Zeros of Polynomials: Beware of Predictions from Plots

Nikos Stylianopoulos
a report of joint work with
Ed Saff
Vanderbilt University

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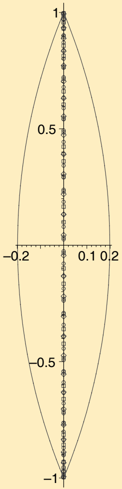
A Betting Game

We are going to introduce five plots for zeros of sequences of polynomials and state conjectures suggested by these plots.

Szegő polynomials S_n for the symmetric lens Λ

Zeros of S_n, for $n = 30$, 40 and 50



Conjecture (I)

All zeros of S_n, for the symmetric lens Λ lie on the imaginary axis, for all n.

Definition

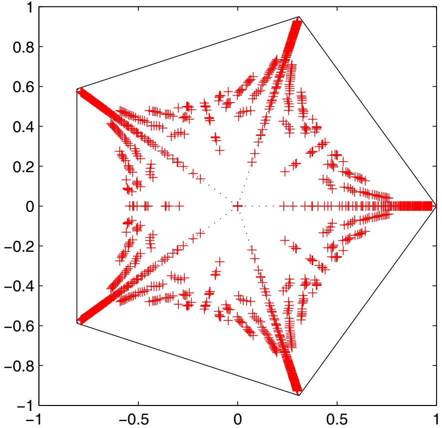
The Szegő polynomials $S_n(z)$ are defined for any rectifiable Jordan curve Γ by

$$\frac{1}{l}\int_\Gamma S_m(z)\overline{S_n(z)}\,|dz| = \delta_{m,n}, \quad S_n(z) = \gamma_n z^n + \cdots, \, \gamma_n > 0,$$

where l denotes the length of Γ.

Bergman polynomials B_n for the canonical pentagon Π

Zeros of B_n, for n up to 50



Conjecture (Eiermann & Stahl, LNM, 1994)

The only points of the canonical pentagon Π *that are limit points of zeros of* B_n, $n = 1, 2, \ldots$, *are its five vertices.*

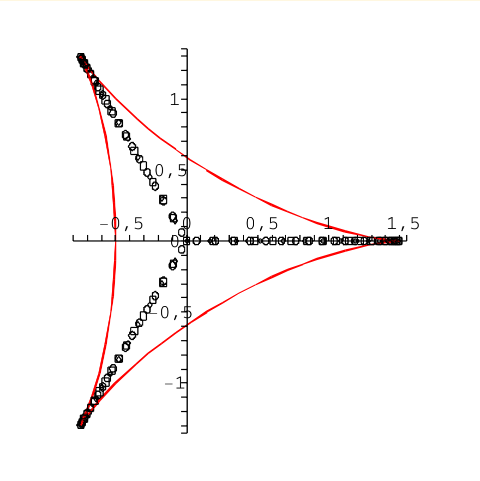
Definition

For any bounded Jordan domain G, the Bergman polynomials $B_n(z)$ are orthonormal with respect to the area measure on G:

$$\int_G B_m(z)\overline{B_n(z)}dA(z) = \delta_{m,n}\,, \quad B_n(z) = \lambda_n z^n + \cdots\,,\ \lambda_n > 0.$$

Bergman polynomials B_n for the hypocycloid Y

Zeros of B_n, for $n = 40$, 50 and 60

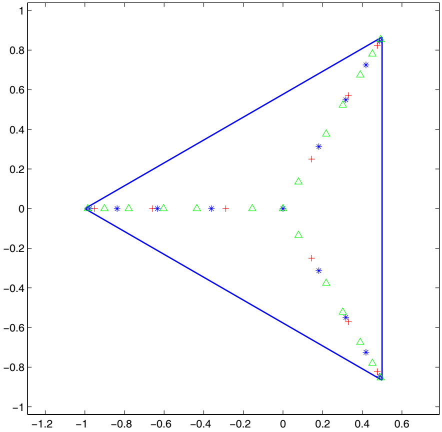


Conjecture (III)

For all n, the zeros of B_n lie on the three radial lines of Y.

Faber polynomials F_n for the equilateral triangle T

Zeros of F_n, for $n = 10$, 15 and 20



Conjecture (IV)

All zeros of F_n, $n = 1, 2, \ldots$, either lie on or are attracted to the radial lines of T.

For any compact set $E \subset \mathbb{C}$ with simply-connected complement $\Omega := \overline{\mathbb{C}} \setminus E$, the Faber polynomials of E are defined as follows.

