



**Since Approximation Theory is already  
there...  
Bring Potential Theory to Operator Theory!**

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# Lebesgue spaces and Orthonormal Polynomials

Let  $\mu$  be a **finite positive Borel measure** having **compact and infinite support**  $S_\mu := \text{supp}(\mu)$  in the complex plane  $\mathbb{C}$ . Then, the measure yields the **Lebesgue spaces**  $L^2(\mu)$  with inner product

$$\langle f, g \rangle_\mu := \int f(z) \overline{g(z)} d\mu(z)$$

and norm

$$\|f\|_{L^2(\mu)} := \langle f, f \rangle_\mu^{1/2}.$$

Let  $\{p_n(\mu, z)\}_{n=0}^\infty$  denote the sequence of **orthonormal polynomials** associated with  $\mu$ . That is, the unique sequence of the form

$$p_n(\mu, z) = \gamma_n(\mu) z^n + \cdots, \quad \gamma_n(\mu) > 0, \quad n = 0, 1, 2, \dots,$$

satisfying  $\langle p_m(\mu, \cdot), p_n(\mu, \cdot) \rangle_\mu = \delta_{m,n}$ .



## Minimal property

The **monic orthogonal** polynomials  $p_n(\mu, z)/\gamma_n(\mu)$ , can be defined by the extremal property

$$\left\| \frac{1}{\gamma_n(\mu)} p_n(\mu, \cdot) \right\|_{L^2(\mu)} := \min_{z^n + \dots} \|z^n + \dots\|_{L^2(\mu)} = \frac{1}{\gamma_n(\mu)}.$$

A related extremal problem leads to the sequence  $\{\lambda_n(\mu, z)\}_{n=0}^{\infty}$  of the **Christoffel functions**. These are defined, for any  $z \in \mathbb{C}$ , by

$$\lambda_n(\mu, z) := \inf\{\|P\|_{L^2(\mu)}^2, P \in \mathbb{P}_n \text{ with } P(z) = 1\},$$

where  $\mathbb{P}_n$  is the space of polynomials of degree  $\leq n$ .



# Christoffel functions

The Cauchy-Schwarz inequality yields that

$$\frac{1}{\lambda_n(\mu, z)} = \sum_{k=0}^n |p_k(\mu, z)|^2, \quad z \in \mathbb{C}.$$

$\lambda_n(\mu, z)$  is the inverse of the diagonal of the **kernel polynomials**

$$K_n(\mu, z, \zeta) := \sum_{k=0}^n \overline{p_k(\mu, \zeta)} p_k(\mu, z)$$

This leads to reconstruction algorithms from a finite set of moments

$$\int z^k \bar{z}^l d\mu(z), \quad k, l = 0, 1, \dots, n.$$

- **Archipelagos**, in Gustafsson, Putinar, Saff & St, Adv. Math. (2009).
- **Archipelagos with Lakes**, in Saff, Stahl, St & Totik, SIAM J. Math. Anal. (2016).



## Distribution of zeros: The tools

For any polynomial  $q_n(z)$ , of degree  $n$ , we denote by  $\nu_{q_n}$  the **normalized counting measure** for the zeros of  $q_n(z)$ ; that is,

$$\nu_{q_n} := \frac{1}{n} \sum_{q_n(z)=0} \delta_z,$$

where  $\delta_z$  is the unit point mass (Dirac delta) at the point  $z$ .  
For any measure  $\mu$  with compact support in  $\mathbb{C}$ ,

$$U^\mu(z) := \int \log \frac{1}{|z-t|} d\mu(t), \quad z \in \mathbb{C}.$$

denotes the **logarithmic potential** on  $\mu$ . Then

$$U^{\nu_{q_n}}(z) = \frac{1}{n} \log \frac{1}{|q_n(z)|}, \quad z \in \mathbb{C}.$$

With  $\mu_K$  we denote the **equilibrium measure** of a compact set  $K$  of positive **logarithmic capacity**.



## Potential Theory: Five theorems

### Theorem (Generalized Minimum Principle)

Let  $G \in \overline{\mathbb{C}}$  be a domain and  $h$  a superharmonic function on  $G$  that is bounded from below and for which

$$\limsup_{z \rightarrow \zeta, z \in G} h(z) \geq m,$$

is satisfied for quasi-every  $\zeta \in \partial G$ . Then,

$$h(z) > m, \quad z \in G,$$

unless  $h$  is constant.

