1-\mu}\sin\left(t-\frac{3\pi}{4}\right)\mathrm{d}t$ and $\int_0^x t^{1-\mu}\sin\left(t-\frac{\pi}{4}\right)\mathrm{d}t$ are non-positive for all $x > 0$. As noted before, $\cos\frac{\theta}{4}/\sin\theta$ and $-\sin\frac{\theta}{4}/\sin\theta$ are decreasing on $(0,\frac{\pi}{2})$ and thus we obtain that for $0<\theta\le b\le\frac{\pi}{2}$

$$\kappa_n(\theta)\ge\frac{1}{\sin b}\int_0^{(2n+1)\theta}\frac{\cos\left(t+\frac{b+\pi}{4}\right)}{t^{1-\mu}}\mathrm{d}t.$$

For $n\ge 3$ this means that in I_1

$$\kappa_n(\theta)\ge\frac{1}{\sin\frac{\pi}{7}}\int_0^{(2n+1)\theta}\frac{\cos\left(t+\frac{2\pi}{7}\right)}{t^{1-\mu}}\mathrm{d}t\ge\frac{1}{\sin\frac{\pi}{7}}\int_0^{\pi}\frac{\cos\left(t+\frac{2\pi}{7}\right)}{t^{1-\mu}}\mathrm{d}t, \tag{18}$$

where the latter inequality is obtained by minimizing the integral over $0<(2n+1)\theta\le\pi$. Likewise, by minimizing the respective integrals over $\pi<(2n+1)\theta$ and $\frac{7\pi}{5}<(2n+1)\theta$, we find that for $n\ge 3$

$$\kappa_n(\theta)\ge\frac{1}{\sin\frac{\pi}{5}}\int_0^{\frac{6\pi}{5}}\frac{\cos\left(t+\frac{3\pi}{10}\right)}{t^{1-\mu}}\mathrm{d}t \quad\text{and}\quad \kappa_n(\theta)\ge\frac{1}{\sin\frac{\pi}{3}}\int_0^{\frac{7\pi}{5}}\frac{\cos\left(t+\frac{\pi}{3}\right)}{t^{1-\mu}}\mathrm{d}t \tag{19}$$

in I_2 and I_3, respectively.

Furthermore, it is straightforward to check that with $f(\theta)$ and $c(\theta)$ as defined above the functions $q(\theta)$ and $r(\theta)$ are positive and increasing on $(0,\frac{\pi}{2})$. Therefore, using the inequality $2\sin\frac{\mu\theta}{2}/\sin\theta\ge\mu$ $(0<\theta<\pi)$, we get that for $0<\theta\le b\le\frac{\pi}{2}$

$$2q(\theta)\frac{\sin\frac{\mu\theta}{2}}{\sin\theta}-r(\theta)s(\theta)\ge\mu q(0)-r(b)s(b). \tag{20}$$

Finally, if $a=\frac{\pi}{4n+1}$ and $b=\frac{\pi}{2n+1}$, then, for $n\ge 3$, we have $b\le\frac{\pi}{7}$, $2an\ge\frac{6\pi}{13}$ and $2a(n+1)\ge\frac{\pi}{2}$ and by Lemma 1 this, together with (18) and (20), shows that

$$\begin{aligned}2^{\mu}\theta^{\mu-1}\Gamma(\mu)U_n(\theta)\ge{}&\frac{1}{\sin\frac{\pi}{7}}\int_0^{\pi}\frac{\cos\left(t+\frac{2\pi}{7}\right)}{t^{1-\mu}}\mathrm{d}t\\&-\frac{\pi}{7\sin\frac{\pi}{7}}\frac{1-\mu}{12\left(\frac{6\pi}{13}\right)^{1-\mu}}-\left(\frac{\pi}{7\sin\frac{\pi}{7}}\right)^2\frac{1-\mu}{9\left(\frac{6\pi}{13}\right)^{1-\mu}}-\frac{\pi\mu(1-\mu)}{\left(\frac{\pi}{2}\right)^{2-\mu}}\\&+\Gamma(\mu)\left(\mu q(0)-r\left(\frac{\pi}{7}\right)s\left(\frac{\pi}{7}\right)\right)=0.0214\ldots\end{aligned}$$

for $\theta\in I_1$ and $n\ge 3$.

