Problems Session

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Hausdorff Geometry Of Polynomials And Polynomial Sequences
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Lebesgue spaces and Orthonormal Polynomials

Let μ be a finite positive Borel measure having compact and infinite support $\boxed{S_\mu := \operatorname{supp}(\mu)}$ in the complex plane $\mathbb{C}$. Then, the measure yields the Lebesgue spaces $L^2(\mu)$ with inner product

$$\langle f, g\rangle_\mu := \int f(z)\overline{g(z)}d\mu(z)$$

and norm

$$\|f\|_{L^2(\mu)} := \langle f, f\rangle_\mu^{1/2}.$$

Let $\{p_n(\mu, z)\}_{n=0}^\infty$ denote the sequence of orthonormal polynomials associated with μ. That is, the unique sequence of the form

$$p_n(\mu, z) = \gamma_n(\mu)z^n + \cdots, \quad \gamma_n(\mu) > 0, \quad n = 0, 1, 2, \ldots,$$

satisfying $\langle p_m(\mu, \cdot), p_n(\mu, \cdot)\rangle_\mu = \delta_{m,n}$.

Distribution of zeros: The tools

For any polynomial $q_n(z)$, of degree n, we denote by ν_{q_n} the normalized counting measure for the zeros of $q_n(z)$; that is,

$$\nu_{q_n} := \frac{1}{n} \sum_{q_n(z)=0} \delta_z,$$

where δ_z is the unit point mass (Dirac delta) at the point z.
For any measure μ with compact support in $\mathbb{C}$,

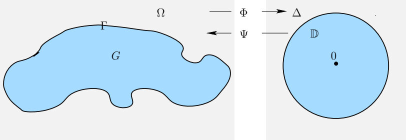
$$U^{\mu}(z) := \int \log \frac{1}{|z-t|} d\mu(t), \quad z \in \mathbb{C}.$$

denotes the logarithmic potential on μ. Then

$$U^{\nu_{q_n}}(z) = \frac{1}{n} \log \frac{1}{|q_n(z)|}, \quad z \in \mathbb{C}.$$

With μ_E we denote the equilibrium measure of a compact set E of positive logarithmic capacity.

Bergman polynomials $\{p_n\}$ on an Jordan domain G



$\Gamma := \partial G$ $\qquad$ $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$

$$\langle f, g \rangle := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle^{1/2}.$$

The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G are the orthonormal polynomials w.r.t. the area measure on G:

$$\langle p_m, p_n \rangle = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots .$$

Shift Operator

Let $L_a^2(G)$ denote the Bergman space of square integrable and analytic functions in G and consider the Bergman shift operator on $L_a^2(G)$. That is,

$$S_z : L_a^2(G) \to L_a^2(G) \quad \text{with} \quad S_z f = zf.$$

Properties of S_z

(i) S_z defines a subnormal operator on $L_a^2(G)$.

(ii) $\sigma(S_z) = \overline{G}$ and $\sigma_{ess}(S_z) = \partial G$ (Axler, Conway & McDonald, Can. J. Math.,1982).

(iii) $S_z^*(f) = P_G(\overline{z}f)$, where P_G denotes the orthogonal projection from $L^2(G)$ to $L_a^2(G)$.

Proof of (iii): For any $f, g \in L_a^2(G)$ it holds that

$$\langle S_z^* f, g \rangle = \langle f, S_z g \rangle = \langle f, zg \rangle = \langle \overline{z} f, g \rangle = \langle P_G(\overline{z} f), g \rangle.$$

Recurrences for Bergman polynomials $\{p_n\}$

In general it holds that

$$zp_n(z) = \sum_{k=0}^{n+1} b_{k,n} p_k(z), \quad \text{where} \quad b_{k,n} := \langle zp_n, p_k \rangle.$$

Matrix representation for S_z

The Bergman operator S_z has the following upper Hessenberg matrix representation with respect to the Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G:

$$\mathcal{M} = \begin{bmatrix} b_{00} & b_{01} & b_{02} & b_{03} & b_{04} & b_{05} & \cdots \\ b_{10} & b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & \cdots \\ 0 & b_{21} & b_{22} & b_{23} & b_{24} & b_{25} & \cdots \\ 0 & 0 & b_{32} & b_{33} & b_{34} & b_{35} & \cdots \\ 0 & 0 & 0 & b_{43} & b_{44} & b_{45} & \cdots \\ 0 & 0 & 0 & 0 & b_{54} & b_{55} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{bmatrix},$$

where $\boxed{b_{k,n} = \langle zp_n, p_k \rangle}$ are the Fourier coefficients of $S_z p_n = zp_n$.

Note

The eigenvalues of the $n \times n$ principal submatrix $\mathcal{M}_n$ of $\mathcal{M}$ coincide with the zeros of p_n.

