The old Grunsky Matrix in Recent Applications

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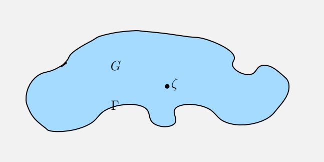
Definition



Γ: bounded Jordan curve, $G := \text{int}(\Gamma)$

$$\langle f, g \rangle := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle^{1/2}$$

The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G are the orthonormal polynomials w.r.t. the area measure:

$$\langle p_m, p_n \rangle = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots.$$

Minimal property

The monic orthogonal polynomials $p_n(z)/\lambda_n$, can be defined by the extremal property

$$\left\|\frac{1}{\lambda_n}p_n\right\|_{L^2(G)} := \min_{z^n+\cdots} \|z^n + \cdots\|_{L^2(G)} = \frac{1}{\lambda_n}.$$

A related extremal problem leads to the sequence $\{\Lambda_n(z)\}_{n=0}^{\infty}$ of the Christoffel functions. These are defined, for any $z \in \mathbb{C}$, by

$$\Lambda_n(z) := \inf\{\|P\|^2_{L^2(G)},\ P \in \mathbb{P}_n \text{ with } P(z) = 1\},$$

where $\mathbb{P}_n$ is the space of polynomials of degree $\leq n$.

Christoffel functions

The Cauchy-Schwarz inequality yields that

$$\frac{1}{\Lambda_n(z)} = \sum_{k=0}^{n} |p_k(z)|^2, \quad z \in \mathbb{C}.$$

Hence, $\Lambda_n(z)$ is the inverse of the diagonal of the kernel polynomials

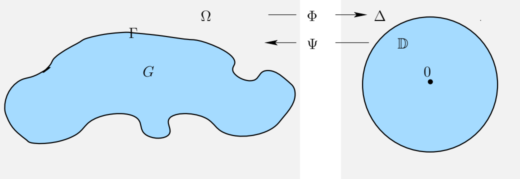
$$K_n(z,\zeta) := \sum_{k=0}^{n} \overline{p_k(\zeta)} p_k(z)$$

This leads to reconstruction algorithms from a finite set of moments

$$\int z^k \overline{z}^l d\mu(z), \quad k, l = 0, 1, \dots, n.$$

- Archipelagos, in Gustafsson, Putinar, Saff & St, Adv. Math. (2009).
- Archipelagos with Lakes, in Saff, Stahl, St & Totik, SIAM J. Math. Anal. (2016).

Exterior Conformal Maps



$$\Omega := \overline{\mathbb{C}} \setminus \overline{G}$$

$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots . \qquad \text{cap}(\Gamma) = 1/\gamma$$

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots . \qquad \text{cap}(\Gamma) = b$$

The Bergman polynomials of G:

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots .$$

Strong asymptotics for Γ non-smooth

Theorem (St, Constr. Approx. (2013))

Assume that Γ *is piecewise analytic without cusps. Then, for* $n \in \mathbb{N}$,

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n}, \quad \text{where} \quad 0 \le \alpha_n \le c(\Gamma)\,\frac{1}{n},$$

and for any $z \in \Omega$,

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\left\{1 + A_n(z)\right\}},$$

where

$$|A_n(z)| \le \frac{c_1(\Gamma)}{\text{dist}(z,\Gamma)\,|\Phi'(z)|}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}.$$

With $\text{dist}(z,\Gamma)$ we denote the Euclidian distance of z from Γ.

Pointwise estimate on Γ

The next theorem gives a pointwise estimate for $|p_n(z)|$, $z \in \Gamma$.

