

Boundary Estimates for Bergman Polynomials in Domains with Corners

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ABSTRACT. Let G be a bounded simply-connected domain in the complex plane \mathbb{C} , whose boundary $\Gamma := \partial G$ is a Jordan curve, and let $\{p_n\}_{n=0}^{\infty}$ denote the sequence of Bergman polynomials of G . This is defined as the unique sequence of polynomials $\{p_n(z)\}_{n=0}^{\infty}$, with positive leading coefficient, that are orthonormal with respect to the area measure on G .

The asymptotic behaviour of $p_n(z)$ in the exterior of Γ , in cases when Γ is a piecewise analytic Jordan curve have been established recently in [15]. The purpose of this note is to derive, for the same class of curves, estimates for the asymptotics of $p_n(z)$ on Γ .

Dedication: To Ed Saff, an outstanding mathematician, a great mentor and collaborator, and a dear friend, on the occasion of his 70th birthday.

1. Introduction and main results

Let G be a bounded simply-connected domain in the complex plane \mathbb{C} , whose boundary $\Gamma := \partial G$ is a Jordan curve and let $\{p_n\}_{n=0}^{\infty}$ denote the sequence of Bergman polynomials of G . This is defined as the unique sequence of polynomials

$$(1.1) \quad p_n(z) = \kappa_n z^n + \cdots, \quad \kappa_n > 0, \quad n = 0, 1, 2, \dots,$$

that are orthonormal with respect to the inner product

$$\langle f, g \rangle_G := \int_G f(z) \overline{g(z)} dA(z),$$

where dA stands for the differential of the area measure. We denote by $L_a^2(G)$ the Hilbert space of functions f analytic in G , for which

$$\|f\|_{L^2(G)} := \langle f, f \rangle_G^{1/2} < \infty,$$

and recall that the sequence of polynomials $\{p_n\}_{n=0}^{\infty}$ forms a complete orthonormal system for $L_a^2(G)$.

Let $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ denote the complement of \overline{G} in $\overline{\mathbb{C}}$ and let Φ denote the conformal map $\Omega \rightarrow \Delta := \{w : |w| > 1\}$, normalized so that near infinity

$$(1.2) \quad \Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots, \quad \gamma > 0,$$

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where $1/\gamma$ gives the (*logarithmic*) capacity $\text{cap}(\Gamma)$ of Γ . We note that Φ has a homeomorphic extension on Γ and we use the same notation Φ for this extension.

Regarding the behaviour of $p_n(z)$ in the unbounded domain Ω , the following strong asymptotics result has been recently establish for non-smooth Γ .

THEOREM 1.1 ([15]). *Assume that Γ is piecewise analytic without cusps. Then, for any $n \in \mathbb{N}$, it holds that*

$$(1.3) \quad p_n(z) = \sqrt{\frac{n+1}{\pi}} \Phi^n(z) \Phi'(z) \{1 + A_n(z)\}, \quad z \in \Omega,$$

where

$$(1.4) \quad |A_n(z)| \leq \frac{c_1(\Gamma)}{|\Phi(z)| - 1} \frac{1}{\sqrt{n}} + c_2(\Gamma) \frac{1}{n}.$$

By Γ being piecewise analytic without cusps we mean that Γ consists of N analytic arcs that meet at points z_j , where they form exterior angles $\omega_j \pi$, with $0 < \omega_j < 2$, $j = 1, \dots, N$. Our first result below is given in term of $\widehat{\omega} := \max\{1, \omega_1, \dots, \omega_N\}$.

Above and in the sequel we use $c(\Gamma)$, $c_1(\Gamma)$, e.t.c., to denote non-negative constants that depend only on Γ . In a similar context we will use $c(\Gamma, z)$ for constants depending on Γ and z . Finally, we will use $\text{dist}(z, B)$ to denote the (Euclidian) distance of z from a set B .