Saff & Totik, Logarithmic Potentials, Springer, 1997.



## Potential Theory: Five theorems

### Theorem (Principle of Descent)

Let  $\mu_n$ ,  $n = 1, 2, \dots$ , be probability measures, supported on the same compact subset of  $\mathbb{C}$ , such that

$$\mu_n \xrightarrow{*} \mu.$$

Suppose that for each  $n$ , a point  $z_n$  is given so that  $z_n \rightarrow z$ , for some  $z \in \mathbb{C}$ . Then,

$$U^\mu(z) \leq \liminf_{n \rightarrow \infty} U^{\mu_n}(z_n).$$



## Potential Theory: Five theorems

### Theorem (Lower Envelope Theorem)

*Let  $\mu_n$ ,  $n = 1, 2, \dots$ , be a sequence of positive unit Borel measures, supported on the same compact subset of  $\mathbb{C}$ , such that*

$$\mu_n \xrightarrow{*} \mu.$$

*Then,*

$$\liminf_{n \rightarrow \infty} U^{\mu_n}(z) = U^{\mu}(z),$$

*for quasi-every  $z \in \mathbb{C}$ .*





## Potential Theory: Five theorems

### Theorem (Unicity Theorem)

*Suppose that the positive measures  $\mu$  and  $\nu$  have compact support and in a region  $D \subset \mathbb{C}$  the potentials  $U^\nu$  and  $U^\mu$  satisfy*

$$U^\mu(z) = U^\nu(z) + u(z),$$

*almost everywhere with respect to two-dimensional Lebesgue measure, where the function  $u$  is harmonic in  $D$ . Then, in  $D$  the measures  $\mu$  and  $\nu$  coincide.*



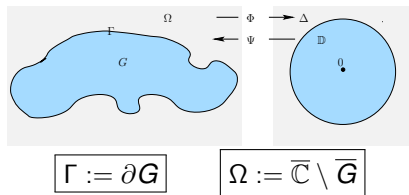
## Potential Theory: Five theorems

### Theorem (Carleson's Unicity Theorem)

*Let  $K$  be a compact set of positive capacity, and  $\Omega$  the unbounded component of  $\overline{\mathbb{C}} \setminus K$ . If  $\mu$  and  $\nu$  are two unit measures supported on  $\partial\Omega$ , and if the potentials  $U^\mu$  and  $U^\nu$  coincide in  $\Omega$ , then  $\mu = \nu$ .*



# Bergman polynomials $\{p_n\}$ on an **Jordan domain** $G$



$$\langle f, g \rangle := \int_G f(z) \overline{g(z)} dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle^{1/2}.$$

The **Bergman polynomials**  $\{p_n\}_{n=0}^{\infty}$  of  $G$  are the orthonormal polynomials w.r.t. the **area measure** on  $G$ :

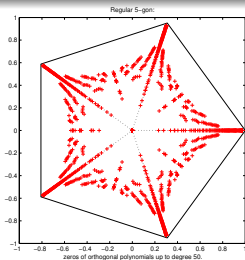
$$\langle p_m, p_n \rangle = \int_G p_m(z) \overline{p_n(z)} dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \dots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \dots$$



## Example: $G$ is the canonical pentagon



Theorem (Levin, Saff & St., Constr. Approx. 2003)

Let  $\varphi$  be a conformal map of  $G$  onto the unit disk  $\mathbb{D}$ . Then, there is a subsequence  $\mathcal{N}$  of  $\mathbb{N}$  such that

$$\nu_{p_n} \xrightarrow{*} \mu_{\Gamma}, \quad n \rightarrow \infty, \quad n \in \mathcal{N},$$

if and only if  $\varphi$  cannot be analytically continued to some open set containing  $\overline{G}$ .



