Since Approximation Theory is already there... Bring Potential Theory to Operator Theory!

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International Conference on Orthogonal Polynomials and Holomorphic Dynamics
Copenhagen, Denmark
August 14–17 2018
Let $\mu$ be a finite positive Borel measure having compact and infinite support $S_\mu := \text{supp}(\mu)$ in the complex plane $\mathbb{C}$. Then, the measure yields the Lebesgue spaces $L^2(\mu)$ with inner product

$$\langle f, g \rangle_\mu := \int f(z) \overline{g(z)} d\mu(z)$$

and norm

$$\|f\|_{L^2(\mu)} := \langle f, f \rangle_\mu^{1/2}.$$

Let $\{p_n(\mu, z)\}_{n=0}^{\infty}$ denote the sequence of orthonormal polynomials associated with $\mu$. That is, the unique sequence of the form

$$p_n(\mu, z) = \gamma_n(\mu) z^n + \cdots, \quad \gamma_n(\mu) > 0, \quad n = 0, 1, 2, \ldots,$$

satisfying $\langle p_m(\mu, \cdot), p_n(\mu, \cdot) \rangle_\mu = \delta_{m,n}$. 
Distribution of zeros: The tools

For any polynomial $q_n(z)$, of degree $n$, we denote by $\nu_{q_n}$ the normalized counting measure for the zeros of $q_n(z)$; that is,

$$\nu_{q_n} := \frac{1}{n} \sum_{q_n(z)=0} \delta_z,$$

where $\delta_z$ is the unit point mass (Dirac delta) at the point $z$. For any measure $\mu$ with compact support in $\mathbb{C}$,

$$U^\mu(z) := \int \log \frac{1}{|z-t|} d\mu(t), \quad z \in \mathbb{C},$$

denotes the logarithmic potential on $\mu$. In particular, if $q_n$ is monic, then

$$U^{\nu_{q_n}}(z) = \frac{1}{n} \log \frac{1}{|q_n(z)|}, \quad z \in \mathbb{C}.$$

With $\mu_K$ we denote the equilibrium measure of a compact set $K$ of positive logarithmic capacity.
Theorem (Generalized Minimum Principle)

Let $G \Subset \mathbb{C}$ be a domain and $h$ a superharmonic function on $G$ that is bounded from below and for which

$$\limsup_{z \to \zeta, z \in G} h(z) \geq m,$$

is satisfied for quasi-every $\zeta \in \partial G$. Then,

$$h(z) > m, \quad z \in G,$$

unless $h$ is constant.

Potential Theory: Five theorems

Theorem (Principle of Descent)

Let \( \mu_n, n = 1, 2, \ldots \), be probability measures, supported on the same compact subset of \( \mathbb{C} \), such that

\[ \mu_n \rightharpoonup \mu. \]

Suppose that for each \( n \), a point \( z_n \) is given so that \( z_n \to z \), for some \( z \in \mathbb{C} \). Then,

\[ U^\mu(z) \leq \liminf_{n \to \infty} U^{\mu_n}(z_n). \]

We say that \( \mu_n \rightharpoonup \mu \), if

\[ \int f \, d\mu_n \to \int f \, d\mu, \quad n \to \infty, \]

for every function \( f \) continuous on \( \mathbb{C} \).
Potential Theory: Five theorems

Theorem (Lower Envelope Theorem)

Let $\mu_n$, $n = 1, 2, \ldots$, be a sequence of positive unit Borel measures, supported on the same compact subset of $\mathbb{C}$, such that

$$
\mu_n \rightharpoonup^* \mu.
$$

Then,

$$
\liminf_{n \to \infty} U^{\mu_n}(z) = U^\mu(z),
$$

for quasi-every $z \in \mathbb{C}$.
Theorem (Unicity Theorem)