In cases when Γ is smooth then (1.3) holds also for all z on Γ , and $A_n(z)$ tends to zero with a rate depending on the smoothness properties of Γ . More precisely, Carleman has shown in [5] that $A_n(z) = O(\varrho^n)$, for some $0 \leq \varrho < 1$, provided that Γ is analytic. Furthermore, it follows from Suetin's results in [16] (see, e.g., [16, Theorem 1.3]) that $A_n(z) = O(1/n^s)$, for some $s > 0$, provided that Γ is sufficiently smooth.

The purpose of this note is to consider the asymptotic behaviour of $p_n(z)$, for $z \in \Gamma$, with Γ as in Theorem 1.1.

A standard argument to derive an estimate for the uniform norm $\|p_n\|_{L^\infty(G)}$ from (1.3)–(1.4) involves moving the asymptotics from the exterior to the boundary Γ and goes as follows. Consider the exterior level lines $L_{1/n} := \{z \in \Omega : |\Phi(z)| = 1 + 1/n\}$, for $n \in \mathbb{N}$, and use the next two results which quantify the behaviour of $\text{dist}(z, \Gamma)$ for $z \in L_{1/n}$:

- (i) The well-known estimate for the distance of $L_{1/n}$ from Γ , see e.g. [11, pp. 688–689]:

$$(1.5) \quad \text{dist}(z, \Gamma) \geq c(\Gamma)n^{-\widehat{\omega}}, \quad z \in L_{1/n}.$$

- (ii) The double inequality, which is a simple consequence of Koebe's 1/4-theorem, see e.g. [3, p. 23]:

$$(1.6) \quad \frac{1}{4} \frac{|\Phi(z)| - 1}{\text{dist}(z, \Gamma)} \leq |\Phi'(z)| \leq 4 \frac{|\Phi(z)| - 1}{\text{dist}(z, \Gamma)}, \quad z \in \Omega \setminus \{\infty\}.$$

The above lead easily to the estimate

$$(1.7) \quad \|p_n\|_{L^\infty(G)} \leq \|p_n\|_{L_{1/n}} \leq c(\Gamma)n^{\widehat{\omega}}.$$

However, as our first result shows, the standard argument has led to an non-optimal exponent of n in the upper bound for $\|p_n\|_{L^\infty(G)}$.

THEOREM 1.2. * Assume that Γ is piecewise analytic without cusps and recall the notation $\widehat{\omega} := \max\{1, \omega_1, \dots, \omega_N\}$. Then,

$$(1.8) \quad \|p_n\|_{L^\infty(G)} \leq c(\Gamma) n^{\widehat{\omega}-1/2}.$$

The next theorem gives a pointwise estimate for the behaviour of $p_n(z)$, $z \in \Gamma$.

THEOREM 1.3. Assume that Γ is piecewise analytic without cusps and let $\omega\pi$, $0 < \omega < 2$, be the opening of the exterior angle at a point $z \in \Gamma$. Then,

$$(1.9) \quad |p_n(z)| \leq c(\Gamma, z) n^{\omega-1/2} \sqrt{\log n}.$$

We clarify that above we take

$$(1.10) \quad \omega := \begin{cases} \omega_j, & \text{if } z = z_j, \\ 1, & \text{otherwise.} \end{cases}$$

It is interesting to note that (1.9) yields the following limit

$$\lim_{n \rightarrow \infty} p_n(z) = 0,$$

provided $0 < \omega < 1/2$.

For the statement of our final theorem in this section, we need a result of Lehman [8] regarding the asymptotic behaviour of both $\Phi(z)$ and $\Phi'(z)$ near z_j :

$$(1.11) \quad \Phi(z) = \Phi(z_j) + a_1(z - z_j)^{1/\omega_j} + o(|z - z_j|^{1/\omega_j}),$$

and

$$(1.12) \quad \Phi'(z) = \frac{1}{\omega_j} a_1(z - z_j)^{1/\omega_j - 1} + o(|z - z_j|^{1/\omega_j - 1}),$$

with $a_1 \neq 0$.

We remark that if $\omega = \widehat{\omega}$, as in the case where Γ is a rectangle and z one of its corners, then Theorem 1.2 shows that the estimate (1.9) holds without the $\sqrt{\log n}$ factor. This is also true, in general, provided z is not a corner point of Γ , as the next result shows. (Note that for such z (1.12) implies that $\Phi'(z) \neq 0$.)

