Relative Asymptotics of Orthogonal Polynomials for Perturbed Measures

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Perturbation of measures

Let μ_0 and μ_1 be two finite Borel measures having compact and infinite supports $\boxed{S_j := \operatorname{supp}(\mu_j)}$ in the complex plane $\mathbb{C}$, with $\mu_0 \geq \mu_1$.
Then, there exists a measure μ_2 such that

$$\mu_0 := \mu_1 + \mu_2,$$

and we denote the support of μ_2 by $\boxed{S_2 := \operatorname{supp}(\mu_2)}$.
We shall regard μ_0 as a perturbation of μ_1 and investigate when such a perturbation is “small".

Lebesgue spaces and Orthonormal Polynomials

The three measures yield three Lebesgue spaces $L^2(\mu_j), j = 0, 1, 2,$ with respective inner products

$$\langle f, g \rangle_{\mu_j} := \int f(z)\overline{g(z)}d\mu_j(z)$$

and norms $\|f\|_{L^2(\mu_j)} := \langle f, f \rangle_{\mu_j}^{1/2}$.
Let $\{p_n(\mu_j, z)\}_{n=0}^{\infty}, j = 0, 1,$ denote the sequence of orthonormal polynomials associated with μ_j. That is, the unique sequence of the form

$$p_n(\mu_j, z) = \gamma_n(\mu_j)z^n + \cdots, \quad \gamma_n(\mu_j) > 0, \quad n = 0, 1, 2, \ldots,$$

satisfying $\langle p_m(\mu_j, \cdot), p_n(\mu_j, \cdot) \rangle_{\mu_j} = \delta_{m,n}$.

Christoffel functions

The monic orthogonal polynomials $p_n(\mu_j, z)/\gamma_n(\mu_j)$, can be defined by the extremal property

$$\left\| \frac{1}{\gamma_n(\mu_j)} p_n(\mu_j, \cdot) \right\|_{L^2(\mu_j)} := \min_{z^n+\cdots} \|z^n + \cdots\|_{L^2(\mu_j)} = \frac{1}{\gamma_n(\mu_j)}.$$

A related extremal problem leads to the sequence $\{\lambda_n(\mu_j, z)\}_{n=0}^{\infty}$ of the Christoffel functions. These are defined, for any $z \in \mathbb{C}$, by

$$\lambda_n(\mu_j, z) := \inf\{\|P\|^2_{L^2(\mu_j)},\ P \in \mathbb{P}_n \text{ with } P(z) = 1\},$$

where $\mathbb{P}_n$ is the space of polynomials of degree $\leq n$.
Since, $\mu_1 \leq \mu_0$ the following inequality is immediate

$$\lambda_n(\mu_1, z) \leq \lambda_n(\mu_0, z), \quad n = 0, 1, \ldots.$$

Christoffel functions

Cauchy-Schwarz inequality yields that

$$\frac{1}{\lambda_n(\mu_j, z)} = \sum_{k=0}^{n} |p_k(\mu_j, z)|^2, \quad z \in \mathbb{C}.$$

This leads to reconstruction algorithms from a finite set of moments

$$\int z^k \overline{z}^l d\mu_j(z), \quad k, l = 0, 1, \ldots, n.$$

- Archipelagos, in Gustafsson, Putinar, Saff & St, Adv. Math. (2009).
- Archipelagos with Lakes, in Saff, Stahl, St & Totik, SIAM J. Math. Anal. (2016).

PS perturbation

Definition (PS perturbation)

With $\boxed{\mu_0 := \mu_1 + \mu_2}$ we say that μ_0 is a polynomially small (PS) perturbation of μ_1 provided that μ_2 is not the zero measure and

$$\lim_{n\to\infty} \|p_n(\mu_1, \cdot)\|_{L^2(\mu_2)} = 0.$$

The next result shows that the fact that μ_0 is a PS perturbation of μ_1 implies certain constraints on the relative position of the support of μ_2. We use $\mathrm{Co}(E)$ to denote the convex hull of a set E.

Proposition

If μ_0 is a PS perturbation of μ_1, then $S_2 \subset \mathrm{Co}(S_1)$.

The result is a simple consequence of Theorem 1.1.4 in Stahl & Totik, *General Orthogonal Polynomials*, CUP 1992.

Examples of PS perturbations

We use $A|_E$ to denote the area measure on a bounded set E and $s|_\Gamma$ to denote the arclength measure on a rectifiable curve Γ.