Suppose that the positive measures $\mu$ and $\nu$ have compact support and in a region $D \subset \mathbb{C}$ the potentials $U^\nu$ and $U^\mu$ satisfy

$$U^\mu(z) = U^\nu(z) + u(z),$$

almost everywhere with respect to two-dimensional Lebesgue measure, where the function $u$ is harmonic in $D$. Then, in $D$ the measures $\mu$ and $\nu$ coincide.
Theorem (Carleson’s Unicity Theorem)

Let $K$ be a compact set of positive capacity, and let $\Omega$ denote the unbounded component of $\mathbb{C} \setminus K$. If $\mu$ and $\nu$ are two unit measures supported on $\partial \Omega$, and if the potentials $U^\mu$ and $U^\nu$ coincide in $\Omega$, then $\mu = \nu$. 
Bergman polynomials \( \{p_n\} \) on an Jordan domain \( G \)

\[ \Gamma := \partial G \quad \text{and} \quad \Omega := \mathbb{C} \setminus \overline{G} \]

\[ \langle f, g \rangle := \int_G f(z) \overline{g(z)} \, dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle^{1/2}. \]

The Bergman polynomials \( \{p_n\}_{n=0}^\infty \) of \( G \) are the orthonormal polynomials w.r.t. the area measure on \( G \):

\[ \langle p_m, p_n \rangle = \int_G p_m(z) \overline{p_n(z)} \, dA(z) = \delta_{m,n}, \]

with

\[ p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots. \]
Example: $G$ is the canonical pentagon

Theorem (Levin, Saff & St., Constr. Approx. 2003)

Let $\varphi$ be a conformal map of $G$ onto the unit disk $\mathbb{D}$. Then, there is a subsequence $\mathcal{N}$ of $\mathbb{N}$ such that

$$\nu_{p_n} \xrightarrow{*} \mu_{\Gamma}, \quad n \to \infty, \quad n \in \mathcal{N},$$

if and only if $\varphi$ cannot be analytically continued to some open set containing $\overline{G}$.
Key results

The above theorem is based on the following facts:

- The area measure on $G$ belongs to the class $\text{Reg}$, that is,
  $$\lim_{n \to \infty} \|p_n\|_{G}^{1/n} = 1.$$  

- The kernel $K(z, \zeta)$, of the Bergman space $L_{a}^{2}(G)$ satisfies,
  $$K(z, \zeta) = \sum_{n=0}^{\infty} p_{n}(\zeta)p_{n}(z), \quad z, \zeta \in G,$$
  and is is related to a normalized conformal map $\varphi_{\zeta} : G \to \mathbb{D}$, $\varphi_{\zeta}(\zeta) = 0, \zeta \in G$, by
  $$K(z, \zeta) = \frac{1}{\pi} \frac{\varphi'_{\zeta}(\zeta)}{\varphi'_{\zeta}(z)}.$$ 

An application of Walsh’s maximal convergence then yields
  $$\limsup_{n \to \infty} |p_{n}(\zeta)|^{1/n} = 1, \quad n \in \mathcal{N},$$
and the result then follows from Theorem III.4.1 in Saff and Totik.
The two intersecting circles

Zeros of $p_n(z)$, with $n = 80, 100, 120$.

**Theorem (Saff & St, JAT 2015)**

*If the boundary $\Gamma$ of $G$ contains an inward corner point, then*

$$\nu_{p_n} \overset{*}{\rightharpoonup} \mu_\Gamma, \quad n \to \infty, \quad n \in \mathbb{N},$$

*where $\mu_\Gamma$ denotes the equilibrium measure on $\Gamma$.*

Based on Gardiner and Pommerenke, Constr, Approx, 2002.