Example: $G \equiv \mathbb{D}$

This example shows why modern text books on Functional Analysis or Operators Theory do not refer to matrices: Indeed, in this case we have:

$$p_n(z) = \sqrt{\frac{n+1}{\pi}} z^n, \quad n = 0, 1, \ldots .$$

Therefore, in the matrix representation $\mathcal{M}$ of S_z the only non-zero diagonals are the main subdiagonal, and hence for any $n \in \mathbb{N}$, $\mathcal{M}_n$ is a nilpotent matrix. As a result, the Caley-Hamilton theorem implies:

$$\sigma(\mathcal{M}_n) = \{0\}.$$

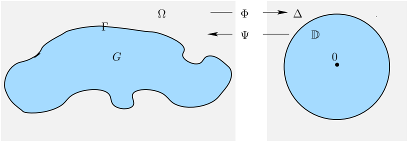
This is in sharp contrast to:

$$\sigma_{ess}(\mathcal{M}) = \sigma_{ess}(S_z) = \{w : |w| = 1\}$$

and

$$\sigma(\mathcal{M}) = \sigma(S_z) = \{w : |w| \leq 1\}.$$

The inverse conformal map Ψ



Recall that

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots,$$

and let $\Psi := \Phi^{-1} : \{w : |w| > 1\} \to \Omega$, denote the inverse conformal map. Then,

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots, \quad |w| < 1,$$

where

$$b = \mathrm{cap}(\Gamma) = 1/\gamma.$$

The Toeplitz matrix with (continuous) symbol Ψ

$$T_\psi = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \cdots \\ b & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \cdots \\ 0 & b & b_0 & b_1 & b_2 & b_3 & b_4 & \cdots \\ 0 & 0 & b & b_0 & b_1 & b_2 & b_3 & \cdots \\ 0 & 0 & 0 & b & b_0 & b_1 & b_2 & \cdots \\ 0 & 0 & 0 & 0 & b & b_0 & b_1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & b & b_0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}.$$

Spectral properties

Theorem (St, Constr, Approx., 2013)

If Γ is piecewise analytic without cusps, then

$$|b_n| \leq c_1(\Gamma)\frac{1}{n^{1+\omega}}, \quad n \in \mathbb{N}, \tag{1}$$

where $\omega\pi$ $(0 < \omega < 2)$ is the smallest exterior angle of Γ.

Therefore, in this case, the symbol Ψ of the Toeplitz matrix T_Ψ belongs to the Wiener algebra. Thus, T_Ψ defines a bounded linear operator on the Hilbert space $l^2(\mathbb{N})$ and

$$\boxed{\sigma_{ess}(T_\Psi) = \Gamma}; \tag{2}$$

see e.g., Bottcher & Grudsky, Toeplitz book, 2005.

Faber polynomials of G

The Faber polynomial of the 2nd kind $G_n(z)$, is the polynomial part of the expansion of the Laurent series expansion of $\Phi^n(z)\Phi'(z)$ at ∞:

$$G_n(z) = \Phi^n(z)\Phi'(z) + O\left(\frac{1}{z}\right), \quad z \to \infty.$$

These polynomials satisfy the recurrence relation:

$$\boxed{zG_n(z) = bG_{n+1}(z) + \sum_{k=0}^{n} b_k G_{n-k}(z)}, \quad n = 0, 1, \ldots,$$

Recall: $zp_n(z) = \sum_{k=0}^{n+1} b_{k,n} p_k(z)$.

Note

The eigenvalues of the $n \times n$ principal submatrix $\mathcal{T}_n$ of T_ψ coincide with the zeros of G_n.

$\mathcal{M} \to T_\psi$ diagonally

The next series of theorems show that the connection between the two matrices $\mathcal{M}$ and T_Ψ is much more substantial.

Theorem (Saff & St., CAOT, 2012 and Beckemann & St., Constr. Approx., 2018)

Assume that Γ is piecewise analytic without cusps. Then, it holds as $n \to \infty$,

$$\sqrt{\frac{n+2}{n+1}}\, b_{n+1,n} = b + O\left(\frac{1}{n}\right), \tag{3}$$

and for $k \geq 0$,

$$\sqrt{\frac{n-k+1}{n+1}}\, b_{n-k,n} = b_k + O\left(\frac{1}{n}\right), \tag{4}$$

where O depends on k but not on n.

$\mathcal{M} \to T_\psi$ diagonally: Smooth curve

Improvements in the order of convergence occur in cases when Γ is smooth.

Theorem (Saff & St., CAOT, 2012 and Beckemann & St., Constr. Approx., 2018)

Assume that $\Gamma \in C(p+1,\alpha)$, *with* $p+\alpha > 1/2$. *Then, it holds as* $n \to \infty$,

$$\sqrt{\frac{n+2}{n+1}}\, b_{n+1,n} = b + O\left(\frac{1}{n^{2(p+\alpha)}}\right), \tag{5}$$

and for $k \geq 0$,

$$\sqrt{\frac{n-k+1}{n+1}}\, b_{n-k,n} = b_k + O\left(\frac{1}{n^{2(p+\alpha)}}\right), \tag{6}$$

where O *depends on* k *but not on* n.