THEOREM 1.4. Assume that Γ is piecewise analytic without cusps and let $z \in \Gamma \setminus \{z_1, \dots, z_N\}$. Then,

$$(1.13) \quad p_n(z) = \sqrt{\frac{n+1}{\pi}} \Phi^n(z) \Phi'(z) \{1 + O(1)\},$$

where $O(1)$ depends on z but not on n .

Clearly, the estimate (1.13) is a restatement of (1.9) without the $\sqrt{\log n}$ factor. Nevertheless, we have it written this way because numerical evidence in Section 3 suggests that (1.13) holds with $o(1)$ in the place of $O(1)$; see Conjecture 3.1 below.

Furthermore, it will become evident from the proof of Theorem 1.4 in Section 2 that (1.13) holds uniformly when z belongs to a closed subarc J of Γ that does not touch any z_j .

The question of sharpness of the exponent of n in the three theorems above is discussed in Section 3, by means of a numerical example. Here we only remark that the exponent $1/2$ is exact in (1.8) in cases when Γ is smooth (hence $\widehat{\omega} = 1$), as the above cited results of Carleman and Suetin show.

*This theorem, along with a sketch of its proof, was presented by the author on the CMFT 2009 conference, held in Ankara, in June 2009.

The inequality (1.8) should be compared with the estimate

$$(1.14) \quad \|p_n\|_{L^\infty(G)} \leq c(\Gamma)n^{\widehat{\omega}},$$

established by Abdullaev in [1, Theorem 1], under the assumption that Γ is a quasiconformal curve and $\Phi \in \text{Lip } 1/\widehat{\omega}$. To see the connection with the results above, observe that if Γ satisfies the assumptions of Theorem 1.2, then Γ is quasiconformal and $\Phi \in \text{Lip } 1/\widehat{\omega}$; see e.g. [10, p. 52]. We recall that a curve Γ is quasiconformal if there exists a constant M such that,

$$\text{diam}\Gamma(z, \zeta) \leq M|z - \zeta|, \text{ for all } z, \zeta \in \Gamma,$$

where $\Gamma(z, \zeta)$ is the arc (of smaller diameter) of Γ between z and ζ . We also recall that a piecewise analytic Jordan curve is quasiconformal if and only if has no cusps. For a fairly recent account on quasiconformal geometry we refer to [10, Chapter 5].

We remark that the comparison estimate for norms of polynomials obtained by Pritsker in [11, Theorem 1.3], when applied to p_n , produces the same weaker estimate (1.14). We note, however, that the cited estimates in both [1] and [11] hold for a wider class of curves than the one considered in Theorem 1.2.

The paper is organised as follows: Section 2 contains the proofs of Theorems 1.2–1.4. In Section 3, we present a numerical example and a conjecture suggested by both theoretical and numerical evidence regarding the pointwise behaviour of $p_n(z)$ on Γ .

2. Proofs

The Faber polynomials $\{F_n\}_{n=0}^\infty$ of G are defined as the polynomial part of the expansion of $\Phi^n(z)$ near infinity, that is,

$$(2.1) \quad \Phi^n(z) = F_n(z) - E_n(z), \quad z \in \Omega,$$

with

$$(2.2) \quad F_n(z) = \gamma^n z^n + \dots \quad \text{and} \quad E_n(z) = O(1/|z|), \quad z \rightarrow \infty.$$

Similarly, we consider the polynomial part of $\Phi^n(z)\Phi'(z)$ and we denote the resulting series by $\{G_n\}_{n=0}^\infty$. Thus,

$$(2.3) \quad \Phi^n(z)\Phi'(z) = G_n(z) - H_n(z), \quad z \in \Omega,$$

with

$$(2.4) \quad G_n(z) = \gamma^{n+1} z^n + \dots \quad \text{and} \quad H_n(z) = O(1/|z|^2), \quad z \rightarrow \infty.$$

We note that $G_n(z)$ is the so-called Faber polynomial of the 2nd kind (of degree n). We also note the relation

$$(2.5) \quad G_n(z) = \frac{F'_{n+1}(z)}{n+1},$$

implied by (2.1) and (2.3).