The reluctance of the zeros to approach the points $\pm i$, is due to the fact that $d\mu_\Gamma(z) = |\Phi'(z)|ds$, where $s$ denotes the arclength on $\Gamma$. 
The circular sector

Figure: Zeros of $p_n$, $n = 50, 100, 150$, for the circular sector with opening angle $\pi/2$. 
Theorem (Mina-Diaz, Saff & St., CMFT 2005)

Let \( E \neq \emptyset \) be a compact subset of \( \mathbb{C} \) such that both \( \mathbb{C} \setminus E \) and \( \mathring{E} := \text{int}(E) \) are connected. Let \( g : \mathbb{C} \setminus \mathring{E} \to \mathbb{C} \) be such that \( g \) is analytic in \( \mathbb{C} \setminus E \), \( |g| \) is continuous and never zero in \( \mathbb{C} \setminus \mathring{E} \), \( g(\infty) = \infty \) and \( g'(\infty) = 1 \). Let \( \{q_n\}_{n=1}^{\infty} \) be a sequence of monic polynomials of respective degrees \( n = 1, 2, \ldots \), such that \( \infty \) is not an accumulation point of the set of zeros of the \( q_n \)'s. Further, assume that

\[
\limsup_{n \to \infty} |q_n(z)|^{1/n} \leq |g(z)| \quad \text{q.e.} \quad z \in \partial E.
\]
Theorem (Mina-Diaz, Saff & St., CMFT 2005, cont.)

Then, any measure $\sigma$ that is a weak*-limit point of the sequence $\{\nu_{q_n}\}_{n=1}^{\infty}$ is supported on $E$ and

$$U^\sigma(z) = \log |g(z)|^{-1} \quad \forall z \in \mathbb{C} \setminus \overset{.}{E}. \quad (1)$$

Moreover, there is a unique measure $\mu_g$ supported on $\partial E$ such that (1) holds with $\sigma = \mu_g$. For such a measure, we have

(a) if $\overset{.}{E} = \emptyset$, then $\nu_{q_n} \overset{*}{\rightharpoonup} \mu_g$ as $n \to \infty$;

(b) if $\overset{.}{E} \neq \emptyset$ and for some $z_0 \in \overset{.}{E}$ and a subsequence $\mathcal{N} \subset \mathbb{N}$

$$\lim_{n \to \infty} |q_n(z_0)|^{1/n} = e^{-U^{\mu_g}(z_0)},$$

then

$$\nu_{q_n} \overset{*}{\rightharpoonup} \mu_g \quad \text{as} \quad n \to \infty, \quad n \in \mathcal{N}.$$
Observe that the assumption of the theorem is equivalent to

\[ \liminf_{n \to \infty} U^{\nu_{q_n}}(z) \geq \log |g(z)|^{-1} \quad \text{q.e.} \quad z \in \partial E. \quad (2) \]

Let \( \sigma \) be a weak*-limit point of the sequence \( \{\nu_{q_n}\}_{n=1}^{\infty} \), so that for some subsequence \( \mathcal{N} \subset \mathbb{N} \)

\[ \nu_{q_n} \xrightarrow{*} \sigma \quad \text{as} \quad n \to \infty, \quad n \in \mathcal{N}. \]

Then \( \sigma \) is a probability measure and by (2) and the Lower Envelope Theorem, we have for q.e. \( z \in \partial E \),

\[ U^{\sigma}(z) = \liminf_{n \to \infty} U^{\nu_{q_n}}(z) \geq \liminf_{n \to \infty} U^{\nu_{q_n}}(z) \geq \log |g(z)|^{-1}. \quad (3) \]
By the assumptions on $g$, the function

$$F^\sigma(z) := U^\sigma(z) - \log |g(z)|^{-1}, \quad z \in \mathbb{C} \setminus E,$$

is superharmonic and lower bounded in $\mathbb{C} \setminus E$, harmonic and equal to zero at $\infty$, and in view of (3) and the lower semicontinuity of $U^\sigma$, it also satisfies for *quasi-every* $z' \in \partial E$

$$\liminf_{z \to z'} F^\sigma(z) \geq \liminf_{z \to z'} U^\sigma(z) - \lim_{z \to z'} \log |g(z)|^{-1} \geq U^\sigma(z') - \log |g(z')|^{-1} \geq 0.$$