Likewise, if $a=\frac{\pi}{2n+1}$, $b=\frac{\pi}{n+2}$ and $n\ge 3$, then $b\le\frac{\pi}{5}$, $2an\ge\frac{6\pi}{7}$ and $2a(n+1)\ge\pi$, and if $a=\frac{\pi}{n+2}$, $b=\frac{\pi}{3}$ and $n\ge 3$, then $2an\ge\frac{6\pi}{5}$ and $2a(n+1)\ge\frac{8\pi}{5}$. Similar reasoning as before now shows that, for $n\ge 3$, $2^{\mu}\theta^{\mu-1}\Gamma(\mu)U_n(\theta)$ is larger than $0.0157\ldots$ and $0.0670\ldots$ in I_2 and I_3, respectively.

2.4. The case $\frac{\pi}{3} < \theta < \frac{5\pi}{8}$ of (4)

In order to prove this case of (4), recall the well-known inequality, see [3, Lemma 3],

$$\left|\sum_{k=n}^{\infty} a_k e^{i2k\theta}\right| \le \frac{a_n}{\sin\theta}, \quad (0 < \theta < \pi)$$

which holds for every positive and decreasing sequence $(a_k)_{k\in\mathbb{N}_0}$. Therefore, since

$$U_n(\theta) \to \frac{\cos\frac{1}{4}((1-4\mu)\theta + (1+2\mu)\pi)}{(2\sin\theta)^{\mu}} \quad (0 < \theta < \pi)$$

as $n \to \infty$ and since the sequence $(d_k)_k$ is decreasing, we obtain for $n \ge 3$ and $0 < \theta < \pi$

$$U_n(\theta) \ge \frac{\cos\frac{1}{4}((1-4\mu)\theta + (1+2\mu)\pi)}{(2\sin\theta)^{\mu}} - \frac{d_4}{\sin\theta}.$$

It is readily verified that the function $\cos\frac{1}{4}((1-4\mu)\theta + (1+2\mu)\pi)$ is positive and increasing on $(\frac{\pi}{3}, \frac{5\pi}{8})$ and so it follows from the above that for $n \ge 3$ and $\frac{\pi}{3} < \theta < \frac{5\pi}{8}$

$$U_n(\theta) \ge \frac{\cos\frac{\pi}{6}(2+\mu)}{2^{\mu}} - \frac{d_4}{\sin\frac{\pi}{3}} = 0.0344\ldots.$$

3. Proof of Theorems 1, 3 and 4

Setting $h(z) = (1+z)/(1-z)$, the function h^{ρ} is univalent in $\mathbb{D}$ for all $0 < \rho \le 2$. Therefore, if (2) holds for some $\mu \in (0, 1]$ and $\rho \in (0, 1]$, then the principle of subordination yields that $s_n^{\mu} \prec h^{2\rho}$. Since for $0 < \rho \le \frac{1}{2}$

$$|\arg h^{2\rho}| < \rho\pi \quad \text{and} \quad |\arg(1-z)^{2\rho-1}| \le \frac{\pi}{2} - \rho\pi \quad \text{in } \mathbb{D},$$

this, in turn, implies the validity of (3) for the same μ and ρ if $0 < \rho \le \frac{1}{2}$. Since $(1-z)^{-\mu} \in \mathcal{S}_{1-\mu/2}$ and since it is shown in [9] that (3) cannot hold for $\mu > \mu^*(\rho)$, it thus follows that in Theorems 1 and 4 the upper bound $\mu^*\left(\frac{1}{4}\right)$ is sharp.

By an application of the convolution theory for starlike functions (see, for instance, [12, p. 55]) it has been shown in [9] that the validity of Conjecture 1 for a $\rho \in (0, 1]$ implies that for all $f \in \mathcal{S}_{1-\mu/2}$ with $0 < \mu \le \mu^*(\rho)$ and all $n \in \mathbb{N}$

$$\frac{s_n(f, z)}{\Phi_{\rho,\mu} * f} \prec \left(\frac{1+z}{1-z}\right)^{\rho}$$

and that this, in turn, implies that for all $f \in \mathcal{S}_{1-\mu/2}$ with $0 < \mu \le \mu^*(\rho)$ and all $n \in \mathbb{N}$

$$\frac{1}{z} s_n(f, z) \prec \left(\frac{1+z}{1-z}\right)^{2\rho}.$$

Hence, Theorems 3 and 4 follow immediately from Theorem 1.

Now, in order to complete the proof of Theorem 1, it only remains to be shown that (2) holds for $\rho = \frac{1}{4}$, $0 < \mu \le \mu^*\left(\frac{1}{4}\right)$ and $n \in \mathbb{N}$ and because of the minimum principle for harmonic functions this will be done once we have proven that

$$\operatorname{Re}\left[(1-e^{2i\theta})\left(s_n^{\mu}(e^{2i\theta})\right)^4\right] > 0 \tag{21}$$

for $0 < \mu \le \mu^*\left(\frac{1}{4}\right)$, $n \in \mathbb{N}$ and $0 < \theta < \pi$.