Example (I)

Let G be a bounded Jordan domain (or the union of finitely many bounded Jordan domains with pairwise disjoint closures) and let B be a compact subset of G. Take $\mu_1 = A|_{G\setminus B}$, $\mu_2 = w(z)A|_B$ where $w(z)$ is integrable on B and $\mu_0 = \mu_1 + \mu_2$.

Then Lemma 2.2 of Stahl, Saff, St & Totik, SIAM, J. Math. Anal. (2015) implies the PS property.

Examples of PS perturbations

We use $A|_E$ to denote the area measure on a bounded set E and $s|_\Gamma$ to denote the arclength measure on a rectifiable curve Γ.

Example (II)

Let Γ be a a closed piecewise analytic Jordan curve without cusps and let B be a compact subset in the interior of Γ. Take $\mu_1 = s|_\Gamma$, $\mu_2 = w(z)A|_B$, where $w(z)$ is integrable on B and $\mu_0 = \mu_1 + \mu_2$.

Then Theorem 2.1 of Pritsker, CMFT (2003) implies the PS property.

Examples

Example (III)

Here we assume μ_1 is in the Szegő class on the unit circle; i.e., the absolutely continuous part $w(\theta)$ of μ_1 with respect to arclength on the unit circle $|z| = 1$ satisfies the condition $\int_0^{2\pi} \log(w(\theta))d\theta > -\infty$, and we let μ_2 be a finite measure supported on a compact set inside the unit circle. Take $\mu_0 = \mu_1 + \mu_2$.

Then Corollary 2.4.10 of B. Simon, *Orthogonal Polynomials on the Unit Circle I*, AMS (2005) implies the PS property.

Examples

Example (IV)

Let Γ be a piecewise analytic Jordan curve without cusps, let G denote its interior let $\mu_1 = A|_G$ and set $\mu_0 = \mu_1 + t\delta_z$, $t > 0$, where z is a point on the boundary Γ and δ_z is the Dirac measure at z.

- If the exterior angle at $z \in \Gamma$ is less than $\pi/2$, then from St, Contemporary Math., (2016) $\boxed{\lim_{n\to\infty} p_n(\mu_1, z) = 0}$, and therefore $\mu_0 = \mu_1 + t\delta_z$, is a PS perturbation of μ_1.
- If the exterior angle at z is π, then from Totik & Varga, Proc. Lond. Math. Soc. (2016) $\boxed{|p_n(\mu_1, z)| \geq Cn^{1/2}}$, for some positive constant C and infinitely many n and thus $\mu_0 = \mu_1 + t\delta_z$ is not a PS perturbation of μ_1,

Examples

Proposition

Let G be a bounded Jordan domain with boundary Γ in the class $C(2,\alpha)$, $\alpha > 1/2$. If $\mu_1 = s|_\Gamma$ is the arclength measure on Γ and $\mu_2 = A|_G$ is the area measure on G, then $\mu_0 = \mu_1 + \mu_2$ is a PS perturbation of μ_1.

A Useful Lemma

In view of the extremal property of the monic orthogonal polynomials the assumption $\mu_0 \geq \mu_1$ implies

$$\gamma_n(\mu_0) \leq \gamma_n(\mu_1), \quad n = 0, 1, \ldots.$$

Lemma (Useful)

For all $n \in \mathbb{N} \cup \{0\}$,

$$\frac{\gamma_n(\mu_1)}{\gamma_n(\mu_0)} = 1 + \beta_n,$$

where β_n is non-negative and such that

$$\frac{1}{\left\{1 - \|p_n(\mu_0, \cdot)\|^2_{L^2(\mu_2)}\right\}^{1/2}} - 1 \leq \beta_n \leq \left\{1 + \|p_n(\mu_1, \cdot)\|^2_{L^2(\mu_2)}\right\}^{1/2} - 1.$$

Our reasoning is guided by the arguments for Bergman polynomials in Saff, Stahl, St & Totik, SIAM J. Math. Anal. (2016).

A Useful Lemma

Lemma (Useful, cont.)

Furthermore,

(i)

$$\|p_n(\mu_0,\cdot) - p_n(\mu_1,\cdot)\|^2_{L^2(\mu_1)} \leq 2\beta_n;$$

(ii) *for any* $z \in \overline{\mathbb{C}} \setminus \mathrm{Co}(S_1)$

$$\left|\frac{p_n(\mu_0,z)}{p_n(\mu_1,z)} - 1\right| \leq \sqrt{2\beta_n}\left[1 + \frac{\mathrm{diam}(S_1)}{\mathrm{dist}(z,\mathrm{Co}(S_1))}\right]^2.$$

The result holds for *any* two positive measures with compact and infinite support, such that $\mu_0 \geq \mu_1$.