Definition

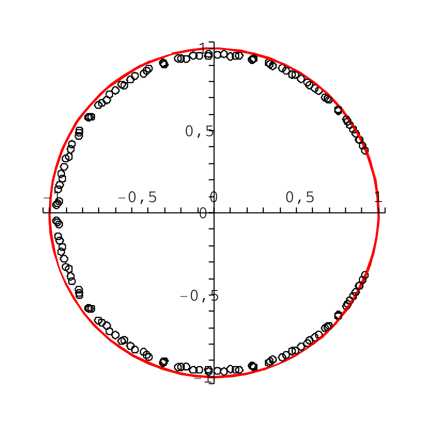
Let

$$w = \Phi(z) = \frac{z}{c} + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \quad c > 0,$$

be the conformal mapping of Ω onto $\Delta := \{w : |w| > 1\}$. Then, the n-th degree Faber polynomial $F_n(z)$ of E is the polynomial part of $\Phi^n(z)$.

Orthogonal polynomials on the unit circle φ_n w.r.t. the measure $d\mu(z) = |\exp\{1/(z-1)^2\}|d\theta,\ z = e^{i\theta}$

Zeros of φ_n, for $n = 40$, 50 and 60



Conjecture (V)

As $n \to \infty$, the zeros of φ_n tend to a proper subarc of the unit circle.

Definition

For a positive finite Borel measure μ with infinite support on $C := \{z : |z| = 1\}$, the orthonormal polynomials w.r.t $d\mu$ are the polynomials $\varphi_n(z)$, $n = 0, 1, \ldots$, that satisfy

$$\int \varphi_m(z)\overline{\varphi_n(z)}d\mu(z) = \delta_{m,n}, \quad \varphi_n(z) = \kappa_n z^n + \cdots, \; \kappa_n > 0, \; z = e^{i\theta}.$$

Evaluation

The surprising fact is that all the above conjectures are false! We wish to emphasize that all the plots are accurate to high precision, so that they represent the truth for the values on n described. The conjectures they suggest fail in the asymptotic sense as $n \to \infty$. In the next two sections we provide the asymptotic theory that disproves the conjectures, as well as give explanations as to why these lower degree plots have the appearance different from the asymptotic truth.

In the sequel we assume that G is a finite simply-connected domain with piecewise analytic Jordan boundary Γ.

Definition

Let Q_n be a polynomial of degree n. The normalized counting measure of the zeros $\nu(Q_n)$ of Q_n is defined for any subset A of $\mathbb{C}$ by

$$\nu(Q_n)(A) := \frac{\text{number of zeros of } Q_n \text{ in } A}{n}.$$

Definition

Given a sequence $\{\sigma_n\}$ of Borel measures, we say that $\{\sigma_n\}$ converges in the weak* sense to a measure σ, symbolically $\sigma_n \xrightarrow{*} \sigma$, if

$$\int f d\sigma_n \longrightarrow \int f d\sigma, \quad n \to \infty,$$

for every function f continuous on $\overline{\mathbb{C}}$.

Let $\boxed{\Omega := \text{ext}(\Gamma)}$, $\Delta := \{w : |w| > 1\}$ and consider the exterior conformal map

$$\Phi : \Omega \to \Delta,$$

with

$$w = \Phi(z) = \frac{z}{c} + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots, \quad c > 0.$$

Note: $\Phi(\Gamma) = \mathbb{T} := \{w : |w| = 1\}$ and $\boxed{c = \text{cap}(\Gamma)}$.

Definition

The equilibrium measure μ_Γ for Γ is defined for any Borel set $A \subset \Gamma$ by

$$\mu_\Gamma(A) := \frac{1}{2\pi} \int_{\Phi(A)} d\theta.$$

Note: $\boxed{\text{supp}(\mu_\Gamma) = \Gamma}$.

Let $\varphi(z)$ be a conformal map from G onto the unit disc $\mathbb{D}$.

Theorem (Levin, Saff & St., Constr Approx, 2003)

A necessary and sufficient condition that there exists a subsequence of $\{\nu(B_n)\}_{n=0}^{\infty}$ (resp. $\{\nu_n(S_n)\}_{n=0}^{\infty}$) which converges in the weak sense to the equilibrium distribution μ_Γ, is that φ (resp. $\sqrt{\varphi'}$) has a singularity on the boundary Γ of G.*

Note: The fact φ or $\sqrt{\varphi'}$ has a singularity on Γ is independent of the choice of φ.

Corollary

If φ (resp. $\sqrt{\varphi'}$) has a singularity on Γ, then every point of Γ is a limit point of zeros of the sequence $\{B_n\}_{n=0}^{\infty}$ (resp. $\{S_n\}_{n=0}^{\infty}$).