Following [15], we consider the sequence of auxiliary polynomials

$$(2.6) \quad q_{n-1}(z) := G_n(z) - \frac{\gamma^{n+1}}{\kappa_n} p_n(z), \quad n \in \mathbb{N}.$$

Observe that $q_{n-1}(z)$ has degree at most $n-1$, but it can be identical to zero, as the special case $G = \{z : |z| < 1\}$ shows.

Our proofs are based on three different estimates for $q_n(z)$, with $z \in \Gamma$, from the same data, namely the norm $\|q_n\|_{L^2(G)}$. These estimates are collected together in the form of a remark.

REMARK 2.1. With the assumptions and the notation of Theorems 1.2–1.4, the following three inequalities hold for *any* polynomial P_n of degree n :

(i) The uniform estimate

$$\|P_n\|_{L^\infty(G)} \leq c(\Gamma)n^{\widehat{\omega}} \|P_n\|_{L^2(G)},$$

cf. Lemma 2.2 and the proof of Theorem 1.2.

(ii) The pointwise estimate

$$|P_n(z)| \leq c(\Gamma, z)n^\omega \sqrt{\log n} \|P_n\|_{L^2(G)},$$

valid for any $z \in \Gamma$, cf. the proof of Theorem 1.3.

(iii) The pointwise estimate

$$|P_n(z)| \leq c(\Gamma, z)n \|P_n\|_{L^2(G)},$$

valid for $z \in \Gamma \setminus \{z_1, \dots, z_N\}$, cf. the proof of Theorem 1.4.

We note that all the above inequalities hold under weaker assumption on the boundary curve Γ , see the conferred proofs for details.

Furthermore, our work is based on the following two asymptotic results

$$(2.7) \quad \frac{\kappa_n}{\gamma^{n+1}} = \sqrt{\frac{n+1}{\pi}} \{1 + \xi_n\}, \quad \text{with } 0 \leq \xi_n \leq c_1(\Gamma) \frac{1}{n},$$

and

$$(2.8) \quad \|q_{n-1}\|_{L^2(G)} \leq c_2(\Gamma) \frac{1}{n},$$

obtained in [15, Theorem 1.1 and Corollary 2.1] under the assumption that Γ is piecewise analytic without cusps.

Finally, for the proof of Theorem 1.2 we need the next lemma, which requires somewhat weaker assumptions.

LEMMA 2.2. *Assume that Γ is a quasiconformal curve and that the conformal map $\Phi : \Omega \rightarrow \Delta$ satisfies a Hölder continuity condition of the form*

$$(2.9) \quad |\Phi(z) - \Phi(\zeta)| \leq L|z - \zeta|^{1/\widehat{\omega}}, \quad z, \zeta \in \overline{\Omega},$$

with $1 \leq \widehat{\omega} < 2$. If for a polynomial P_n , of degree n , it holds that

$$(2.10) \quad |P_n(z)| \leq \frac{M_n}{\text{dist}(z, \Gamma)}, \quad z \in G,$$

for some positive constant M_n , then

$$(2.11) \quad \|P_n\|_{L^\infty(G)} \leq c(\Gamma)n^{\widehat{\omega}} M_n.$$

Proof of Lemma 2.2. Our assumption on Γ implies the existence of a K -quasiconformal reflection $y : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ defined, for some $K \geq 1$, by Γ and a fixed point a in G . Here, we collect together from [2, pp. 25–27 & 103–104] (see also [4, §6]) some well-known properties of $y(z)$ which will be used hereafter:

- (i) \overline{y} is a K -quasiconformal mapping $\overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$.
- (ii) $y(G) = \Omega$, $y(\Omega) = G$, with $y(a) = \infty$ and $y(\infty) = a$.
- (iii) $y(\zeta) = \zeta$, for every $\zeta \in \Gamma$ and $y(y(z)) = z$, for all $z \in \mathbb{C}$.