To this end, note first that

$$T_n(\theta) := \sum_{k=0}^{n} \frac{(\mu)_k}{k!} \cos\left(2k + \frac{1}{4}\right)\theta > 0 \quad \text{for } 0 \le \theta < 2\pi, \quad 0 < \mu \le \mu^*\left(\frac{1}{4}\right) \tag{22}$$

and $n \in \mathbb{N}$. For $0 \le \theta < \pi$ this is a consequence of [5, (6.4)], while for $\pi \le \theta < 2\pi$ it follows from the fact that $T_n(2\pi - \theta) = U_n(\pi - \theta) > 0$ (this latter relation also shows the sharpness of the bound $\mu^*\left(\frac{1}{4}\right)$ for the positivity of the T_n in $[0, 2\pi)$). We also have

$$V_n(\theta) := \sum_{k=0}^{n} \frac{(\mu)_k}{k!} \cos\left(2k\theta + \frac{\pi}{4}\right) > 0 \quad \text{for } \theta \in \mathbb{R},\ 0 < \mu \le \mu^*\left(\frac{1}{4}\right) \tag{23}$$

and $n \in \mathbb{N}$. Applying elementary trigonometric identities (e.t.i.) we get

$$V_n(\theta) = \cos\frac{\theta}{4}\, U_n(\theta) + \sin\frac{\theta}{4}\, T_n(\pi - \theta),$$

and thus (23) follows from the positivity of the U_n and T_n in $[0, \pi)$. The sharpness of $\mu^*\left(\frac{1}{4}\right)$ for the positivity of the V_n in $\mathbb{R}$ can be seen by an asymptotic analysis similar to the one presented in the proof of [9, Lemma 1].

Now, in order to prove (21), set $s_n^{\mu}(e^{2i\theta}) =: C_n(\theta) + iS_n(\theta)$, i.e.

$$C_n(\theta) := \sum_{k=0}^{n} \frac{(\mu)_k}{k!} \cos 2k\theta \quad \text{and} \quad S_n(\theta) := \sum_{k=0}^{n} \frac{(\mu)_k}{k!} \sin 2k\theta,$$

for $0 < \mu \le 1$, $\theta \in \mathbb{R}$ and $n \in \mathbb{N}$. Then e.t.i. and (23) show that

$$\begin{aligned} X_n(\theta) := C_n(\theta)^2 - S_n(\theta)^2 &= (C_n(\theta) + S_n(\theta))(C_n(\theta) - S_n(\theta)) \\ &= 2\, V_n(\theta) \cdot V_n(\pi - \theta) > 0 \end{aligned} \tag{24}$$

for $\theta \in \mathbb{R}$, $0 < \mu \le \mu^*\left(\frac{1}{4}\right)$ and $n \in \mathbb{N}$ (in particular, since $\operatorname{Re}(s_n^{\mu}(e^{2i\theta}))^2 = C_n^2(\theta) - S_n^2(\theta)$, we see that (6) holds). Setting $Z_n(\theta) := C_n(\theta)S_n(\theta)/X_n(\theta)$ we find that the left-hand side of (21) is equal to

$$2\sin\theta\, X_n^2(\theta)\, p(Z_n(\theta)), \quad \text{where } p(x) := -4x^2\sin\theta + 4x\cos\theta + \sin\theta.$$

For $0 < \theta < \pi$ we have $p(x) > 0$ if, and only if,

$$-1 + \cos\theta < 2x\sin\theta < 1 + \cos\theta,$$

and thus, since $\sin(\pi - \theta)Z_n(\pi - \theta) = -\sin\theta\, Z_n(\theta)$ and $\cos(\pi - \theta) = -\cos\theta$, it follows that (21) holds if, and only if,

$$2\sin\theta\, Z_n(\theta) < 1 + \cos\theta \quad \text{for } 0 < \theta < \pi. \tag{25}$$