$\mathcal{M} \to T_\psi$ diagonally: Analytic curve

For the case of an analytic boundary Γ further improved asymptotic results can be obtained.

Theorem (Saff & St., CAOT, 2012 and Beckemann & St., Constr. Approx., 2018)

Assume that the boundary Γ *is analytic and let* $\rho < 1$ *be the smallest index for which* Φ *is conformal in the exterior of* L_ρ*. Then, it holds as* $n \to \infty$,

$$\sqrt{\frac{n+2}{n+1}}\, b_{n+1,n} = b + O(\rho^{2n}), \tag{7}$$

and for $k \geq 0$,

$$\sqrt{\frac{n-k+1}{n+1}}\, b_{n-k,n} = b_k + O(\rho^{2n}), \tag{8}$$

where O *depends on* k *but not on* n.

Is $\mathcal{M} - T_\psi$ compact?

Corollary

If the upper Hessenberg matrix $\mathcal{M}$ is banded, with constant bandwidth, then $\mathcal{M} - T_\psi$ defines a compact operator on $l^2(\mathbb{N})$.

Theorem (Putinar & St, CAOT, 2007)

If the Bergman polynomials $\{p_n\}$ satisfy a 3-term recurrence relation, then $\Gamma = \partial G$ is an ellipse.

Theorem (Khavinson & St, Springer, 2009 (St, CRAS, 2010))

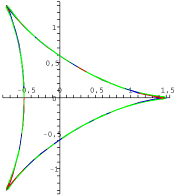
Assume that:

(i) *$\Gamma = \partial G$ is C^2 continuous (piecewise analytic without cusps).*

(ii) *The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy an $m+1$-term recurrence relation, with some $m \geq 2$.*

Then $m = 2$ and Γ is an ellipse.

Example: G is a 3-cusped hypocycloid



Note that $\text{supp}(\mu_\Gamma) = \Gamma$ and recall $\sigma_{ess}(\mathcal{M}) = \Gamma = \sigma_{ess}(T_\Psi)$.

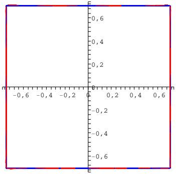
- Levin, Saff & St., Constr. Approx. (2003):

$$\boxed{\nu(p_n) \xrightarrow{*} \mu_\Gamma, \quad n \to \infty,\ n \in \mathcal{N}}, \quad \mathcal{N} \subset \mathbb{N}.$$

- He & Saff, JAT (1994):

$$\boxed{\sigma(\mathcal{T}_n) \subset [0, 1.5] \cup [0, 1.5e^{i2\pi/3}] \cup [0, 1.5e^{i4\pi/3}]}.$$

Example: G is the square



$$\sigma_{ess}(\mathcal{M}) = \Gamma = \sigma_{ess}(T_\Psi).$$

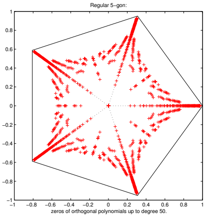
- Maymeskul & Saff, JAT (2003):

$$\boxed{\sigma(\mathcal{M}_n) \subset \text{the two diagonals}}.$$

- Kuijlaars & Saff, Math. Proc. Cambrigde Phil. Soc. (1995):

$$\boxed{\nu(G_n) \xrightarrow{*} \mu_\Gamma, \quad n \to \infty,\ n \in \mathcal{N}}, \quad \mathcal{N} \subset \mathbb{N}$$

Example: G is the canonical pentagon



$$\sigma_{ess}(\mathcal{M}) = \Gamma = \sigma_{ess}(T_\Psi).$$

Levin, Saff & St., Constr. Approx. (2003):

$$\boxed{\nu(p_n) \xrightarrow{*} \mu_\Gamma, \quad n \to \infty,\ n \in \mathcal{N}}, \quad \mathcal{N} \subset \mathbb{N}$$

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$$\boxed{\nu(G_n) \xrightarrow{*} \mu_\Gamma, \quad n \to \infty,\ n \in \mathcal{N}}, \quad \mathcal{N} \subset \mathbb{N}$$

The challenge

Problem

Describe the three distinct behaviours in the spectral properties of $\mathcal{M}_n$ and $\mathcal{T}_n$, by using the two infinite matrices $\mathcal{M}$ and T_ψ ONLY!

Note that each of the matrix alone, carries all the information of the domain G, because it contains, either as limits, or explicitly, all the coefficients of the inverse conformal mapping $\psi : \Delta \to \Omega$.