## Key results

The above theorem is based on the following facts:

- The area measure on  $G$  belongs to the class  $\text{Reg}$ , that is,

$$\lim_{n \rightarrow \infty} \|\rho_n\|_{\overline{G}}^{1/n} = 1.$$

- The kernel  $K(z, \zeta)$ , of the Bergman space  $L_a^2(G)$  satisfies,

$$K(z, \zeta) = \sum_{n=0}^{\infty} \overline{\rho_n(\zeta)} \rho_n(z), \quad z, \zeta \in G,$$

and is related to a normalized conformal map  $\varphi_\zeta : G \rightarrow \mathbb{D}$ ,  $\varphi_\zeta(\zeta) = 0$ ,  $\zeta \in G$ , by

$$K(z, \zeta) = \frac{1}{\pi} \overline{\varphi'_\zeta(\zeta)} \varphi'_\zeta(z).$$

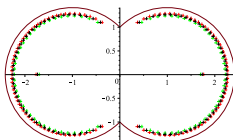
An application of Walsh's maximal convergence then yields

$$\limsup_{n \rightarrow \infty} |\rho_n(\zeta)|^{1/n} = 1, \quad n \in \mathcal{N},$$

and the result then follows from Theorem III.4.1 in Saff and Totik.



## The two intersecting circles



Zeros of  $p_n(z)$ , with  $n = 80, 100, 120$ .

Theorem (Saff & St, JAT 2015)

If the boundary  $\Gamma$  of  $G$  contains an inward corner point, then

$$\nu_{p_n} \xrightarrow{*} \mu_{\Gamma}, \quad n \rightarrow \infty, \quad n \in \mathbb{N},$$

where  $\mu_{\Gamma}$  denotes the *equilibrium measure* on  $\Gamma$ .

Based on Gardiner and Pommerenke, Constr, Approx, 2002.

The reluctance of the zeros to approach the points  $\pm i$ , is due to the

fact that  $d\mu_{\Gamma}(z) = |\Phi'(z)| ds$ , where  $s$  denotes the arclength on  $\Gamma$ .



# The circular sector

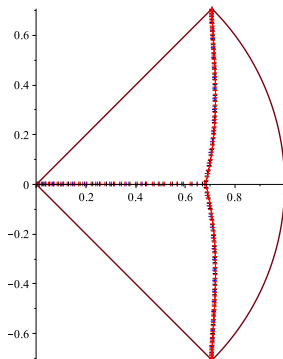
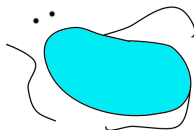


Figure: Zeros of  $p_n$ ,  $n = 50, 100, 150$ , for the circular sector with opening angle  $\pi/2$ .



# Explanation



## Theorem (Mina-Diaz, Saff & St., CMFT 2005)

Let  $E \neq \emptyset$  be a compact subset of  $\mathbb{C}$  such that both  $\overline{\mathbb{C}} \setminus E$  and  $\mathring{E} := \text{int}(E)$  are connected. Let  $g : \overline{\mathbb{C}} \setminus \mathring{E} \rightarrow \overline{\mathbb{C}}$  be such that  $g$  is analytic in  $\mathbb{C} \setminus E$ ,  $|g|$  is continuous and never zero in  $\overline{\mathbb{C}} \setminus \mathring{E}$ ,  $g(\infty) = \infty$  and  $g'(\infty) = 1$ . Let  $\{q_n\}_{n=1}^{\infty}$  be a sequence of monic polynomials of respective degrees  $n = 1, 2, \dots$ , such that  $\infty$  is not an accumulation point of the set of zeros of the  $q_n$ 's. Further, assume that

$$\limsup_{n \rightarrow \infty} |q_n(z)|^{1/n} \leq |g(z)| \quad \text{q.e. } z \in \partial E.$$





### Theorem (Mina-Diaz, Saff & St., CMFT 2005, cont.)

Then, any measure  $\sigma$  that is a weak\*-limit point of the sequence  $\{\nu_{q_n}\}_{n=1}^{\infty}$  is supported on  $E$  and

$$U^{\sigma}(z) = \log |g(z)|^{-1} \quad \forall z \in \mathbb{C} \setminus \overset{\circ}{E}. \quad (1)$$

Moreover, there is a unique measure  $\mu_g$  supported on  $\partial E$  such that (1) holds with  $\sigma = \mu_g$ . For such a measure, we have