- (iv) There exists a neighborhood U of Γ , such that for all ζ on Γ and $z \in U$ it holds:

$$(2.12) \quad c_1(\Gamma)|z - \zeta| \leq |y(z) - \zeta| \leq c_2(\Gamma)|z - \zeta|.$$

- (v) The function

$$(2.13) \quad \tilde{\Phi}(z) := \begin{cases} \frac{\Phi(z)}{1/\Phi(y(z))} & \text{for } z \in \overline{\Omega}, \\ 1/\Phi(y(z)) & \text{for } z \in G, \end{cases}$$

defines a quasiconformal extension of Φ in $\overline{\mathbb{C}}$. We keep the same notation Φ for this extension.

We take $z \in G \cap U$ and ζ on Γ and observe that $|\Phi(y(z))| > 1$ and $|\Phi(y(\zeta))| = |\Phi(\zeta)| = 1$. Therefore, using (ii)–(v) and the assumption (2.9) we obtain,

$$(2.14) \quad \begin{aligned} |\Phi(z) - \Phi(\zeta)| &= \left| \frac{1}{\Phi(y(z))} - \frac{1}{\Phi(y(\zeta))} \right| \leq |\Phi(y(z)) - \Phi(y(\zeta))| \\ &\leq L|y(z) - y(\zeta)|^{1/\hat{\omega}} \leq c_3(\Gamma)L|z - \zeta|^{1/\hat{\omega}}. \end{aligned}$$

Consider now the level lines of Φ in G ,

$$L_{1/n}^* := \{z \in G : |\Phi(z)| = 1 - 1/n\},$$

for all $n \geq n_0$, such that $L_{1/n}^* \subset G \cap U$. Then, it is easy to check that (2.14) yields

$$(2.15) \quad \text{dist}(z, \Gamma) \geq c_4(\Gamma) \frac{1}{n^{\hat{\omega}}}, \quad z \in L_{1/n}^*.$$

Consequently, the assumption (2.10) implies that

$$\|P_n\|_{L^\infty(L_{1/n}^*)} \leq c_5(\Gamma)n^{\hat{\omega}}M_n$$

and the required result (2.11) follows immediately from the inequality

$$\|P_n\|_{L^\infty(G)} \leq c_6(\Gamma)\|P_n\|_{L^\infty(L_{1/n}^*)},$$

see [3, pp. 337–338]. ■

Proof of Theorem 1.2. We recall that our assumptions imply that Γ is quasi-conformal and that Φ satisfies (2.9), cf. also [11, p. 688–689]. Thus, by applying the result of the lemma above to the well-known estimate

$$|q_n(z)| \leq \frac{\|q_n\|_{L^2(G)}}{\sqrt{\pi} \text{dist}(z, \Gamma)}, \quad z \in G,$$

we obtain, in view of (2.8),

$$(2.16) \quad \|q_n\|_{L_\infty(G)} \leq c_1(\Gamma)n^{\hat{\omega}-1}.$$

Next, we recall the estimate

$$\|F_n\|_{L^\infty(G)} \leq c_2(\Gamma), \quad n \in \mathbb{N},$$

derived in [7], under the assumption that Γ is piecewise Dini-smooth; see also [12, Theorem 2.1]. This in view of Markov's inequality [17, p. 51] leads to

$$\|F'_n\|_{L^\infty(G)} \leq c_3(\Gamma)n^{\hat{\omega}}.$$

(We refer to [11] for a comprehensive discussion regarding Markov's inequality and the influence of $\text{dist}(z, \Gamma)$ under various assumptions on Γ .) From this latter estimate and (2.5) we get

$$(2.17) \quad \|G_n\|_{L^\infty(G)} \leq c_4(\Gamma)n^{\hat{\omega}-1}.$$

The required result (1.8) for $n \geq n_0$, and thus for all $n \in \mathbb{N}$ with a possibly bigger constant, follows easily from (2.6) by using the estimates (2.7), (2.16) and (2.17). \blacksquare

