Strong Asymptotics for Bergman and Szegő Polynomials over Domains with Corners

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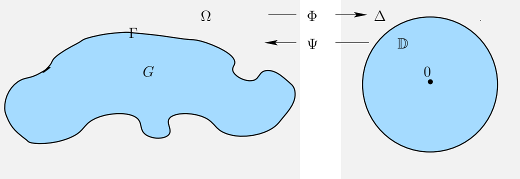
Exterior Conformal Maps



$$\Omega := \overline{\mathbb{C}} \setminus \overline{G}$$

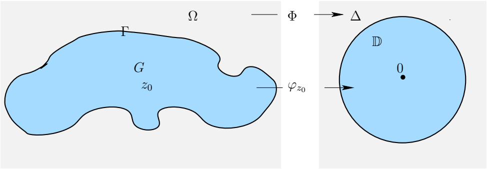
$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots . \qquad \boxed{\operatorname{cap}(\Gamma) = 1/\gamma}$$

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots . \qquad \boxed{\operatorname{cap}(\Gamma) = b}$$

The Bergman polynomials of G:

$$p_n(z) = \lambda_n z^n + \cdots , \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots .$$

The Conformal Mapping Problem



For Γ a bounded Jordan curve, set $G := \text{int}(\Gamma)$ and $\Omega := \text{ext}(\Gamma)$.

Fix $z_0 \in G$ and consider the normalized interior map: $\varphi_{z_0} : G \to \mathbb{D}$,
so that $\varphi_{z_0}(z_0) = 0$ and $\varphi'_{z0}(z_0) > 0$.

We want to compute the mapping $f_0 : G \to \mathbb{D}_r$, $r := 1/\varphi'_{z0}(z_0)$

$$f_0(z) := \frac{\varphi_{z0}(z)}{\varphi'_{z0}(z_0)}, \text{ so that } f_0(z_0) = 0 \text{ and } f'_0(z_0) = 1.$$

Note that f_0 extents homeomorphically to Γ.

The Bergman space $L_a^2(G)$

$$L_a^2(G) := \{f : f \text{ analytic in } G,\ \langle f, f\rangle_G < \infty\},$$

where, $\boxed{\langle f, g\rangle_G := \int_G f(z)\overline{g(z)}dA(z)}$ and dA denotes area measure.

$L_a^2(G)$: is a Hilbert space with corresponding norm $\|f\|_{L^2(G)} := \langle f, f\rangle_G^{\frac{1}{2}}$.

The Bergman polynomials $\{p_n\}$ of G

The orthonormal polynomials w.r.t. the area measure on G:

$$\langle p_m, p_n\rangle_G = \delta_{m,n}, \quad p_n(z) = \lambda_n z^n + \cdots, \ \lambda_n > 0,\ n = 0, 1, 2, \ldots.$$

The Bergman kernel $K(\cdot, z_0)$ of G

The reproducing kernel of $L_a^2(G)$, w.r.t. the point evaluation at z_0:

$$\langle g, K(\cdot, z_0)\rangle_G = g(z_0), \text{ for all } g \in L_a^2(G).$$

Series representation for the Bergman kernel, Bergman 1920's

The function $K(z, z_0)$ has the following Fourier series expansion

$$K(z, z_0) = \sum_{j=0}^{\infty} \overline{p_j(z_0)} p_j(z), \;\; z, z_0 \in G,$$

where, for each fixed $z_0 \in G$ the series convergence uniformly on each compact subset B of G.

Connection with the conformal mapping

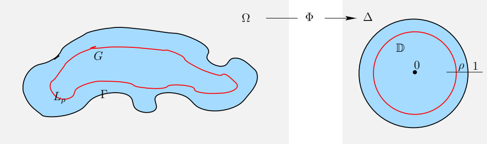
The Bergman kernel $K(\cdot, z_0)$ is related to the mapping function f_0 by means of

$$f_0'(z) = \frac{K(z, z_0)}{K(z_0, z_0)}.$$

Hence

$$f_0(z) = \frac{1}{K(z_0, z_0)} \int_{z_0}^{z} K(\zeta, z_0) d\zeta.$$

Strong asymptotics when Γ is analytic



T. Carleman, Ark. Mat. Astr. Fys. (1922)

If $\rho < 1$ is the smallest index for which Φ is conformal in ext(L_ρ), then

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n}, \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\rho^{2n},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\{1 + A_n(z)\}}, \quad n \in \mathbb{N},$$