(a) if  $\overset{\circ}{E} = \emptyset$ , then  $\nu_{q_n} \xrightarrow{*} \mu_g$  as  $n \rightarrow \infty$ ;

(b) if  $\overset{\circ}{E} \neq \emptyset$  and for some  $z_0 \in \overset{\circ}{E}$  and a subsequence  $\mathcal{N} \subset \mathbb{N}$

$$\lim_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}}} |q_n(z_0)|^{1/n} = e^{-U^{\mu_g}(z_0)},$$

then

$$\nu_{q_n} \xrightarrow{*} \mu_g \quad \text{as } n \rightarrow \infty, \quad n \in \mathcal{N}.$$



## Used in the proof

Observe that the assumption of the theorem is equivalent to

$$\liminf_{n \rightarrow \infty} U^{\nu_{q_n}}(z) \geq \log |g(z)|^{-1} \quad \text{q.e. } z \in \partial E. \quad (2)$$

Let  $\sigma$  be a weak\*-limit point of the sequence  $\{\nu_{q_n}\}_{n=1}^{\infty}$ , so that for some subsequence  $\mathcal{N} \subset \mathbb{N}$

$$\nu_{q_n} \xrightarrow{*} \sigma \quad \text{as } n \rightarrow \infty, \quad n \in \mathcal{N}.$$

Then  $\sigma$  is a probability measure and by (2) and the **Lower Envelope Theorem**, we have for q.e.  $z \in \partial E$ ,

$$U^{\sigma}(z) = \liminf_{\substack{n \rightarrow \infty \\ n \in \mathcal{N}}} U^{\nu_{q_n}}(z) \geq \liminf_{n \rightarrow \infty} U^{\nu_{q_n}}(z) \geq \log |g(z)|^{-1}. \quad (3)$$



## Used in the proof

By the assumptions on  $g$ , the function

$$F^\sigma(z) := U^\sigma(z) - \log |g(z)|^{-1}, \quad z \in \mathbb{C} \setminus E,$$

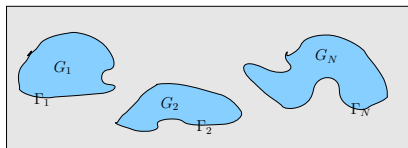
is superharmonic and lower bounded in  $\mathbb{C} \setminus E$ , harmonic and equal to zero at  $\infty$ , and in view of (3) and the lower semicontinuity of  $U^\sigma$ , it also satisfies for *quasi-every*  $z' \in \partial E$

$$\liminf_{\substack{z \rightarrow z' \\ z \in \mathbb{C} \setminus E}} F^\sigma(z) \geq \liminf_{z \rightarrow z'} U^\sigma(z) - \lim_{\substack{z \rightarrow z' \\ z \in \mathbb{C} \setminus E}} \log |g(z)|^{-1} \geq U^\sigma(z') - \log |g(z')|^{-1} \geq 0.$$

Then, by the **generalized minimum principle for superharmonic functions** we conclude that  $F^\sigma \equiv 0$ , which implies that (1) holds in  $\mathbb{C} \setminus E$ . It also implies that  $U^\sigma$  is harmonic in  $\mathbb{C} \setminus E$  and therefore, in view of the **Unicity Theorem**  $\text{supp}(\sigma)$  must be contained in  $E$ . It is a direct consequence of **Carleson's Unicity Theorem** that there can be at most one measure  $\mu_g$  supported on  $\partial E$  that satisfies (1) with  $\sigma = \mu_g$ .



# Bergman polynomials on an archipelago



$\Gamma_j, j = 1, \dots, N$ , a system of disjoint and mutually exterior Jordan

curves in  $\mathbb{C}$ ,  $G_j := \text{int}(\Gamma_j)$ ,  $\Gamma := \cup_{j=1}^N \Gamma_j$ ,  $G := \cup_{j=1}^N G_j$ .

$$\langle f, g \rangle_G := \int_G f(z) \overline{g(z)} dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle_G^{1/2}$$

The **Bergman polynomials**  $\{p_n\}_{n=0}^\infty$  of  $G$  are the unique orthonormal polynomials w.r.t. the **area measure** on  $G$ :

$$\langle p_m, p_n \rangle_G = \int_G p_m(z) \overline{p_n(z)} dA(z) = \delta_{m,n},$$