Hence, Conjectures (I)–(III) regarding Szegő and Bergman polys are false!

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Concerning Faber polynomials we use to the following result:

Theorem (Kuijlaars & Saff, Math Proc Cam Philos Soc, 1995)

If Γ is a piecewise analytic curve with a singularity other than an outward pointing cusp, then there is a subsequence of $\{\nu(F_n)\}_{n=0}^{\infty}$ that converges weakly* to the equilibrium distribution μ_Γ of Γ. In such a case, every point of Γ attracts the zeros of $\{F_n\}_{n=0}^{\infty}$.

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Regarding OPUC we appeal to the following result:

Theorem (Mhaskar & Saff, JAT, 1990)

Let

$$\Phi_n(z) := \frac{\varphi_n(z)}{\kappa_n} = z^n + \cdots, \quad n = 0, 1, \ldots.$$

If the Verblunsky coefficients $\Phi_n(0)$ satisfy

$$\limsup_{n\to\infty} |\Phi_n(0)|^{1/n} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n} |\Phi_k(0)| = 0,$$

then, there exists a subsequence of $\{\nu(\varphi_n)\}_{n=0}^{\infty}$ that converges weak to the normalized Lebesgue measure on the unit circle C.*

The above combined with the result $\boxed{\mu' > 0 \implies \lim_{n\to\infty} \Phi_n(0) = 0,}$ shows that every point of the unit circle C attracts the zeros of $\{\varphi_n\}_{n=0}^{\infty}$, which disproves Conjecture (V).

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A Useful Lemma (Miña-Díaz, Saff & St., CMFT, 2005)

Let $\{Q_n\}_{n=1}^{\infty}$ be a sequence of polynomials of respective degrees $n = 1, 2, \ldots$ with positive leading coefficients β_n, such that

$$\lim_{n\to\infty} \beta_n^{1/n} = \frac{1}{\mathrm{cap}(\Gamma)} \quad \textit{and} \quad \limsup_{n\to\infty} |Q_n(z)|^{1/n} \leq 1, \quad z \in \Gamma. \tag{1}$$

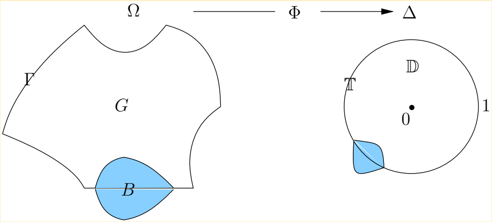
Extend Φ by reflection across a part of Γ, so that Φ is analytic in $\Omega \cup B$, where B is a continuum with points in both G and Ω. Assume that

$$\limsup_{n\to\infty} |Q_n(z)|^{1/n} \leq |\Phi(z)|, \quad z \in G \cap B. \tag{2}$$

Then,

$$\nu(Q_n)(B) \xrightarrow{*} 0, \quad n \to \infty.$$

Illustrating the Useful Lemma



Szegő, Bergman and Faber polynomials satisfy the condition (1). This leads to:

Case

(i) $\forall z \in G \cap B$: $\boxed{\limsup_{n\to\infty} |Q_n(z)|^{1/n} \le |\Phi(z)|} \Longrightarrow \boxed{\nu(Q_n)(B) \xrightarrow{*} 0.}$

(ii) $\exists z \in G$: $\boxed{\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} |Q_n(z)|^{1/n} = 1} \Longrightarrow \boxed{\nu(Q_n) \xrightarrow{*} \mu_\Gamma.}$

Szegő polynomials for the lens Λ

Recall:

$$G := \mathrm{int}(\Lambda), \quad \Omega := \mathrm{ext}(\Lambda) \quad \text{and} \quad \Phi : \Omega \to \Delta := \{w : |w| > 1\}.$$

By the reflection principle we obtain an analytic and conformal extension of Φ to $\mathbb{C} \setminus [-i, i]$, such that $|\Phi|$ is continuous in $\overline{\mathbb{C}}$. Then:

Theorem

For any ζ in the lens G, there exist positive constants κ_1 and κ_2, such that

$$|S_{m+n}(\zeta)| \le \kappa_1 \, |\Phi(\zeta)|^n + \kappa_2 \, \frac{1}{m^{7/2}}, \quad m, n = 1, 2, \dots .$$

Note: $\boxed{|\Phi(\zeta)| < 1,}$ for any $\zeta \in G$.