where

$$|A_n(z)| \le c_2(\Gamma)\sqrt{n}\,\rho^n, \quad z \in \overline{\Omega}.$$

Strong asymptotics when Γ is smooth

We say that $\Gamma \in C(p,\alpha)$, for some $p \in \mathbb{N}$ and $0 < \alpha < 1$, if Γ is given by $z = g(s)$, where s is the arclength, with $g^{(p)} \in \text{Lip}\alpha$. Then both Φ and $\Psi := \Phi^{-1}$ are p times continuously differentiable in $\overline{\Omega} \setminus \{\infty\}$ and $\overline{\Delta} \setminus \{\infty\}$ respectively, with $\Phi^{(p)}$ and $\Psi^{(p)} \in \text{Lip}\alpha$.

P.K. Suetin, Proc. Steklov Inst. Math. AMS (1974)

Assume that $\Gamma \in C(p+1,\alpha)$, with $p+\alpha > 1/2$. Then, for $n \in \mathbb{N}$,

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n,} \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\frac{1}{n^{2(p+\alpha)}},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\,\Phi^n(z)\Phi'(z)\{1 + A_n(z)\}},$$

where

$$|A_n(z)| \le c_2(\Gamma)\,\frac{\log n}{n^{p+\alpha}}, \quad z \in \overline{\Omega}.$$

Strong asymptotics for Γ non-smooth

Theorem (St, C. R. Acad. Sci. Paris (2010) & Constr. Approx. (2013))

Assume that Γ is piecewise analytic without cusps. Then, for $n \in \mathbb{N}$,

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n}, \quad \textit{where} \quad 0 \le \alpha_n \le c(\Gamma)\,\frac{1}{n},$$

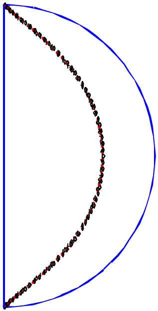
and for any $z \in \Omega$,

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\left\{1 + A_n(z)\right\}},$$

where

$$|A_n(z)| \le \frac{c_1(\Gamma)}{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

Strong asymptotics for Γ non-smooth: An example



$$\gamma = \frac{1}{\text{cap}(\Gamma)} = \frac{3\sqrt{3}}{4}$$

We compute, by using the Gram-Schmidt process (in finite precision), the Bergman polynomials $p_n(z)$ for the unit half-disk, for n up to 60 and test the hypothesis

$$\alpha_n := 1 - \frac{n+1}{\pi} \frac{\gamma^{2(n+1)}}{\lambda_n^2} \approx C \frac{1}{n^s}.$$

Strong asymptotics for Γ non-smooth: Numerical data

n	α_n	s
51	0.003 263 458 678	-
52	0.003 200 769 764	0.998 887
53	0.003 140 444 435	0.998 899
54	0.003 082 351 464	0.998 911
55	0.003 026 369 160	0.998 923
56	0.002 972 384 524	0.998 934
57	0.002 920 292 482	0.998 946
58	0.002 869 952 027	0.998 957
59	0.002 821 401 485	0.998 968
60	0.002 774 426 207	0.998 979

The numbers indicate clearly that $\alpha_n \approx C\frac{1}{n}$. Accordingly, we have made the conjecture that the order $O(1/n)$ for α_n, $n \in \mathbb{N}$, is sharp. Recently E. Mina-Diaz (Numer. Algorithms, 2015) has studied a special case, where this is so for a subsequence of natural numbers.

Strong asymptotics for Γ non-smooth: Numerical data

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Ratio asymptotics for λ_n

Corollary

$$\boxed{\sqrt{\frac{n+1}{n}}\frac{\lambda_{n-1}}{\lambda_n} = \mathrm{cap}(\Gamma) + \xi_n}, \quad \textit{where} \quad |\xi_n| \le c(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}.$$

The above relation provides the means for computing approximations to the capacity of Γ, by using only the leading coefficients of the Bergman polynomials.