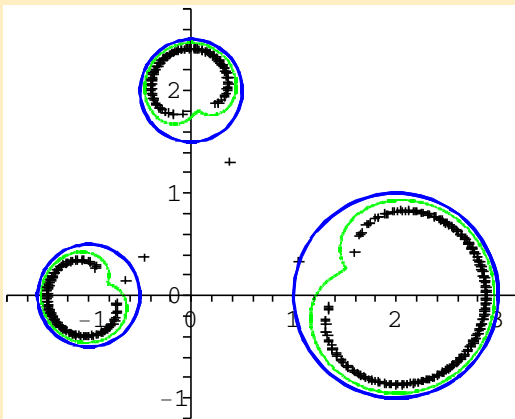
with

$$p_n(z) = \lambda_n z^n + \dots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \dots$$



# Three-disks

Zeros of the Bergman polynomials  $p_{140}$ ,  $p_{150}$  and  $p_{160}$ .



Theory in: Gustafsson, Putinar, Saff & St, Adv. Math., 2009.



## The basic tool for the distribution of zeros

- $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$ .
- $K(z, \zeta)$ : the Bergman (reproducing) kernel function of  $L_a^2(G)$ .
- $L_R := \{z : g_\Omega(z, \infty) = \log R\}$  the level lines of the Green function.
- $\varrho(\zeta) := \sup\{R : K(z, \zeta) \text{ has an analytic continuation inside } L_R\}$ .
- 

$$h(z) := \begin{cases} g_\Omega(z, \infty), & z \in \overline{\Omega}, \\ -\log \varrho(z), & z \in G, \end{cases}$$

- $\beta := \frac{1}{2\pi} \Delta h$ , in the sense of distributions.
- $\nu_{p_n}$ : the *normalized counting measure* of zeros of  $p_n$ .
- $\mathcal{C}$ : the set of *weak-star cluster points* of the counting measures  $\{\nu_{p_n}\}_{n=1}^\infty$ , i.e., the set of measures  $\sigma$  for which there exists a subsequence  $\mathcal{N}_\sigma \subset \mathbb{N}$  such that  $\nu_{p_n} \xrightarrow{*} \sigma$ , as  $n \rightarrow \infty$ ,  $n \in \mathcal{N}_\sigma$ .
- $\mu_\Gamma$ : the *equilibrium measure* on the boundary  $\Gamma$ .



# The basic result for the distribution of zeros

Theorem (Gustafsson, Putinar, Saff & St, Advances in Math, 2009)

- (i)  $\beta$  is a positive unit measure with support contained in  $\overline{G}$ .
- (ii) The balayage of  $\beta$  onto  $\Gamma$  gives the equilibrium measure  $\mu_\Gamma$ :

$$\begin{cases} U^\beta \geq U^{\mu_\Gamma} & \text{in } \mathbb{C}, \\ U^\beta = U^{\mu_\Gamma} & \text{in } \Omega. \end{cases}$$

- (iii)  $\mathcal{C}$  is nonempty, and for any  $\sigma \in \mathcal{C}$ ,

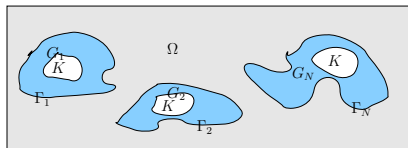
$$\begin{cases} U^\sigma \geq U^\beta & \text{in } \mathbb{C}, \\ U^\sigma = U^\beta & \text{in the unbounded component of } \overline{\mathbb{C}} \setminus \text{supp } \beta. \end{cases}$$

- (iv) The measure  $\beta$  is the lower envelope of  $\mathcal{C}$ :  $U^\beta = \text{lsc}(\inf_{\sigma \in \mathcal{C}} U^\sigma)$ .
- (v) If  $\mathcal{C}$  has only one element, then this is  $\beta$  and

$$\nu_{p_n} \xrightarrow{*} \beta, \quad n \rightarrow \infty, \quad n \in \mathbb{N}.$$