Theorem (St, Contemp. Math., 2016)

Assume that Γ is piecewise analytic without cusps. Then, for any $z \in \Gamma$ away from corners,

$$|p_n(z)| \leq c(\Gamma, z) n^{1/2}.$$

If z_j is a corner of Γ with exterior angle $\omega_j \pi$, $0 < \omega_j < 2$, then

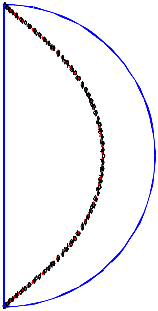
$$|p_n(z_j)| \leq c(\Gamma, z) n^{\omega_j - 1/2} \sqrt{\log n}.$$

It is interesting to note that the above yields the following limit

$$\lim_{n \to \infty} p_n(z_j) = 0,$$

provided $0 < \omega_j < 1/2$.

Sharpness of α_n for Γ non-smooth: An example



$$\gamma = \frac{1}{\text{cap}(\Gamma)} = \frac{3\sqrt{3}}{4}$$

We compute, by using the Gram-Schmidt process (in finite precision), the Bergman polynomials $p_n(z)$ for the unit half-disk, for n up to 60 and test the hypothesis

$$\alpha_n := 1 - \frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} \approx C\,\frac{1}{n^s}.$$

Sharpness of α_n for Γ non-smooth: Numerical data

n	α_n	s
51	0.003 263 458 678	-
52	0.003 200 769 764	0.998 887
53	0.003 140 444 435	0.998 899
54	0.003 082 351 464	0.998 911
55	0.003 026 369 160	0.998 923
56	0.002 972 384 524	0.998 934
57	0.002 920 292 482	0.998 946
58	0.002 869 952 027	0.998 957
59	0.002 821 401 485	0.998 968
60	0.002 774 426 207	0.998 979

The numbers indicate clearly that $\alpha_n \approx C\frac{1}{n}$. Accordingly, we have made the conjecture that the order $O(1/n)$ for α_n, is sharp. This has been verified by E. Mina-Diaz (Numer. Algorithms, 2015).

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Sharpness of A_n for Γ non-smooth: An example

Note that $A_n(\infty) = \alpha_n$, hence the estimate $A_n(\infty) = O(1/n)$ is sharp.

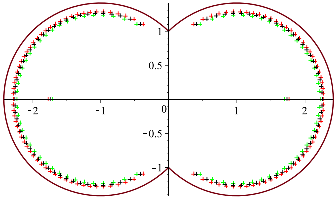
Question

Is the order $A_n(z) = O(1/\sqrt{n})$ sharp in compact subsets of Ω?

Consider the case where G is defined by the two intersecting circles $|z-1| = \sqrt{2}$ and $|z+1| = \sqrt{2}$. Then,

$$\Phi(z) = \frac{1}{2}\left(z - \frac{1}{z}\right).$$

The two intersecting circles



Zeros of the Bergman polynomials $p_n(z)$, with $n = 80, 100, 120$.

Theorem (Saff & St, JAT 2015)

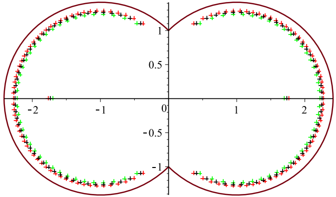
Let ν_n denote the normalised counting measure of zeros of p_n. Then

$$\nu_n \xrightarrow{*} \mu_\Gamma, \quad n \to \infty, \quad n \in \mathbb{N},$$

where μ_Γ denotes the equilibrium measure on Γ.

The reluctance of the zeros to approach the points $\pm i$, is due to the fact that $d\mu_\Gamma(z) = |\Phi'(z)|ds$, where s denotes the arclength on Γ.

The two intersecting circles



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Sharpness of A_n for Γ non-smooth: Numerical data

We test the hypothesis $|A_n(3)| \approx C\, 1/n^s$.

n	$\lvert A_n(3)\rvert$	s_n	n	$\lvert A_n(3)\rvert$	s_n
100	8.120e-5	1.02301	101	7.210e-5	0.9537
102	7.958e-5	1.02284	103	7.077e-5	0.9543
104	7.801e-5	1.02266	105	6.948e-5	0.9549
106	7.651e-5	1.02249	107	6.824e-5	0.9555
108	7.506e-5	1.02233	109	6.704e-5	0.9561
110	7.366e-5	1.02216	111	6.589e-5	0.9567
112	7.232e-5	1.02200	113	6.477e-5	0.9572
114	7.102e-5	1.02184	115	6.369e-5	0.9577
116	6.977e-5	1.02169	117	6.265e-5	0.9582
118	6.856e-5	1.02154	119	6.164e-5	—
120	6.739e-5	—			

Computed values for $|A_n(3)|$ and s_n.