Proof of Theorem 1.3. Fix a point $z_0 \in G$ and consider the polynomial Q_n defined as an anti-derivative of q_{n-1} by

$$Q_n(t) := \int_z^t q_{n-1}(\zeta) d\zeta - \int_z^{z_0} q_{n-1}(\zeta) d\zeta.$$

Since $Q_n(z_0) = 0$ and $Q'_n = q_{n-1}$, it follows from Andrievskii's lemma (which is valid for quasiconformal curves, see e.g. [6]) that

$$\|Q_n\|_{L^\infty(G)} \leq c(\Gamma) \sqrt{\log n} \|Q'_n\|_{L^2(G)} \leq c(\Gamma) \frac{\sqrt{\log n}}{n},$$

where we made use of (2.8). From this estimate and Markov's inequality

$$|Q'_n(z)| \leq c(\Gamma, z) n^\omega \|Q_n\|_{L^\infty(G)},$$

we obtain

$$(2.18) \quad |q_{n-1}(z)| \leq c(\Gamma, z) n^{\omega-1} \sqrt{\log n}.$$

Next we use the asymptotics for the derivative of the Faber polynomials

$$(2.19) \quad F'_{n+1}(z) = \frac{\omega(n+1)^\omega a_1^\omega \Phi^{n+1-\omega}(z)}{\Gamma(\omega+1)} \{1 + o(1)\},$$

where $\Gamma(x)$ denotes the Gamma function with argument x , which follows from [12, Theorem 1.1] (see also [17, pp. 56–57]) and Lehman's asymptotic expansions (1.11)–(1.12) and refer to [14, pp. 875–876] for a proof of the corresponding result

$$(2.20) \quad F_n(z) = \omega \Phi^n(z) + o(1),$$

regarding the Faber polynomials. Thus, from (2.19) we obtain at once the asymptotics

$$(2.21) \quad G_n(z) = \frac{\omega(n+1)^{\omega-1} a_1^\omega \Phi^{n+1-\omega}(z)}{\Gamma(\omega+1)} \{1 + o(1)\},$$

for the Faber polynomials of the 2nd kind.

The result of the theorem then follows easily from (2.6) by using the estimates (2.18), (2.21) and (2.7). \blacksquare

Proof of Theorem 1.4. Take $z \in \Gamma \setminus \{z_1, \dots, z_N\}$ and set $r := \min\{|z - z_j| : j = 1, \dots, N\}$. Then Φ' can be extended by reflection to become analytic in a small disk D_z with center at z and radius less than r , so that (2.3) is valid in D_z . In particular,

$$(2.22) \quad \Phi^n(z) \Phi'(z) = G_n(z) - H_n(z).$$

Therefore, from (2.6),

$$(2.23) \quad \frac{\gamma^{n+1}}{\kappa_n} p_n(z) = \Phi^n(z) \Phi'(z) + H_n(z) - q_{n-1}(z), \quad n \in \mathbb{N}.$$

The result of the theorem will emerge from this and suitable estimates for $H_n(z)$ and $q_{n-1}(z)$.

Regarding $H_n(z)$, it follows easily from the proof of Theorem 2.3 in [15, p. 80] that

$$(2.24) \quad |H_n(z)| \leq \frac{c(\Gamma)}{r} \frac{1}{n}.$$

Next, we claim that

$$(2.25) \quad |q_n(z)| \leq C(\Gamma, z), \quad n \in \mathbb{N}.$$

This can be obtained by means of a recent result of Totik in [18], regarding the behavior of the Christoffel functions $\lambda_n(z)$ at boundary points contained in C^2 arcs. We recall that $\lambda_n(z)$ is defined, for any $z \in \mathbb{C}$, as the solution to the minimal problem:

$$(2.26) \quad \lambda_n(z) := \inf \{ \|P\|_{L^2(G)}^2, P \in \mathbb{P}_n \text{ with } P(z) = 1 \},$$

where \mathbb{P}_n denotes the space of polynomials of degree n or less. As it can be readily verified, the result of [18, Theorem 1.3], in conjunction with the discussion on page 2058 of the cited opus, implies that

$$(2.27) \quad \lim_{n \rightarrow \infty} n^2 \lambda_n(z) = \frac{2\pi}{|\Phi'(z)|^2},$$

uniformly if z is contained in a closed subarc J of Γ that does not touch any z_j . (Note that (1.12) asserts $\Phi'(z) \neq 0$, for such z .) The minimal property (2.26) yields

$$|q_n(z)|^2 \lambda_n(z) \leq \|q_n\|_{L^2(G)}^2$$

and this in conjunction with (2.8) and (2.27) implies (2.25).