# Bergman polynomials on archipelago with lakes



With  $K$  is a compact subset of  $G$ , set  $G^* := G \setminus K$  and consider

$$\langle f, g \rangle_{G^*} := \int_{G^*} f(z) \overline{g(z)} dA(z), \quad \|f\|_{L^2(G^*)} := \langle f, f \rangle_{G^*}^{1/2}.$$

The **Bergman polynomials**  $\{p_n^*\}_{n=0}^\infty$  of  $G^*$  are the unique orthonormal polynomials w.r.t. the **area measure** on  $G^*$ :

$$\langle p_m^*, p_n^* \rangle_{G^*} = \int_{G^*} p_m^*(z) \overline{p_n^*(z)} dA(z) = \delta_{m,n},$$

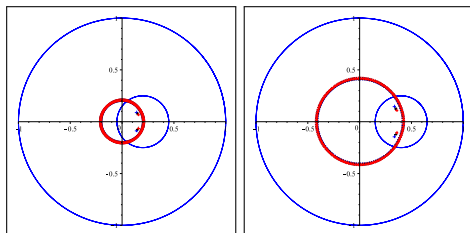
with

$$p_n^*(z) = \gamma_n^* z^n + \dots, \quad \gamma_n^* > 0, \quad n = 0, 1, 2, \dots$$





## The annular case



Plots of the zeros of  $p_n^*(z)$ , for  $n = 120, 140$  and  $160$ .

Let  $G = \mathbb{D}$ ,  $\mathcal{K} := \{z : |z - a| \leq \varrho\}$ ,  $|a| + \varrho < 1$ ,  $\varrho > 0$ ,  $G^* = \mathbb{D} \setminus \mathcal{K}$ . We recall that there exists a unique pair of points  $z_1$  and  $z_2$  that are mutually inverse points with respect to the two circles  $\mathbb{T} := \partial\mathbb{D}$  and  $\{z : |z - a| = \varrho\}$ , that is

$$z_1 \overline{z_2} = 1 \quad \text{and} \quad (z_1 - a)(\overline{z_2 - a}) = \varrho^2.$$

Let  $z_1$  denote the point that lies in  $\mathcal{K}$  ( $z_2$  will then lie outside  $\mathbb{D}$ ).



## The annular case: Explanation

Proposition (Saff & St, Mat. Sbornik, 2018)

*With the above notation, there exists a subsequence  $\mathcal{N} \subset \mathbb{N}$  such that the normalized zero counting measures for  $p_n^*(z)$  satisfy*

$$\nu_{p_n^*} \xrightarrow{*} \mu_{|z_1|}, \quad n \rightarrow \infty, \quad n \in \mathcal{N},$$

*where  $\mu_{|z_1|}$  denotes the normalized arclength measure on the circle  $|z| = |z_1|$ .*

Thus, no matter what the relative position of  $\mathcal{K}$ , a weak limit of  $\nu_n$  will invariably be the arclength measure on a specific circle in  $\mathbb{D}$ , always centered at the origin.



## Shift Operator on $L^2(\mu)$

Let  $N_z$  denote the **shift operator** on  $L^2(\mu)$ . That is,

$$N_z : L^2(\mu) \rightarrow L^2(\mu) \quad \text{with} \quad N_z f = z f.$$

$N_z$  defines a normal operator on  $L^2(\mu)$ . Furthermore,

$$p_n(\mu, z) = \lambda_n(\mu) \det(z - \pi_n N_z \pi_n),$$

where  $\pi_n$  is the projection onto the  $n$ -dimensional subspace onto  $\mathbb{P}_{n-1}$ .

**Theorem (B. Simon, Duke Math. J., 2009)**

Let

$$N(\mu) := \sup\{|z| : z \in \mathcal{S}_\mu\}.$$

Then, for any  $k \in \mathbb{N}$ ,

$$\pi_n N_z^k \pi_n - (\pi_n N_z \pi_n)^k,$$

is an operator of rank at most  $k$  and norm at most  $2N(\mu)^k$ .



## Shift Operator on $L^2(\mu)$

Let  $\mu_n$  denote the unit measures  $d\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} |p_n(\mu, z)|^2 d\mu(z)$ .