The computed values in the table indicate clearly $|A_n(3)| \approx C\,\frac{1}{n}$ rather than the order $|A_n(3)| \approx C\,\frac{1}{\sqrt{n}}$, predicted by the theory above.

Faber polynomials of the second kind

Consider the polynomial part of $\Phi^n(z)\Phi'(z)$ and denote the resulting series by $\{G_n\}_{n=0}^{\infty}$. Thus,

$$\Phi^n(z)\Phi'(z) = G_n(z) - H_n(z), \quad z \in \Omega,$$

with

$$G_n(z) = \gamma^{n+1} z^n + \cdots \quad \text{and} \quad H_n(z) = O\left(1/|z|^2\right), \quad z \to \infty.$$

$G_n(z)$ is the so-called Faber polynomial of the 2nd kind (of degree n). We also consider the auxiliary polynomial

$$q_{n-1}(z) := G_n(z) - \frac{\gamma^{n+1}}{\lambda_n} p_n(z).$$

Observe that $q_{n-1}(z)$ has degree at most $n-1$, but it can be identical to zero, as the special case $G = \{z : |z| < 1\}$ shows.

The links

Set

$$\varepsilon_n := \frac{n+1}{\pi} \|H_n\|^2_{L^2(\Omega)}$$

and

$$\beta_n := \frac{n+1}{\pi} \|q_{n-1}\|^2_{L^2(G)}.$$

Theorem (St, Constr. Approx. 2013)

It holds $H_n \in L^2(\Omega)$ and

$$\frac{n+1}{\pi} \|G_n\|^2_{L^2(G)} + \frac{n+1}{\pi} \|H_n\|^2_{L^2(\Omega)} = 1. \tag{1}$$

$$\frac{n+1}{\pi} \frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - (\beta_n + \varepsilon_n). \tag{2}$$

Note: Equations (1) and (2) hold for any bounded simply connected domain G, provided that ∂G has zero area.

Quasiconformal curves

Definition

A Jordan curve Γ is quasiconformal if there exists a constant $K > 0$, such that

$$\operatorname{diam}\Gamma(a,b) \le K|a-b|, \text{ for all } a,b \in \Gamma,$$

where $\Gamma(a,b)$ is the arc (of smaller diameter) of Γ between a and b.

Note: A piecewise analytic Jordan curve is quasiconformal if and only if has no cusps (0 and 2π angles).

Estimating $\alpha_n := \varepsilon_n + \beta_n$

Theorem (St, Constr. Approx. 2013)

If Γ is quasiconformal, then for any $n \in \mathbb{N}$,

$$0 \leq \beta_n \leq c(\Gamma)\, \varepsilon_n,$$

If in addition Γ is piecewise analytic, then for any $n \in \mathbb{N}$,

$$0 \leq \varepsilon_n \leq c(\Gamma)\, \frac{1}{n}.$$

These two results lead to the estimate,

$$0 \leq \alpha_n \leq c(\Gamma)\, \frac{1}{n}, \quad n \in \mathbb{N}.$$

Estimating β_n

The estimate for β_n is based on the following

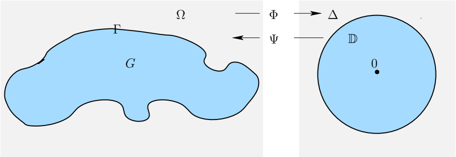
Lemma (St, Constr. Approx. 2013)

Assume that Γ is quasiconformal and rectifiable. Then, for any f analytic in G, continuous on $\overline{G}$ and g analytic in Ω, continuous on $\overline{\Omega}$, with $g' \in L^2(\Omega)$, there holds that

$$\left| \frac{1}{2i} \int_\Gamma f(z)\overline{g(z)}dz \right| \leq \frac{k}{\sqrt{1-k^2}} \|f\|_{L^2(G)} \|g'\|_{L^2(\Omega)},$$

where $k \geq 0$ is a reflection factor of Γ.