Ratio asymptotics for $p_n(z)$

Corollary

For any $z \in \Omega$, and sufficiently large $n \in \mathbb{N}$,

$$\boxed{\sqrt{\frac{n}{n+1}}\,\frac{p_n(z)}{p_{n-1}(z)} = \Phi(z)\{1 + D_n(z)\}},$$

where

$$|D_n(z)| \leq \frac{c_1(\Gamma)}{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

The above relation, provides an efficient method for computing approximations to $\Phi : \Omega \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$.

Note: The kernel polynomials $K_n(z, z_0) := \sum_{j=0}^{n} \overline{p_j(z_0)} p_j(z)$ are used in the Bergman kernel method for computing approximations to the interior conformal map $\varphi : G \to \mathbb{D}$.

Faber polynomials of the second kind

We consider the polynomial part of $\Phi^n(z)\Phi'(z)$ and denote the resulting series by $\{G_n\}_{n=0}^{\infty}$. Thus,

$$\Phi^n(z)\Phi'(z) = G_n(z) - H_n(z), \quad z \in \Omega,$$

with

$$G_n(z) = \gamma^{n+1} z^n + \cdots \quad \text{and} \quad H_n(z) = O\left(1/|z|^2\right), \quad z \to \infty.$$

$G_n(z)$ is the so-called Faber polynomial of the 2nd kind (of degree n). We also consider the auxiliary polynomial

$$q_{n-1}(z) := G_n(z) - \frac{\gamma^{n+1}}{\lambda_n} p_n(z).$$

Observe that $q_{n-1}(z)$ has degree at most $n-1$, but it can be identical to zero, as the special case $G = \{z : |z| < 1\}$ shows.

The links

The strong asymptotic results are based on the following relations:

St, Constr. Approx. (2013)

$$\frac{n+1}{\pi}\|G_n\|^2_{L^2(G)} + \frac{n+1}{\pi}\|H_n\|^2_{L^2(\Omega)} = 1. \quad (1)$$

$$\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \frac{n+1}{\pi}\|q_{n-1}\|^2_{L^2(G)} - \frac{n+1}{\pi}\|H_n\|^2_{L^2(\Omega)}. \quad (2)$$

$$\|q_{n-1}\|_{L^2(G)} \le c_1(\Gamma)\,\|H_n\|_{L^2(\Omega)}, \quad \text{for } \Gamma \text{ quasiconformal.}$$

$$\|H_n\|^2_{L^2(\Omega)} \le c_2(\Gamma)\frac{1}{n^2}, \quad \text{for } \Gamma \text{ piecewise analytic no cusps.}$$

Note: Equations (1) and (2) hold for any bounded simply connected domain G.

Theorem (B. Beckermann & St, arXiv 2016)

Let Γ *be quasiconformal, and set*

$$\varepsilon_n := \frac{n+1}{\pi}\|H_n\|^2_{L^2(\Omega)}.$$

If $\varepsilon_n = \mathcal{O}(1/n^\beta)$, *for some* $\beta > 0$. *Then*

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\left\{1 + O(\frac{\sqrt{\varepsilon_n}}{n^{\beta/2}})\right\}}, \quad n \to \infty,$$

uniformly on compact subsets of Ω.

Note that:

$$\varepsilon_n = \begin{cases} \mathcal{O}(\rho^{2n}), & \text{if } \Gamma \in U(\rho), \text{ (T. Carleman)} \\ \mathcal{O}(1/n^{2(p+\alpha)}), & \text{if } \Gamma \in \mathcal{C}(p+1,\alpha) \text{ (P.K. Suetin)}, \\ \mathcal{O}(1/n), & \text{if } \Gamma \text{ is piecewise analytic no cusps (St)}, \end{cases}$$

A uniform estimate on Γ

By Γ being piecewise analytic without cusps we mean that Γ consists of N analytic arcs that meet at points z_j, where they form exterior angles $\omega_j\pi$, with $0 < \omega_j < 2$, $j = 1, \dots, N$.
Our result below is be given in terms of

$$\boxed{\widehat{\omega} := \max\{1, \omega_1, \dots, \omega_N\}}.$$

Theorem (St, Contemp. Math., 2015)

Assume that Γ is piecewise analytic without cusps. Then,

$$\|p_n\|_{L^\infty(G)} \leq c(\Gamma) n^{\widehat{\omega}-1/2}.$$

Pointwise estimate on Γ

The next theorem gives a pointwise estimate for $|p_n(z)|$, $z \in \Gamma$.