Because of (24) the inequality (25) is equivalent to

$$\begin{aligned} 0 &< (1+\cos\theta)(C_n^2(\theta) - S_n^2(\theta)) - 2\sin\theta\, C_n(\theta) S_n(\theta) \\ &= \mathrm{Re}\left[(1+\mathrm{e}^{\mathrm{i}\theta})\left(s_n^{\mu}(\mathrm{e}^{2\mathrm{i}\theta})\right)^2\right] = 2\cos\frac{\theta}{2}\,\mathrm{Re}\left[\left(\mathrm{e}^{\mathrm{i}\theta/4} s_n^{\mu}(\mathrm{e}^{2\mathrm{i}\theta})\right)^2\right] \end{aligned}$$

for $0 < \theta < \pi$. E.t.i. show that

$$\mathrm{Re}\left[\left(\mathrm{e}^{\mathrm{i}\theta/4} s_n^{\mu}(\mathrm{e}^{2\mathrm{i}\theta})\right)^2\right] = 2\,U_n(\theta)\cdot T_n(\pi-\theta)$$

for $0 < \theta < \pi$. By Theorem 2 and (22) the product $U_n(\theta)T_n(\pi-\theta)$ is positive for $0 < \theta < \pi$, $0 < \mu \le \mu^*\left(\frac{1}{4}\right)$ and $n \in \mathbb{N}$. Hence, (21) holds for the required set of parameters. The proof of Theorem 1 is thus complete.

4. Proof of Theorem 5

We will first show that $\xi'(x)$ cannot be completely monotonic on $(0,\infty)$ when $0 < \mu < 1/3$. To see this, we observe that

$$\xi'(x) = 1 - \frac{\Gamma(x+\mu)}{\Gamma(x+1)}\, x^{1-\mu}\Big(x\left(\psi(x+\mu) - \psi(x+1)\right) + 2 - \mu\Big), \tag{26}$$

where $\psi(x) := \Gamma'(x)/\Gamma(x)$, and therefore $\xi'(0) = 1$. On the other hand, using the asymptotic formulae

$$\frac{\Gamma(x+\mu)}{\Gamma(x+1)}\, x^{1-\mu} = 1 - \frac{\mu(1-\mu)}{2\,x} + \frac{\mu(1-\mu)(2-\mu)(3\mu-1)}{24\,x^2} + O\left(\frac{1}{x^3}\right), \tag{27}$$

$$\psi(x+\mu) - \psi(x+1) = \frac{\mu-1}{x} + \frac{\mu(1-\mu)}{2\,x^2} + \frac{\mu(1-\mu)(1-2\mu)}{6\,x^3} + O\left(\frac{1}{x^4}\right), \tag{28}$$

as $x \to \infty$ (cf. [1, p. 257]; the second formula follows by considering $\frac{\mathrm{d}}{\mathrm{d}x}\log(\Gamma(x+\mu)/\Gamma(x+1))$ and applying the first one), we obtain

$$\lim_{x\to\infty}\left(x^2\,\xi'(x)\right) = \frac{\mu(1-\mu)(2-\mu)(3\mu-1)}{24}, \tag{29}$$

which is negative when $0 < \mu < 1/3$. Thus, for this range of μ, the function $\xi'(x)$ changes sign on $(0,\infty)$.

Now, for $0 < \alpha < 1$, consider the function

$$f(x) := \frac{\mathrm{e}^{\alpha x} - 1}{\mathrm{e}^{x} - 1}.$$

The convexity of this function on $(0,\infty)$ was crucial in the proof of [6, Thm. 1(i)]. In fact, it is proved in [6] that the function $f(x)$ is strictly decreasing on $(0,\infty)$ when $0 < \alpha < 1$. For those α it is convex on $(0,\infty)$ if and only if $0 < \alpha \le 1/2$. See [6, Lemma 1, (1)] (to see that the condition $0 < \alpha \le 1/2$ is necessary, observe that, by (30) and (33) below, $f''(0) = \alpha(\alpha-1)(2\alpha-1)$). However, in the case $1/2 < \alpha \le 2/3$ we have the following:

Lemma 2. *Suppose that* $1/2 < \alpha \le 2/3$. *Then*

(i) $f''(x) > 0$, *for* $x \in [1, \infty)$.
(ii) $f'''(x) > 0$, *for* $x \in [0, 1]$.

Proof. We have

$$f''(x) = \frac{1}{(e^x - 1)^3} \left[(\alpha - 1)^2 e^{(\alpha+2)x} + (-2\alpha^2 + 2\alpha + 1)e^{(\alpha+1)x} + \alpha^2 e^{\alpha x} - e^{2x} - e^x\right].$$

Thus

$$f''(x) = \left(\frac{x}{e^x - 1}\right)^3 \sum_{n=3}^{\infty} \frac{P_n(\alpha)}{n!} x^{n-3}, \tag{30}$$

where

$$P_n(\alpha) := (\alpha - 1)^2 (\alpha + 2)^n + (-2\alpha^2 + 2\alpha + 1) (\alpha + 1)^n + \alpha^{n+2} - 2^n - 1, \quad n \geq 3.$$