The main perturbation result

The main purpose of the talk is to show that the following two associated pairs of sequences

$$\{\gamma_n(\mu_0), \gamma_n(\mu_1)\}, \quad \{p_n(\mu_0, z), p_n(\mu_1, z)\},$$

have comparable asymptotics when the measure μ_0 is a polynomially small perturbation of μ_1. Also, the pair

$$\{\lambda_n(\mu_0, z), \lambda_n(\mu_1, z)\},$$

is comparable on a somewhat stronger condition.

- We let Ω denote the unbounded component of $\overline{\mathbb{C}} \setminus S_1$.
- We use $\mathrm{cap}(E)$ to denote the logarithmic capacity of a compact set E.

The main perturbation result

Theorem (Main)

If the measure μ_0 is a PS perturbation of the measure μ_1, then:

(i)

$$\lim_{n\to\infty} \gamma_n(\mu_1)/\gamma_n(\mu_0) = 1;$$

(ii)

$$\lim_{n\to\infty} \|p_n(\mu_1,\cdot) - p_n(\mu_0,\cdot)\|_{L^2(\mu_0)} = 0;$$

(iii) *uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \mathrm{Co}(S_1)$:*

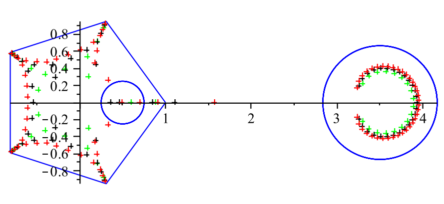
$$\lim_{n\to\infty} p_n(\mu_1,z)/p_n(\mu_0,z) = 1;$$

Furthermore, if $\mathrm{cap}(S_1) > 0$ and $\lim_{m\to\infty} \sum_{j=m}^{\infty} \|p_j(\mu_1,z)\|^2_{L^2(\mu_2)} = 0$, then uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \Omega$,

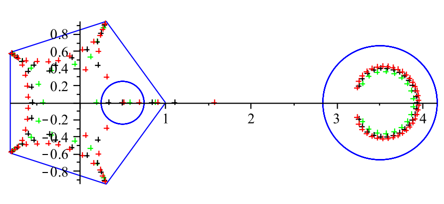
$$\lim_{n\to\infty} \lambda_n(\mu_0,z)/\lambda_n(\mu_1,z) = 1,$$

Motivation

Zeros of p_{40}, p_{60} and p_{80}, for pentagon G_1, disk lake K, disk G_2

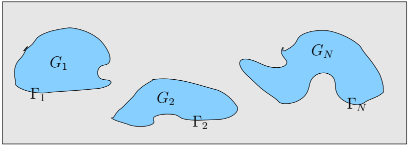


$$d\mu = dA|_{G_1} + dA|_{G_2} - dA|_K$$



$$d\mu = dA|_{G_1} + dA|_{G_2} + dA|_K$$

Bergman polynomials on an archipelago



Γ_j, $j = 1, \ldots, N$, a system of disjoint and mutually exterior Jordan curves in $\mathbb{C}$, $\boxed{G_j := \text{int}(\Gamma_j)}$, $\boxed{\Gamma := \cup_{j=1}^N \Gamma_j}$, $\boxed{G := \cup_{j=1}^N G_j}$.

$$\langle f, g\rangle_G := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f\rangle_G^{1/2}$$

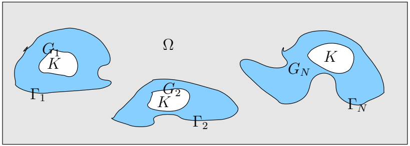
The Bergman polynomials $\{p_n\}_{n=0}^\infty$ of G are the unique orthonormal polynomials w.r.t. the area measure on G:

$$\langle p_m, p_n\rangle_G = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \gamma_n z^n + \cdots, \quad \gamma_n > 0, \quad n = 0, 1, 2, \ldots.$$

Bergman polynomials on archipelago with lakes



With K is a compact subset of G, set $\boxed{G^* := G \setminus K}$ and consider

$$\langle f, g\rangle_{G^*} := \int_{G^*} f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G^*)} := \langle f, f\rangle_{G^*}^{1/2}.$$