Then, by the generalized minimum principle for superharmonic functions we conclude that $F^\sigma \equiv 0$, which implies that (1) holds in $\mathbb{C} \setminus E$. It also implies that $U^\sigma$ is harmonic in $\mathbb{C} \setminus E$ and therefore, in view of the Unicity Theorem $\text{supp}(\sigma)$ must be contained in $E$. It is a direct consequence of Carleson’s Unicity Theorem that there can be at most one measure $\mu_g$ supported on $\partial E$ that satisfies (1) with $\sigma = \mu_g$. 
Bergman polynomials on an archipelago

\[ G_j, \ j = 1, \ldots, N, \ \text{a system of disjoint and mutually exterior Jordan curves in} \ \mathbb{C}, \ \begin{align*} \Gamma_j & := \text{int}(\Gamma_j), \quad \Gamma := \bigcup_{j=1}^{N} \Gamma_j, \quad G := \bigcup_{j=1}^{N} G_j. \end{align*} \]

\[ \langle f, g \rangle_G := \int_{G} f(z) \overline{g(z)} \, dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle_G^{1/2} \]

The Bergman polynomials \( \{p_n\}_{n=0}^{\infty} \) of \( G \) are the unique orthonormal polynomials w.r.t. the area measure on \( G \):

\[ \langle p_m, p_n \rangle_G = \int_{G} p_m(z) \overline{p_n(z)} \, dA(z) = \delta_{m,n}, \]

with

\[ p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots. \]
Three-disks

Zeros of the Bergman polynomials $p_{140}$, $p_{150}$ and $p_{160}$.

The basic tool for the distribution of zeros

- $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$.
- $K(z, \zeta)$: the Bergman (reproducing) kernel function of $L^2_a(G)$.
- $L_R := \{z : g_\Omega(z, \infty) = \log R\}$ the level lines of the Green function.
- $\varrho(z) := \sup\{R : K(z, \zeta) \text{ has an analytic continuation inside } L_R\}$.
- $h(z) := \begin{cases} g_\Omega(z, \infty), & z \in \Omega, \\ -\log \varrho(z), & z \in G, \end{cases}$
- $\beta := \frac{1}{2\pi} \Delta h$, in the sense of distributions.
- $\nu_{p_n}$: the normalized counting measure of zeros of $p_n$.
- $\mathcal{C}$: the set of weak-star cluster points of the counting measures $\{\nu_{p_n}\}_{n=1}^\infty$, i.e., the set of measures $\sigma$ for which there exists a subsequence $\mathcal{N}_\sigma \subset \mathbb{N}$ such that $\nu_{p_n} \rightharpoonup^* \sigma$, as $n \to \infty$, $n \in \mathcal{N}_\sigma$.
- $\mu_\Gamma$: the equilibrium measure on the boundary $\Gamma$. 
The basic result for the distribution of zeros

**Theorem (Gustafsson, Putinar, Saff & St, Advances in Math, 2009)**

(i) \( \beta \) is a positive unit measure with support contained in \( \overline{G} \).

(ii) The balayage of \( \beta \) onto \( \Gamma \) gives the equilibrium measure \( \mu_\Gamma \):

\[
\begin{align*}
U^\beta &\geq U^{\mu_\Gamma} \text{ in } \mathbb{C}, \\
U^\beta &= U^{\mu_\Gamma} \text{ in } \Omega.
\end{align*}
\]

(iii) \( \mathcal{C} \) is nonempty, and for any \( \sigma \in \mathcal{C} \),

\[
\begin{align*}
U^\sigma &\geq U^\beta \text{ in } \mathbb{C}, \\
U^\sigma &= U^\beta \text{ in the unbounded component of } \overline{\mathbb{C}} \setminus \text{supp}\beta.
\end{align*}
\]

(iv) The measure \( \beta \) is the lower envelope of \( \mathcal{C} \): \( U^\beta = \text{lsc}(\inf_{\sigma \in \mathcal{C}} U^\sigma) \).