Grunsky coefficients



Recall

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots .$$

The corresponding Grunsky coefficients $b_{\ell,n} = b_{n,\ell}$ are defined by the generating series

$$\log\left(\frac{\Psi(w) - \Psi(t)}{\Psi'(\infty)(w-t)}\right) = -\sum_{n=1}^{\infty}\sum_{\ell=1}^{\infty} b_{n,\ell} w^{-\ell} t^{-n},$$

which is analytic and absolutely convergent for $|w| > 1, |t| > 1$.

Grunsky coefficients: Connection to quasi-conformal

It is more convenient to work with the normalized Grunsky coefficients

$$C_{n,k} = C_{k,n} = \sqrt{n+1}\,\sqrt{k+1}\,b_{n+1,k+1}, \quad \text{for } n,k = 0,1,2,\ldots,$$

Then, we have Grunsky inequality: For any integer $m \geq 0$ and any complex numbers $y_0, y_1, \ldots, y_m$ there holds

$$\sum_{n=0}^{\infty}\left|\sum_{k=0}^{m} C_{n,k}\,y_k\right|^2 \leq \sum_{k=0}^{m} |y_k|^2.$$

Theorem (Pommerenke, Univalent Functions, 1975)

Let $C = (C_{n,k})_{n,k=0,1,\ldots}$ denote the infinite Grunsky matrix. Then $\|C\| < 1$, if and only if Γ is quasiconformal.

Grunsky coefficients

Pommerenke, Univalent Functions, 1975

Many difficulties in the application of Grunsky inequality are connected with the fact that the Grunsky coefficients $C_{k,l}$ are already defined by $C_{k,0} = b_k$ in an very complicated way. It is not clear how one could put this information onto a functional analytic or algebraic formulation.

Lemma (B. Beckermann & St, Constr. Approx., 2018)

It holds for $n \in \mathbb{N}$ that

$$\varepsilon_n = \sum_{k=0}^{\infty} |C_{k,n}|^2.$$

Set

$$f_n(z) := \frac{F'_{n+1}(z)}{\sqrt{\pi}\sqrt{n+1}} = \sqrt{\frac{n+1}{\pi}}\gamma^{n+1} z^n + \text{ terms of smaller degree.}$$

and express f_n in the orthonormal basis $\{p_n\}$, that is,

$$f_n(z) = \sum_{j=0}^{n} p_j(z) R_{j,n}, \quad R_{j,n} = \langle f_n, p_j \rangle_{L^2(G)}.$$

Corollary

If Γ is quasiconformal, then the infinite Gram matrix $M = (\langle f_n, f_k \rangle_{L^2(G)})_{k,n=0,1,\dots}$ can be decomposed as

$$M = R^* R = I - C^* C,$$

where M and R represent two bounded linear operators on ℓ^2 with norms ≤ 1. Furthermore, both M and R are boundedly invertible, with $\|M^{-1}\| = \|R^{-1}\|^2 \leq (1 - \|C\|^2)^{-1}$.

Theorem (B. Beckermann & St, Constr. Approx., 2018)

- *If Γ has a corner (with angle different than π), then the corresponding Grunsky operator C is not compact.*
- *Γ is an analytic Jordan curve, if and only if there exists $\rho \in [0,1)$ such $\varepsilon_n = \mathcal{O}(\rho^{2n})$.*
- *There exists a quasiconformal curve Γ such that, for infinitely many $n \in \mathbb{N}$,*

$$\varepsilon_n \geq \frac{\gamma^2}{(n+1)^{1-1/25}}.$$

Proposition (B. Beckermann & St, Constr. Approx., 2018)

C is compact (of p-Schatten class), if and only if $I - M$ and/or $I - M^{-1}$ are compact (of $p/2$-Schatten class). Furthermore, if C is Hilbert-Schmidt then so are both $I - R$ and $I - R^{-1}$.