Theorem (B. Simon, Duke Math. J., 2009)

$$\frac{1}{n} \operatorname{Tr}(\pi_n N_z \pi_n)^k = \int z^k d\nu_{p_n}.$$

$$\frac{1}{n} \operatorname{Tr}(\pi_n N_z^k \pi_n) = \int z^k d\mu_n.$$

Thus, from the previous theorem, for any  $k = 0, 1, 2, \dots$ ,

$$\left| \int z^k d\nu_{p_n} - \int z^k d\mu_n \right| \leq \frac{2kN^k(\mu)}{n}.$$

Furthermore, if  $K$  is a compact set containing the supports of all  $\nu_n$  and  $\mu$ , such that  $\{z_k\}_{k=0}^{\infty} \cup \{\bar{z}_k\}_{k=0}^{\infty}$  are  $\|\cdot\|_{\infty}$ -total in  $\mathcal{C}(K)$ , then for

any subsequence  $\{n_j\}$ ,  $\boxed{\mu_{n_j} \xrightarrow{*} \nu}$  if and only if  $\boxed{\mu_{n_j} \xrightarrow{*} \nu}$ .



## Krylov subspaces

Let  $A \in \mathcal{L}(H)$  be a linear bounded operator acting on the complex Hilbert space  $H$  and let  $\xi \in H$  be a non-zero vector. We denote  $H_n(A, \xi)$  the linear span of the vectors  $\xi, A\xi, \dots, A^{n-1}\xi$  and let  $\pi_n$  be the orthogonal projection of  $H$  onto  $H_n(A, \xi)$ . Let  $a_n$  denote the counting measures of the spectra of the *finite central truncations*  $A_n = \pi_n A \pi_n$ . Note that for any complex polynomial  $p(z)$  it holds that

$$\int p(z) da_n(z) = \frac{\text{tr } p(A_n)}{n}.$$

The orthogonal monic polynomials  $P_n$  in this case are defined as minimizers of the functional (semi-norm):

$$\|q\|_{A, \xi}^2 = \|q(A)\xi\|^2, \quad q \in \mathbb{C}[z],$$

and the zeros of  $P_n$  (whenever  $P_n$  exists) coincide with the spectrum of  $A_n$ .



### Theorem (Gustafsson & Putinar, Springer 2017)

Let  $A, B \in \mathcal{L}(H)$  with  $A - B$  of finite trace:  $A - B \in \mathcal{C}_1(H)$ . Then for every polynomial  $p \in \mathbb{C}[z]$  we have

$$\lim_{n \rightarrow \infty} \frac{\text{Tr}(p(A_n)) - \text{Tr}(p(B_n))}{n} = 0.$$

### Corollary

Let  $a_n, b_n$  denote the counting measures of the spectra of  $A_n$  and  $B_n$ , respectively. Then,

$$\lim_{n \rightarrow \infty} \left[ \int \frac{da_n(\zeta)}{\zeta - z} - \int \frac{db_n(\zeta)}{\zeta - z} \right] = 0,$$

uniformly on compact subsets which are disjoint of the convex hull of  $\sigma(A) \cup \sigma(B)$ .



## Conclusion

All the results in this section yield information for the analytic moments:

$$\lim_{n \rightarrow \infty} \int z^k d\nu_n = \int z^k d\nu, \quad k = 0, 1, 2, \dots,$$

where  $\nu$  is a known positive measure and  $\{\nu_n\}$  are a sequence of positive measures (all supported on the same compact set  $K$  in the complex plane) we want to describe its weak limit points. Note that the measures being positive implies the same information for the anti-analytic moments:

$$\lim_{n \rightarrow \infty} \int \bar{z}^k d\nu_n = \int \bar{z}^k d\nu, \quad k = 1, 2, \dots$$



## Conclusion

However, according to the complex Stone-Weierstrass theorem, in order to establish

$$\nu_n \xrightarrow{*} \nu,$$

we need the limits of all the complex moments

$$\lim_{n \rightarrow \infty} \int z^k \bar{z}^j d\nu_n = \int z^k \bar{z}^j d\nu, \quad k, j = 0, 1, 2, \dots,$$

unless  $K$  is of a special form (Mergelyan, Walsh), where the analytic moments constitute sufficient information.