(v) If \( \mathcal{C} \) has only one element, then this is \( \beta \) and

\[
\nu_{p_n} \overset{*}{\longrightarrow} \beta, \quad n \to \infty, \quad n \in \mathbb{N}.
\]
Bergman polynomials on archipelago with lakes

With $K$ is a compact subset of $G$, set $G^* := G \setminus K$ and consider

$$\langle f, g \rangle_{G^*} := \int_{G^*} f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G^*)} := \langle f, f \rangle_{G^*}^{1/2}.$$ 

The Bergman polynomials $\{p^*_n\}_{n=0}^\infty$ of $G^*$ are the unique orthonormal polynomials w.r.t. the area measure on $G^*$:

$$\langle p^*_m, p^*_n \rangle_{G^*} = \int_{G^*} p^*_m(z)\overline{p^*_n(z)}dA(z) = \delta_{m,n},$$

with

$$p^*_n(z) = \gamma^*_nz^n + \cdots, \quad \gamma^*_n > 0, \quad n = 0, 1, 2, \ldots.$$
The annular case

Plots of the zeros of $p_n^*(z)$, for $n = 120, 140$ and $160$.

Let $G = \mathbb{D}$, $\mathcal{K} := \{z : |z - a| \leq \varrho\}$, $|a| + \varrho < 1$, $\varrho > 0$, $G^* = \mathbb{D} \setminus \mathcal{K}$, $\mathbb{T} := \partial \mathbb{D}$ and $\{z : |z - a| = \varrho\}$, that is

$$z_1 \overline{z_2} = 1 \quad \text{and} \quad (z_1 - a)(\overline{z_2} - a) = \varrho^2.$$ 

Let $z_1$ denote the point that lies in $\mathcal{K}$ ($z_2$ will then lie outside $\mathbb{D}$).
Proposition (Saff & St, Mat. Sbornik, 2018)

With the above notation, there exists a subsequence $N \subset \mathbb{N}$ such that the normalized zero counting measures for $p_n^*(z)$ satisfy

$$\nu_{p_n^*} \xrightarrow{\ast} \mu_{|z_1|}, \quad n \to \infty, \quad n \in N,$$

where $\mu_{|z_1|}$ denotes the normalized arclength measure on the circle $|z| = |z_1|$.  

Thus, no matter what the relative position of $K$, a weak limit of $\nu_n$ will invariably be the arclength measure on a specific circle in $\mathbb{D}$, always centered at the origin.
Shift Operator on $L^2(\mu)$

Let $N_z$ denote the shift operator on $L^2(\mu)$. That is,

$$N_z : L^2(\mu) \rightarrow L^2(\mu) \quad \text{with} \quad N_z f = zf.$$  

$N_z$ defines a normal operator on $L^2(\mu)$. Furthermore,

$$p_n(\mu, z) = \lambda_n(\mu) \det(z - \pi_n N_z \pi_n),$$

where $\pi_n$ is the projection onto the $n$-dimensional subspace onto $\mathbb{P}_{n-1}$.


Let

$$N(\mu) := \sup\{|z| : z \in S_{\mu}\}.$$  

Then, for any $k \in \mathbb{N}$,

$$\pi_n N_z^k \pi_n - (\pi_n N_z \pi_n)^k,$$

is an operator of rank at most $k$ and norm at most $2N(\mu)^k$. 