(cf. [6, (2.1)]). We first prove that

$$P_n(\alpha) > 0, \quad \text{for all } n \geq 5 \text{ when } 1/2 \leq \alpha \leq 2/3. \tag{31}$$

Clearly,

$$P_n(\alpha) > Q_n(\alpha),$$

where

$$Q_n(\alpha) := (\alpha - 1)^2 (\alpha + 2)^n - 2^n.$$

We shall show that

$$Q_n(\alpha) > 0, \quad \text{for all } n \geq 8,\ 1/2 \leq \alpha \leq 2/3. \tag{32}$$

First,

$$Q_n(\alpha) \geq (\alpha - 1)^2 \left(\frac{5}{2}\right)^n - 2^n \geq \frac{1}{9} \left(\frac{5}{2}\right)^n - 2^n > 0, \quad n \geq 10.$$

On the other hand, since

$$Q_n'(\alpha) = (\alpha + 2)^{n-1} (\alpha - 1) \Big((n + 2)\alpha - n + 4\Big),$$

$Q_n'(\alpha) < 0$, for $5 \leq n \leq 9$. Since $Q_8(2/3) = 28.1236\ldots$ and $Q_9(2/3) = 245.6630\ldots$, the proof of (32) is complete. In order to prove (31), first observe that $P_7(\alpha) > Q_7(\alpha) + q(\alpha)$, where $q(\alpha) := (-2\alpha^2 + 2\alpha + 1) (\alpha + 1)^7 - 1$. It is easy to see that $q(\alpha)$ is an increasing function of α and therefore $P_7(\alpha) > Q_7(2/3) + q(1/2) = 3.1752\ldots$. Also,

$$P_5(\alpha) = \alpha(\alpha - 1) (12\alpha^3 + 27\alpha^2 + 7\alpha - 23),$$
$$P_6(\alpha) = \alpha(\alpha - 1) (20\alpha^4 + 68\alpha^3 + 73\alpha^2 - 17\alpha - 72).$$

A straightforward computation shows that $P_5(\alpha) > 0$ and $P_6(\alpha) > 0$ for $1/2 \leq \alpha \leq 2/3$ and therefore the proof of (31) is complete.

Note that

$$P_3(\alpha) = \alpha (\alpha - 1) (2\alpha - 1) < 0, \quad 1/2 < \alpha \leq 2/3, \tag{33}$$

while the polynomial

$$P_4(\alpha) = 6\alpha (\alpha - 1) (\alpha^2 + \alpha - 1) \tag{34}$$

has a unique root $\alpha_0 := \frac{-1+\sqrt{5}}{2} = 0.618\ldots$ in the interval $(1/2,\ 2/3)$ so that $P_4(\alpha) > 0$ for $1/2 < \alpha < \alpha_0$ and $P_4(\alpha) < 0$ for $\alpha_0 < \alpha < 2/3$.

In order to prove part (i) of the lemma we use (30) and set

$$S(x) := \sum_{n=3}^{\infty} \frac{P_n(\alpha)}{n!}\, x^{n-3}.$$

It follows from (31) that $S''(x) > 0$ for $x > 0$ and $1/2 \le \alpha \le 2/3$. Therefore $S'(x)$ is a strictly increasing function of x in $[1,\ \infty)$ and hence

$$\begin{aligned} S'(x) \ge S'(1) &= \sum_{n=4}^{\infty}(n-3)\, \frac{P_n(\alpha)}{n!} > \frac{P_4(\alpha)}{4!} + 2\, \frac{P_5(\alpha)}{5!} \\ &= \frac{1}{30}\, \alpha(\alpha-1)(6\alpha^3 + 21\alpha^2 + 11\alpha - 19) > 0, \end{aligned}$$

for $1/2 \le \alpha \le 2/3$. $S(x)$ is thus strictly increasing on $[1, \infty)$ and consequently

$$\begin{aligned} S(x) \ge S(1) &= \sum_{n=3}^{\infty} \frac{P_n(\alpha)}{n!} > \sum_{k=3}^{7} \frac{P_k(\alpha)}{k!} \\ &= \frac{1}{5040}\, \alpha\,(\alpha-1)\,(30\alpha^5 + 275\alpha^4 + 1220\alpha^3 + 3040\alpha^2 + 2977\alpha - 3771) > 0, \end{aligned}$$

for $1/2 \le \alpha \le 2/3$. This in combination with (30) completes the proof of part (i).