Szegő polynomials for the lens Λ

Discussion

Since $\min_{\zeta \in G} |\Phi(\zeta)| = |\Phi(0)| \approx 0.797$, and

$$(0.797)^{50} \approx 1.18 \times 10^{-5}, \quad \frac{1}{50^{7/2}} \approx 1.13 \times 10^{-6}$$

it follows from the last theorem that for $n \leq 100$, essentially,

$$|S_n(\zeta)| \leq \kappa |\Phi(\zeta)|^n, \quad \zeta \in G.$$

This suggest that $S_n(\zeta)$, for small values of n, “thinks" it belongs to a sequence for which (2) holds for any $n \in \mathbb{N}$. Hence it places its zeros according to Case (i), i.e.

$$\nu(Q_n)(B) \xrightarrow{*} 0,$$

for any weak* limit σ of $\nu(S_n)$ and any compact $B \subset \mathbb{C} \setminus [-i, i]$.

Bergman polynomials for the pentagon Π

Discussion

This case is similar to Szegő, with singularities at the vertices of the pentagon yielding a decay of order $1/n^s$, with $s = 7/3$ and the locations of the poles of the extension of the interior conformal map φ (relatively to the level lines of the exterior map Φ) contributing a geometric term, which is dominant for at least all n up to 60.

Bergman polynomials for the hypocycloid Y

Discussion

Let $G = \text{int}(Y)$. It follows from a result of Andrievskii & Pritsker (J Anal Math, 2000) that for any $\zeta \in G$ there exist positive constants κ, c and r, with $\boxed{0 < r < 1,}$ such that

$$|B_n(\zeta)| \leq \kappa \exp(-cn^r), \quad n = 1, 2, \ldots .$$

For ζ inside G, near the boundary apart from the vertices, and n not sufficiently large we have,

$$|B_n(\zeta)|^{1/n} \leq \{\kappa \exp(-cn^r)\}^{1/n} \leq |\Phi(\zeta)|,$$

where $\Phi(\zeta)$ is defined in G by the reflection principle across the three segments of Y. Thus, once more the result of the Useful Lemma can be employed to explain the position of the zeros of B_n, in the plot.

Faber polynomials for the equilateral triangle T

Discussion

First, we extend the exterior conformal map Φ inside G, by reflection across the three sides of T, so that Φ becomes analytic in $\mathbb{C}$ apart from the three radial lines, and $|\Phi|$ is continuous in $\overline{\mathbb{C}}$. Fix a $\zeta \in G$ and let ρ be such that $\boxed{|\Phi(\zeta)| < \rho < 1}$. Then one can show that there exist positive constants κ_1 and κ_2, such that

$$|F_n(\zeta)| \leq \kappa_1 \, \rho^n + \kappa_2 \, \frac{1}{n^{5/3}}, \quad n = 1, 2, \ldots .$$

Again, for n not sufficiently large the geometric term dominates and discourages the zeros of F_n from getting to the boundary.
Note: $\min_{\zeta \in G} |\Phi(\zeta)| = |\Phi(0)| \approx 0.66$.

OPUC φ_n w.r.t. $d\mu(z) = |\exp\{1/(z-1)^2\}|d\theta$

Assume for the moment that $d\mu$ is such that the resulting monic polynomials $\{\Phi_n\}_{n=0}^{\infty}$ have constant Verblunsky coefficients $\boxed{\Phi_{n+1}(0) = \alpha}$, with $0 < |\alpha| < 1$. These are the so-called Geronimus polynomials. In such a case we have the following:

Result

1. The support of $d\mu$ consists of

$$C_\beta := \{e^{i\theta} : \beta \le \theta \le 2\pi - \beta\},$$

where $\beta := 2\arcsin(|\alpha|)$, with one possible mass point on $C \setminus C_\beta$. (Golinskii, Nevai, Pintér & Van Assche, JAT, 1999.)

2. z_0 is a limit point of the zeros of φ_n if and only if z_0 lies in the support of $d\mu$. (B. Simon, Comm Pure Appl Math, to appear.)

OPUC φ_n w.r.t. $d\mu(z) = |\exp\{1/(z-1)^2\}|d\theta$

n	$\Phi_{n+1}(0)$
55	-0.129 883
56	-0.129 129
57	-0.128 392
58	-0.127 672
59	-0.126 968

Discussion

In the above table we list the values of $\Phi_{n+1}(0)$, for $n = 55, \ldots, 59$. The theory predicts that they should tend to zero. However the numbers in the table indicate a very slow convergence. As a result, for low values of n the polynomial Φ_n thinks it is actually a member of a sequence of Geronimus polynomials with $\boxed{\alpha \approx 0.127.}$ Hence it places its zeros according to the Results (1) and (2), with $\boxed{\beta \approx 2\arcsin(0.127) \approx 0.25,}$ as a close inspection of the zero-free region in the respective plot shows.