Theorem (St, Contemp. Math., 2015)

Assume that Γ is piecewise analytic without cusps. Then, for any $z \in \Gamma$ away from corners,

$$|p_n(z)| \leq c(\Gamma, z) n^{1/2}.$$

If z_j is a corner of Γ with exterior angle $\omega_j \pi$, $0 < \omega_j < 2$, then

$$|p_n(z_j)| \leq c(\Gamma, z_j) n^{\omega_j - 1/2} \sqrt{\log n}.$$

It is interesting to note that the above yields the following limit

$$\lim_{n \to \infty} p_n(z_j) = 0,$$

provided $0 < \omega_j < 1/2$.

The following result settles, in a certain sense, the question of sharpness of the pointwise, and hence of the uniform, estimate.

Theorem (V. Totik & T. Varga, Proc. London Math. Soc., 2015)

Assume that Γ has a C^{1+} smooth corner of exterior angle $\omega\pi$, with $1 \leq \omega < 2$ at the point z. Then, for infinitely many n,

$$|p_n(z)| \geq c(\Gamma, z)n^{\omega-1/2}.$$

A result of Lehman

For the statement of a conjecture regarding the behaviour of $p_n(z)$ on Γ, we need a result of Lehman, for the asymptotics of both Φ and Φ'. Assume that $\omega\pi$, $0 < \omega < 2$, is the opening of the exterior angle at a point $z \in \Gamma$. Then, for any ζ near z:

$$\Phi(\zeta) = \Phi(z) + a_1(\zeta - z)^{1/\omega} + o(|\zeta - z|^{1/\omega}),$$

and

$$\Phi'(\zeta) = \frac{1}{\omega} a_1(\zeta - z)^{1/\omega - 1} + o(|\zeta - z|^{1/\omega - 1}),$$

with $a_1 \neq 0$.

A Conjecture for the asymptotics of p_n on Γ

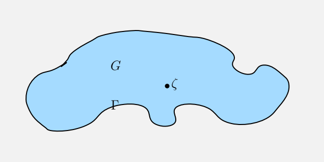
Conjecture (St, Contemp. Math., 2015)

Assume that Γ *is piecewise analytic without cusps. Then, at any point* z *of* Γ *with exterior angle* $\omega\pi$, $0 < \omega < 2$, *it holds that*

$$p_n(z) = \frac{\omega(n+1)^{\omega-1/2} a_1^{\omega} \Phi^{n+1-\omega}(z)}{\sqrt{\pi}\,\Gamma(\omega+1)}\{1+\beta_n(z)\},$$

with $\lim_{n\to\infty} \beta_n(z) = 0$.

Definition: Szegő polynomials $\{P_n\}$



Γ: rectifiable Jordan curve.

$$\langle f, g\rangle_\Gamma := \frac{1}{2\pi}\int_\Gamma f(z)\overline{g(z)}|dz|, \quad \|f\|_{L^2(\Gamma)} := \langle f, f\rangle_\Gamma^{1/2}$$

The Szegő polynomials $\{P_n\}_{n=0}^\infty$ of Γ are the orthonormal polynomials w.r.t. the normalized arc length measure:

$$\langle P_m, P_n\rangle_\Gamma = \frac{1}{2\pi}\int_\Gamma P_m(z)\overline{P_n(z)}|dz| = \delta_{m,n},$$

with

$$P_n(z) = \mu_n z^n + \cdots, \quad \mu_n > 0, \quad n = 0, 1, 2, \ldots.$$

The Smirnov space

$$E^2(G) := \{f \text{ analytic in } G,\ \|f\|_{L^2(\Gamma)} < \infty\},$$

is a Hilbert space with reproducing kernel $K_S(z,\zeta)$: For any $\zeta \in G$,

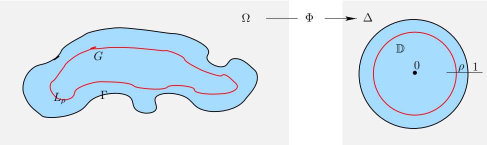
$$f(\zeta) = \langle f, K_S(\cdot,\zeta)\rangle,\ \forall\ f \in E^2(G).$$