The result of the theorem then follows easily from (2.23), by using the estimates (2.24), (2.25) and (2.7). \blacksquare

3. Discussion and Numerical results

The first result in this section is a refinement of the asymptotics (2.21), in cases when z is not a corner point of Γ .

PROPOSITION 3.1. *Under the assumptions of Theorem 1.4 it holds that*

$$(3.1) \quad G_n(z) = \Phi^n(z) \Phi'(z) \{1 + B_n(z)\},$$

where, with $r := \min\{|z - z_j| : j = 1, \dots, N\}$,

$$(3.2) \quad |B_n(z)| \leq \frac{c(\Gamma)}{r} \frac{1}{n}.$$

Proof. The result is a simple consequence of the discussion at the beginning of the proof of Theorem 1.4. More precisely, it follows from the relation (2.22), by using (2.24). Note that $\Gamma(2) = 1$ and $a_1 = \Phi'(z) \neq 0$. \blacksquare

Motivated by this proposition, the relation (2.21) and Theorems 1.2–1.4, we propose the following, as an extension of Theorem 1.1 when $z \in \Gamma$:

CONJECTURE 3.1. Assume that Γ is piecewise analytic without cusps. Then, at any point z of Γ with exterior angle $\omega\pi$, $0 < \omega < 2$, it holds that

$$(3.3) \quad p_n(z) = \frac{\omega(n+1)^{\omega-1/2} a_1^\omega \Phi^{n+1-\omega}(z)}{\sqrt{\pi} \Gamma(\omega+1)} \{1 + \beta_n(z)\},$$

with $\lim_{n \rightarrow \infty} \beta_n(z) = 0$.

Below, we test the conjecture numerically by constructing a finite sequence of Bergman polynomials associated with a very simple geometry. More precisely, we choose G to be defined by the two intersecting circles $|z - 1| = \sqrt{2}$ and $|z + 1| = \sqrt{2}$, which meet orthogonally at the points i and $-i$, as shown in Figure 1, and consider the two cases $z = i$ (corner) and $z = 1 + \sqrt{2}$ (non-corner).

We note that, this particular type of domain was suggested in [15, pp. 62–63] as suitable for showing sharpness in the strong asymptotics for the leading coefficients given by (2.7). This fact was proved very recently by Miña-Díaz in [9] by means of the same geometry.

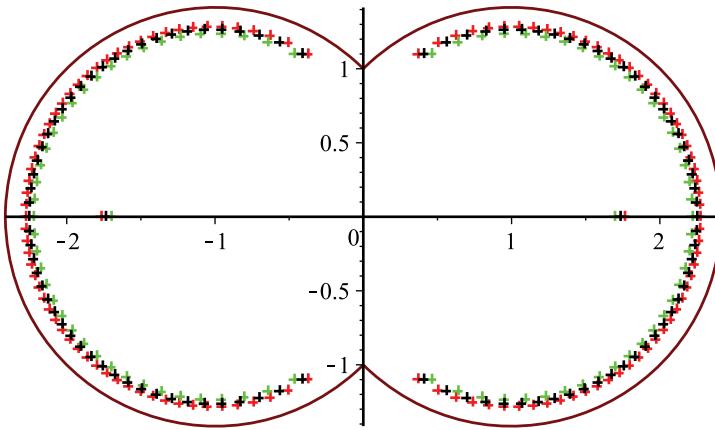


FIGURE 1. Zeros of the Bergman polynomials p_n , with $n = 80, 100, 120$, for the two-intersecting-circles domain.