Approximation Theory  Operator Theory  Shift
Shift Operator on $L^2(\mu)$

Let $\mu_n$ denote the unit measures $d\mu_n = \frac{1}{n} \sum_{j=0}^{n-1} |p_n(\mu, z)|^2 d\mu(z)$.


$$\frac{1}{n} \text{Tr}(\pi_n N_z \pi_n)^k = \int z^k d\nu_{p_n}.$$  

$$\frac{1}{n} \text{Tr}(\pi_n N_z^k \pi_n) = \int z^k d\mu_n.$$  

**Thus, from the previous theorem, for any $k = 0, 1, 2, \ldots$,**

$$\left| \int z^k d\nu_{p_n} - \int z^k d\mu_n \right| \leq \frac{2kN^k(\mu)}{n}.$$  

**Furthermore, if $K$ is a compact set containing the supports of all $\nu_n$ and $\mu$, such that $\{z_k\}_{k=0}^{\infty} \cup \{\bar{Z}_k\}_{k=0}^{\infty}$ are $\|\cdot\|_\infty$-total in $\mathcal{C}(K)$, then for any subsequence $\{n_j\}$, $\mu_{n_j} \overset{*}{\rightharpoonup} \nu$ if and only if $\mu_{n_j} \rightharpoonup \nu$.**
Krylov subspaces

Let $A \in \mathcal{L}(H)$ be a linear bounded operator acting on the complex Hilbert space $H$ and let $\xi \in H$ be a non-zero vector. We denote $H_n(A, \xi)$ the linear span of the vectors $\xi, A\xi, ..., A^{n-1}\xi$ and let $\pi_n$ be the orthogonal projection of $H$ onto $H_n(A, \xi)$. Let $a_n$ denote the counting measures of the spectra of the finite central truncations $A_n = \pi_n A \pi_n$. Note that for any complex polynomial $p(z)$ it holds that

$$\int p(z) \, da_n(z) = \frac{\text{tr} \, p(A_n)}{n}.$$ 

The orthogonal monic polynomials $P_n$ in this case are defined as minimizers of the functional (semi-norm):

$$\|q\|_{A, \xi}^2 = \|q(A)\xi\|^2, \quad q \in \mathbb{C}[z],$$

and the zeros of $P_n$ (whenever $P_n$ exists) coincide with the spectrum of $A_n$. 
Theorem (Gustafsson & Putinar, Springer 2017)

Let $A, B \in \mathcal{L}(H)$ with $A - B$ of finite trace: $A - B \in \mathcal{C}_1(H)$. Then for every polynomial $p \in \mathbb{C}[z]$ we have

$$\lim_{n \to \infty} \frac{\text{Tr}(p(A_n)) - \text{Tr}(p(B_n))}{n} = 0.$$ 

Corollary

Let $a_n, b_n$ denote the counting measures of the spectra of $A_n$ and $B_n$, respectively. Then,

$$\lim_{n \to \infty} \left[ \int \frac{da_n(\zeta)}{\zeta - z} - \int \frac{db_n(\zeta)}{\zeta - z} \right] = 0,$$

uniformly on compact subsets which are disjoint of the convex hull of $\sigma(A) \cup \sigma(B)$. 

Approximation Theory  Operator Theory  Shift
Conclusion

All the results in this section yield information for the analytic moments:

$$\lim_{n \to \infty} \int z^k d\nu_n = \int z^k d\nu, \quad k = 0, 1, 2, \ldots,$$

where $\nu$ is a known positive measure and $\{\nu_n\}$ are a sequence of positive measures (all supported on the same compact set $K$ in the complex plane) we want to describe its weak limit points. Note that the measures being positive implies the same information for the anti-analytic moments:

$$\lim_{n \to \infty} \int \overline{z}^k d\nu_n = \int \overline{z}^k d\nu, \quad k = 1, 2, \ldots.$$
Conclusion

However, according to the complex Stone-Weierstrass theorem, in order to establish

$$\nu_n \xrightarrow{\ast} \nu,$$

we need the limits of all the complex moments

$$\lim_{n \to \infty} \int z^k \overline{z}^j d\nu_n = \int z^k \overline{z}^j d\nu, \quad k, j = 0, 1, 2, \ldots,$$

unless $K$ is of a special form (Mergelyan, Walsh), where the analytic moments constitute sufficient information.