We now turn to the proof of (ii). Differentiating (30) we see that inequality $f'''(x) > 0$ is equivalent to

$$3\, \frac{\rho'(x)}{\rho(x)}\, S(x) + S'(x) > 0, \tag{35}$$

where $S(x)$ as above and

$$\rho(x) = \frac{x}{\mathrm{e}^x - 1}.$$

Inequality (35) is true for $x = 0$, because it reduces to

$$P_4(\alpha) - 6P_3(\alpha) = 6\alpha^2\,(\alpha-1)^2 > 0.$$

Assume that $0 < x \le 1$. Writing

$$\frac{\rho'(x)}{\rho(x)} = \frac{1}{x}\left(1 - \frac{x}{\mathrm{e}^x - 1}\right) - 1$$

and using the known inequalities

$$1 - \frac{x}{2} < \frac{x}{\mathrm{e}^x - 1} < 1 - \frac{x}{2} + \frac{x^2}{12}$$

(see [7] for a more general result), we get

$$-\frac{1}{2} - \frac{x}{12} < \frac{\rho'(x)}{\rho(x)} < -\frac{1}{2}. \tag{36}$$

Let $1/2 \le \alpha \le \alpha_0$, where $\alpha_0 = \frac{-1+\sqrt{5}}{2} = 0.618\ldots$ is the unique root in the interval $(1/2,\, 2/3)$ of the polynomial $P_4(\alpha)$. Since in this case $P_4(\alpha) \ge 0$, $P_3(\alpha) \le 0$, we obtain, using (31) and (36),

$$3\,\frac{\rho'(x)}{\rho(x)}\,S(x) + S'(x) > -\left(\frac{3x}{2} + \frac{x^2}{4}\right)\sum_{k=4}^{\infty}\frac{P_k(\alpha)}{k!}\,x^{k-4} + \sum_{k=4}^{\infty}\frac{P_k(\alpha)}{k!}\,(k-3)\,x^{k-4}. \quad (37)$$

When $0 < x \le x_0 := -3 + \sqrt{13} = 0.6055\ldots$ we have $\frac{3x}{2} + \frac{x^2}{4} \le 1$. Hence, from (37) we get

$$3\,\frac{\rho'(x)}{\rho(x)}\,S(x) + S'(x) > \sum_{k=5}^{\infty}\frac{P_k(\alpha)}{k!}\,(k-4)\,x^{k-4} > 0, \quad (38)$$

and the last inequality is obtained using once more (31). When $x_0 < x \le 1$ we obviously have $\frac{3x}{2} + \frac{x^2}{4} \le \frac{7}{4}$ and thus, because of (37) and (31), we get

$$\begin{aligned}
3\frac{\rho'(x)}{\rho(x)}\,S(x) + S'(x) &> -\frac{3}{4}\,\frac{P_4(\alpha)}{4!} + \sum_{k=5}^{\infty}\left(k - \frac{19}{4}\right)\frac{P_k(\alpha)}{k!}\,x^{k-4} \\
&> -\frac{3}{4}\,\frac{P_4(\alpha)}{4!} + \frac{1}{4}\,\frac{P_5(\alpha)}{5!}\,x + \frac{5}{4}\,\frac{P_6(\alpha)}{6!}\,x^2 \\
&> -\frac{3}{4}\,\frac{P_4(\alpha)}{4!} + \frac{1}{4}\,\frac{P_5(\alpha)}{5!}\,x_0 + \frac{5}{4}\,\frac{P_6(\alpha)}{6!}\,x_0^2 \\
&= \frac{1}{1440}\,(11 - 3\sqrt{13})\,\alpha(1-\alpha)\big(-100\,\alpha^4 - (394 + 18\sqrt{13})\alpha^3 \\
&\quad + (256 + 162\sqrt{13})\alpha^2 + (796 + 192\sqrt{13})\alpha - 279 - 168\sqrt{13}\big).
\end{aligned}$$

Since it is straightforward to check that the last expression is positive for $1/2 \le \alpha \le \alpha_0$, this and (38) establish (35) in the case where $1/2 \le \alpha \le \alpha_0$.