Approximation Property

If G is a Smirnov domain then $\{P_n\}_{n=0}^{\infty}$ is a complete ON system of $E^2(G)$ and

$$K_S(z,\zeta) = \sum_{n=0}^{\infty} \overline{P_n(\zeta)} P_n(z), \quad z,\zeta \in G.$$

Strong asymptotics when Γ is analytic



G. Szegő, Math. Z. (1921)

If $\rho < 1$ is the smallest index for which Φ is conformal in $\text{ext}(L_\rho)$, then for any $n \in \mathbb{N}$,

$$\frac{\gamma^{2n+1}}{\mu_n^2} = 1 + O(\rho^{2n}),$$

and for any $z \in \overline{\Omega}$,

$$P_n(z) = \Phi^n(z)\sqrt{\Phi'(z)}\{1 + O(\sqrt{n}\,\rho^n)\}.$$

Strong asymptotics when Γ is smooth

Under various degrees of smoothness on Γ, strong asympotics were studied by a number of great Russian Mathematicians: Smirnov, Keldysh, Lavrent'ev, Korovkin, Geronimus,...

P.K. Suetin, (1964)

Assume that $\Gamma \in C(p+1, \alpha)$, with $0 < \alpha < 1$. Then, for any $n \in \mathbb{N}$,

$$\frac{\gamma^{2n+1}}{\mu_n^2} = 1 + O(\frac{1}{n^{2(p+\alpha)}}),$$

and for any $z \in \overline{\Omega}$,

$$\boxed{P_n(z) = \Phi^n(z)\sqrt{\Phi'(z)}\{1 + O(\frac{\log n}{n^{p+\alpha}})\}.}$$

Strong asymptotics for Γ non-smooth

Theorem

Assume that Γ *is piecewise analytic without cusps. Then, for* $n \in \mathbb{N}$,

$$\boxed{\frac{\gamma^{2n+1}}{\mu_n^2} = 1 + \alpha_n}, \quad \textit{where} \quad 0 \leq \alpha_n \leq c(\Gamma)\,\frac{1}{n},$$

and for any $z \in \Omega$,

$$\boxed{P_n(z) = \Phi^n(z)\sqrt{\Phi'(z)}\,\{1 + A_n(z)\}},$$

where

$$|A_n(z)| \leq \frac{c(\Gamma)}{\sqrt{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}}\,\frac{1}{\sqrt{n}}, \quad n \in \mathbb{N}$$

Generalised Faber polynomials

We consider the polynomial part of $\Phi^n(z)\sqrt{\Phi'(z)}$ and denote the resulting series by $\{B_n\}_{n=0}^{\infty}$. Thus,

$$\Phi^n(z)\sqrt{\Phi'(z)} = B_n(z) - V_n(z), \quad z \in \Omega,$$

with

$$B_n(z) = \gamma^{n+1/2} z^n + \cdots \quad \text{and} \quad V_n(\infty) = 0.$$

We also consider the auxiliary polynomial

$$Q_{n-1}(z) := B_n(z) - \frac{\gamma^{n+1/2}}{\mu_n} P_n(z).$$

Observe that $Q_{n-1}(z)$ has degree at most $n-1$, but it can be identical to zero, as the special case $G = \{z : |z| < 1\}$ shows.

The links

The strong asymptotic results are based on the following relations:

$$\|B_n\|^2_{L^2(\Gamma)} - \|V_n\|^2_{L^2(\Gamma)} = 1. \tag{3}$$

$$\frac{\gamma^{2n+1}}{\mu_n^2} = 1 + \|V_n\|^2_{L^2(\Gamma)} - \|Q_{n-1}\|^2_{L^2(\Gamma)} \tag{4}$$

$$\|V_n\|^2_{L^2(\Gamma)} \leq c(\Gamma)\frac{1}{n}, \quad \text{for } \Gamma \text{ piecewise analytic no cusps.}$$

Note: Equations (3) and (4) hold for any rectifiable Jordan curve Γ.

Two conjectures for strong asymptotics

Conjecture (Bergman polynomials)

Assume that Γ is a quasiconformal curve. Then

$$\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 + o(1),$$

Conjecture (Szegő polynomials)

Assume that G is a Smirnov domain. Then

$$\frac{\gamma^{2n+1}}{\mu_n^2} = 1 + o(1),$$

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