The Bergman polynomials $\{p_n^*\}_{n=0}^{\infty}$ of G^* are the unique orthonormal polynomials w.r.t. the area measure on G^*:

$$\langle p_m^*, p_n^*\rangle_{G^*} = \int_{G^*} p_m^*(z)\overline{p_n^*(z)}dA(z) = \delta_{m,n},$$

with

$$p_n^*(z) = \gamma_n^* z^n + \cdots, \quad \gamma_n^* > 0, \quad n = 0, 1, 2, \ldots.$$

The SSST Lemma

Take $\mu_1 = A|_{G\setminus K}$ and $\mu_2 = A|_K$. Then $\mu_0 = A|_G$ is a PS perturbation of μ_1, in view of the following result

Lemma (Saff, Stahl, St & Totik, SIAM, J. Math. Anal., 201))

We have

$$\sum_{n=0}^{\infty} |p_n^*(z)|^2 < \infty,$$

uniformly on compact subsets of G. In particular, $p_n^*(z) \to 0$ *uniformly on compact subsets of G.*

The distribution of zeros of p_n^*

Our task is to describe the asymptotic behaviour of the zeros of the polynomials p_n^*.

The behaviour of the zeros of p_n was clarified in Gustafsson, Putinar, Saff & St, Adv. Math. (2009).

Our tool is the normalized counting measure ν_n for the zeros of a the polynomial p_n^*:

$$\nu_n := \frac{1}{n} \sum_{p_n^*(z)=0} \delta_z,$$

where δ_z is the unit point mass (Dirac delta) at the point z.

We denote by μ_E the equilibrium measure for a compact set E.

The balayage theorem

Theorem

If μ is any weak-star limit measure of the sequence $\{\nu_n\}_{n\in\mathbb{N}}$, then μ is a Borel probability measure supported on $\overline{\mathbb{C}}\setminus\Omega$ and $\mu^b=\mu_\Gamma$, where μ^b is the balayage of μ out of $\overline{\mathbb{C}}\setminus\Omega$ onto $\partial\Omega$. Similarly, the sequence of balayaged counting measures converges to μ_Γ:

$$\nu_n^b \xrightarrow{*} \mu_\Gamma, \quad n\to\infty, \quad n\in\mathbb{N}.$$

By the weak-star convergence of a sequence of measures τ_n to a measure τ we mean that, for any continuous f with compact support in $\mathbb{C}$, there holds

$$\int f d\tau_n \to \int f d\tau, \quad \text{as } n\to\infty.$$

The result is a consequence of the fact that $A|_{G^*}$ belongs to the class **Reg** and Theorem 2.3 Mhaskar & Saff, JAT (1991).

The IC-point theorem

A point z_0 on the boundary Γ_j of G_j is said to be an (inward-corner) IC point, if there exists a circular sector of the form $S := \{z : 0 < |z - z_0| < r,\ \alpha\pi < \arg(z - z_0) < \beta\pi\}$ with $\beta - \alpha > 1$ whose closure is contained in G_j except for z_0.

Theorem

Assume that for each $j = 1, \ldots, k$ the boundary Γ_j of G_j, contains an IC point. Then

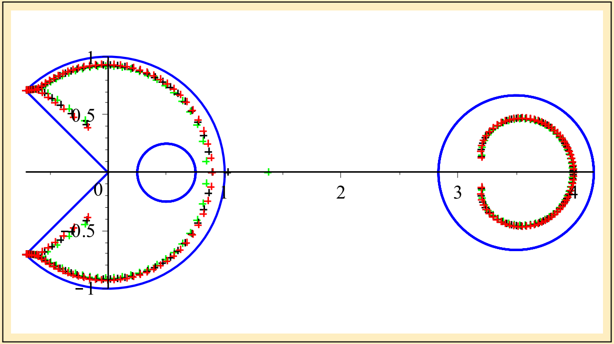
$$\nu_n|_{\mathcal{V}} \xrightarrow{*} \mu_\Gamma|_{\mathcal{V}}, \quad n \to \infty, \quad n \in \mathbb{N},$$

where $\mathcal{V}$ is an open set containing $\bigcup_{j=1}^{k} \overline{G}_j$, such that if $k < m$ the distance of $\overline{\mathcal{V}}$ from $\bigcup_{j=k+1}^{m} \overline{G}_j$ is positive.

The result is a consequence of Corollary 2.2 of Saff & St, JAT (2015).

Example

Zeros of p_{120}, p_{140} and p_{160}, for sector G_1, disk lake K, disk G_2



$$d\mu = dA|_{G_1} + dA|_{G_2} - dA|_K$$