Theorem (B. Beckermann & St, Constr. Approx., 2018)

Let Γ be quasiconformal, and set

$$\varepsilon_n := \frac{n+1}{\pi}\|H_n\|^2_{L^2(\Omega)}.$$

If $\varepsilon_n = \mathcal{O}(1/n^\beta)$, *for some* $\beta > 0$. *Then*

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\left\{1 + O(\frac{\sqrt{\varepsilon_n}}{n^{\beta/2}})\right\}}, \quad n \to \infty,$$

uniformly on compact subsets of Ω.

Note that:

$$\varepsilon_n = \begin{cases} \mathcal{O}(\rho^{2n}), & \text{if } \Gamma \in U(\rho), \text{ (T. Carleman)}, \\ \mathcal{O}(1/n^{2(p+\alpha)}), & \text{if } \Gamma \in \mathcal{C}(p+1,\alpha) \text{ (P.K. Suetin)}, \\ \mathcal{O}(1/n), & \text{if } \Gamma \text{ is piecewise analytic without cusps (St)}. \end{cases}$$

For Γ is piecewise analytic, the estimate $A_n(z) = O(1/n)$ is sharp.

The estimates above are based on:

Theorem (B. Beckermann & St, Constr. Approx., 2018)

If Γ is quasiconformal, then for all $j \geq 0$*,*

$$\max\Big(\|e_j^*(I-R^*)\|, \|e_j^*(R^{-1}-I)\|\Big) \leq \|e_j^*(R^{-1}-R^*)\| \leq \sqrt{\varepsilon_j \frac{\|C\|^2}{1-\|C\|^2}}.$$

Furthermore, for all $0 \leq j \leq n$*,*

$$\max\Big(|R_{j,n}-\delta_{j,n}|, |R_{j,n}^{-1}-\delta_{j,n}|\Big) \leq \frac{\sqrt{\varepsilon_j\,\varepsilon_n}}{1-\|C\|^2}.$$

The upper Hessenberg matrix $\mathcal{M}$

Consider the (infinite) recurrence relation satisfied by the p_n's

$$zp_n(z) = \sum_{k=0}^{n+1} a_{k,n} p_k(z), \quad n = 0, 1, \ldots,$$

and note that $a_{k,n}$ are Fourier coefficients: $\boxed{a_{k,n} = \langle zp_n, p_k \rangle}$. This induces the (infinite) upper Hessenberg matrix

$$\mathcal{M} = \begin{bmatrix} a_{00} & a_{01} & a_{02} & a_{03} & a_{04} & a_{05} & a_{06} & \cdots \\ a_{10} & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & \cdots \\ 0 & a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & \cdots \\ 0 & 0 & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & \cdots \\ 0 & 0 & 0 & a_{43} & a_{44} & a_{45} & a_{46} & \cdots \\ 0 & 0 & 0 & 0 & a_{54} & a_{55} & a_{56} & \cdots \\ 0 & 0 & 0 & 0 & 0 & a_{65} & a_{66} & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$

The Toeplitz matrix $\mathcal{G}$

Similarly, the Faber polys of the 2nd kind satisfy the (infinite) recurrence relation

$$zG_n(z) = bG_{n+1}(z) + \sum_{k=0}^{n} b_k G_{n-k}(z), \quad n = 0, 1, \ldots,$$

and induce the upper Hessenberg Toeplitz matrix

$$\mathcal{G} = \begin{bmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & b_6 & \cdots \\ b & b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \cdots \\ 0 & b & b_0 & b_1 & b_2 & b_3 & b_4 & \cdots \\ 0 & 0 & b & b_0 & b_1 & b_2 & b_3 & \cdots \\ 0 & 0 & 0 & b & b_0 & b_1 & b_2 & \cdots \\ 0 & 0 & 0 & 0 & b & b_0 & b_1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & b & b_0 & \ddots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}$$