Next, suppose that $\alpha_0 < \alpha \le 2/3$. In this case $P_4(\alpha) < 0$. We have

$$\begin{aligned}
3\,\frac{\rho'(x)}{\rho(x)}\,S(x) + S'(x) &= 3\frac{\rho'(x)}{\rho(x)}\Big(\frac{P_3(\alpha)}{3!} + \frac{P_4(\alpha)}{4!}\,x\Big) + \frac{P_4(\alpha)}{4!} \\
&\quad + 3\frac{\rho'(x)}{\rho(x)}\sum_{k=5}^{\infty}\frac{P_k(\alpha)}{k!}\,x^{k-3} + \sum_{k=5}^{\infty}\frac{P_k(\alpha)}{k!}\,(k-3)\,x^{k-4}.
\end{aligned} \quad (39)$$

Since $P_3(\alpha) < 0$, using the second inequality of (36) we obtain

$$3\,\frac{\rho'(x)}{\rho(x)}\left(\frac{P_3(\alpha)}{3!} + \frac{P_4(\alpha)}{4!}\,x\right) + \frac{P_4(\alpha)}{4!} > -\frac{P_3(\alpha)}{4} + \frac{P_4(\alpha)}{24} = \frac{1}{4}\alpha^2\,(1-\alpha)^2 > 0. \quad (40)$$

On the other hand, using the first inequality of (36) together with (31), we obtain

$$3\,\frac{\rho'(x)}{\rho(x)}\sum_{k=5}^{\infty}\frac{P_k(\alpha)}{k!}\,x^{k-3} + \sum_{k=5}^{\infty}\frac{P_k(\alpha)}{k!}(k-3)\,x^{k-4} > \sum_{k=5}^{\infty}\left(k - \frac{19}{4}\right)\frac{P_k(\alpha)}{k!}\,x^{k-4} > 0. \quad (41)$$

Combining (40) with (41) we deduce that the expression in (39) is positive and this establishes (35) in the case where $\alpha_0 < \alpha \le 2/3$.

The proof of Lemma 2 is complete. □

We can now give a proof of Theorem 5.

First observe that

$$\xi''(x) = -\frac{\Gamma(x+\mu)}{\Gamma(x+1)}\, x^{2-\mu}\, \Phi(x), \tag{42}$$

where

$$\Phi(x) := \left(\psi(x+\mu) - \psi(x+1) + \frac{2-\mu}{x}\right)^2 + \left(\psi(x+\mu) - \psi(x+1) + \frac{2-\mu}{x}\right)'.$$

Then, as in the proof of [6, Thm. 1] we find that

$$\Phi(x) = \int_0^\infty \mathrm{e}^{-xu}\, F(u)\, \mathrm{d}u, \tag{43}$$

where, for $u > 0$,

$$F(u) := \int_0^u \sigma(u-v)\,\sigma(v)\,\mathrm{d}v - u\,\sigma(u)$$

and

$$\sigma(u) := 2 - \mu - \phi(u)$$

with

$$\phi(u) := \frac{\mathrm{e}^{(1-\mu)u} - 1}{\mathrm{e}^u - 1}, \qquad \phi(0) = 1 - \mu.$$

Then

$$F'(u) = \int_0^u \sigma'(u-v)\,\sigma(v)\,\mathrm{d}v - u\,\sigma'(u),$$

and

$$F''(u) = u\phi''(u) + \int_0^u \phi'(u-v)\phi'(v)\,\mathrm{d}v. \tag{44}$$

It is shown in [6, Lemma 1] that when $0 < \mu < 1$ we have $\phi'(u) < 0$ for $u \in [0, \infty)$. In addition, when $1/2 \le \mu < 1$ we have $\phi''(u) \ge 0$ for $u \in [0, \infty)$. In view of (44), the combination of these results implies that $F''(u) > 0$ for $u > 0$.

Using Lemma 2, we shall prove that for $1/3 \le \mu < 1/2$, we also have $F''(u) > 0$ for $u > 0$, although, for this range of μ the function $\phi''(u)$ assumes negative values. In fact,

$$\phi''(0) = \frac{1}{6}\mu\,(1-\mu)\,(2\mu - 1) < 0.$$

Note also that

$$\phi'(0) = -\frac{1}{2}\mu\,(1-\mu) < 0.$$

It follows from Lemma 2, that when $1/3 \le \mu < 1/2$ the function $\phi''(u)$ has a unique root in the interval $(0, 1)$ which we denote by ω_μ. Clearly, $\phi''(u) \ge 0$ for $u \in [\omega_\mu, \infty)$ and therefore, by (44), $F''(u) > 0$ for $u \in [\omega_\mu, \infty)$.