All the computations presented here were carried out in Maple 16 with 128 significant figures on a MacBook Pro. The construction of the Bergman polynomials was made by using the Arnoldi Gram-Schmidt algorithm; see [15, Section 7.4] for a discussion regarding the stability of the algorithm.

In order to support the claim that the presented computations are accurate, we depict in Figure 1 the set of the computed zeros of the Bergman polynomials p_n , for $n = 80, 100$ and 120 . The zeros in the plot follow remarkably close the theoretical distribution predicted by a recent result in [13, Corollary 3.1], which shows that the equilibrium measure on the boundary Γ of G is the *only* limit of the sequence of the normalized counting measures of the zeros of p_n , as $n \rightarrow \infty$. We also refer to [13, Section 3] for an explanation of the evident reluctance of the zeros in the figure to accumulate near the two corners i and $-i$.

By the simple geometry of G , it is trivial to check that the associated conformal map $\Phi : G \rightarrow \Omega$ is given by

$$(3.4) \quad \Phi(z) = \frac{1}{2} \left(z - \frac{1}{z} \right).$$

Case (a): $z = i$. Then, $\Phi(i) = i$, $\omega = 1/2$, $\Gamma(3/2) = \sqrt{\pi}/2$ and $a_1 = 1/(2i)$, as it is readily seen by using the expansion (1.11) in conjunction with the exact

formula for $\Phi(z)$. Thus, in this case (3.3) takes the form

$$(3.5) \quad p_n(i) = \frac{i^n}{\sqrt{2\pi}} \{1 + \beta_n\}.$$

n	$ \beta_n $	n	$ \beta_n $
100	0.057 121	101	0.037 299
102	0.056 990	103	0.037 428
104	0.056 864	105	0.037 554
106	0.056 741	107	0.037 675
108	0.056 623	109	0.037 793
110	0.056 508	111	0.037 907
112	0.056 396	113	0.038 017
114	0.056 288	115	0.038 125
116	0.056 183	117	0.038 229
118	0.056 081	119	0.038 312
120	0.055 981		

TABLE 3.1. Computed values for $|\beta_n|$ for Case (a).

In Table 3.1 we report the computed values for $|\beta_n|$ in six decimal places, which we believe to be correct, for $n = 100, \dots, 120$, in two columns of even and odd values of n .

Case (b): $z = 1 + \sqrt{2}$. Now $\omega = 1$, $\Phi(z) = 1$ and (3.3) takes the form

$$(3.6) \quad p_n(1 + \sqrt{2}) = \sqrt{\frac{n+1}{\pi}} \frac{2 + \sqrt{2}}{(1 + \sqrt{2})^2} \{1 + \beta_n\},$$

where, due to the reflective symmetry of G , $\beta_n \in \mathbb{R}$.

n	$ \beta_n $	n	$ \beta_n $
100	0.000 596	111	0.000 986
101	0.001 095	112	0.000 784
102	0.000 930	113	0.000 557
103	0.001 410	114	0.000 466
104	0.001 163	115	0.000 184
105	0.001 557	116	0.000 261
106	0.001 246	117	0.000 429
107	0.001 525	118	0.000 447
108	0.001 224	119	0.000 822
109	0.001 325	120	0.000 722
110	0.001 054		

TABLE 3.2. Computed values of $|\beta_n|$ for Case (b).

In Table 3.2, we report the computed values for $|\beta_n|$ in six decimal places, which we believe to be correct, for $n = 100, \dots, 120$, in two columns of consecutive values of n .

The presented values of β_n in both tables seem to support the hypothesis that $\lim_{n \rightarrow \infty} \beta_n = 0$. Moreover, they provide a numerical confirmation of the upper bounds for $p_n(i)$ and $p_n(1 + \sqrt{2})$ given in Theorems 1.2–1.4.

Note Added in Proof. The recent results of Totik and Varga on p. 1097 in [19] permit us to conclude that Inequality (1.9) in Theorem 1.3 is “sharp” when $1 \leq \omega < 2$, in the sense that in this case,

$$|p_n(z)| \geq c(\Gamma, z) n^{\omega-1/2},$$

for infinitely many n .

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