Suppose that $0 < u < \omega_\mu$. Consider the function of v

$$\delta(v) := \phi'(u-v)\,\phi'(v), \quad 0 \le v \le u < \omega_\mu < 1.$$

Differentiating with respect to v we get

$$\delta'(v) = -\phi''(u-v)\,\phi'(v) + \phi'(u-v)\,\phi''(v)$$

and also

$$\delta''(v) = \phi'''(u-v)\,\phi'(v) - 2\,\phi''(u-v)\,\phi''(v) + \phi'(u-v)\,\phi'''(v).$$

Lemma 2 ensures that $\delta''(v) < 0$ when $v \in [0, u]$ with $0 < u < \omega_\mu$. Therefore, the function $\delta(v)$ is concave when $v \in [0, u]$ and thus we obtain the estimate

$$\delta(v) \ge \phi'(0)\,\phi'(u), \quad v \in [0, u]. \tag{45}$$

It is perhaps of interest to note that the function $\delta(v)$ has a graph that is symmetric with respect to line $v = u/2$ and that $\delta(v)$ is increasing on $[0, u/2]$ and decreasing on $[u/2, u]$ and therefore the estimate (45) can also be obtained in this way.

It follows from (44) and (45) that

$$F''(u) \ge u\left[\phi''(u) + \phi'(0)\,\phi'(u)\right], \tag{46}$$

for $0 < u < \omega_\mu$. From Lemma 2 we deduce that $\phi'''(u) + \phi'(0)\,\phi''(u) > 0$ for $u \in (0, \omega_\mu)$. Therefore $\phi''(u) + \phi'(0)\,\phi'(u)$ increases in this interval, and hence

$$\phi''(u) + \phi'(0)\phi'(u) \ge \phi''(0) + \phi'(0)^2 = \frac{1}{12}\mu\,(1-\mu)\,(2-\mu)\,(3\mu-1) \ge 0, \tag{47}$$

for $1/3 \le \mu < 1$.

It follows from (46) and (47) that $F''(u) > 0$ for $0 < u < \omega_\mu$ and thus this inequality holds for all $u > 0$. Hence the function $F(u)$ satisfies the following: $F''(u) > 0$, $F'(u) > 0 = F'(0)$ and $F(u) > F(0) = 0$. Taking into consideration (43), it follows from [6, Lemma 2] (see also [8, Thm. 1.3] for a more general result), that the function $x^2\,\Phi(x)$ is completely monotonic on $(0, \infty)$.

Because of (42) we have

$$-\xi''(x) = \frac{\Gamma(x+\mu)}{\Gamma(x+1)}\,\frac{1}{x^\mu}\,x^2\,\Phi(x).$$

It is straightforward to check that $x^{-\mu}$ is completely monotonic on $(0, \infty)$ for $\mu > 0$; using Bernstein's Theorem (cf. Section 1) and the well-known formula (cf. [2, p. 615])

$$\frac{\Gamma(x+a)}{\Gamma(x+b)} = \frac{1}{\Gamma(b-a)}\int_0^\infty \mathrm{e}^{-xu}\,\mathrm{e}^{-au}\,(1-\mathrm{e}^{-u})^{b-a-1}\,\mathrm{d}u, \quad b > a,$$

we see that also $\Gamma(x+\mu)/\Gamma(x+1)$ is completely monotonic. Since it follows readily from the Leibniz product rule that the product of two completely monotonic functions is again completely monotonic, we have thus shown that the function $-\xi''(x)$ is completely monotonic on $(0, \infty)$. Finally, from (29) we get $\lim_{x\to\infty}\xi'(x) = 0$, and thus $\xi'(x) > 0$ for $x > 0$. The relation (9) shows that the function $\xi'(x)$ is completely monotonic on $(0, \infty)$.

Thus the function $\xi(x)$ is strictly increasing and concave on $(0, \infty)$ and, by (27),

$$\lim_{x\to\infty}\xi(x) = \frac{\mu\,(1-\mu)}{2},$$

which gives the second inequality of (8).

This completes the proof of Theorem 5.

As a consequence of the above we also have the following remarkable result.

Corollary 2. *Let*

$$\Phi(x) := \left(\psi(x+\mu) - \psi(x+1) + \frac{2-\mu}{x}\right)^2 + \left(\psi(x+\mu) - \psi(x+1) + \frac{2-\mu}{x}\right)'.$$

The function $x^2\,\Phi(x)$ *is completely monotonic on* $(0, \infty)$ *if and only if* $1/3 \le \mu < 1$.

Proof. The proof is contained in the proof of Theorem 5. In order to see that the result is sharp with respect to μ, observe that by (28),

$$\lim_{x\to\infty} x^4\,\Phi(x) = \frac{1}{12}\mu\,(1-\mu)\,(2-\mu)\,(3\mu-1) < 0$$

for $0 < \mu < 1/3$, while a direct calculation yields

$$\lim_{x\to 0^+} x^2\,\Phi(x) = (1-\mu)\,(2-\mu) > 0$$

for $0 < \mu < 1$. Therefore the function $x^2\,\Phi(x)$ changes sign on $(0, \infty)$ when $0 < \mu < 1/3$. □

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