# Since Approximation Theory is already there...
# Bring Potential Theory to Operator Theory!

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# Lebesgue spaces and Orthonormal Polynomials

Let $\mu$ be a finite positive Borel measure having compact and infinite support $\boxed{S_\mu := \operatorname{supp}(\mu)}$ in the complex plane $\mathbb{C}$. Then, the measure yields the Lebesgue spaces $L^2(\mu)$ with inner product

$$\langle f, g\rangle_\mu := \int f(z)\overline{g(z)}d\mu(z)$$

and norm

$$\|f\|_{L^2(\mu)} := \langle f, f\rangle_\mu^{1/2}.$$

Let $\{p_n(\mu, z)\}_{n=0}^\infty$ denote the sequence of orthonormal polynomials associated with $\mu$. That is, the unique sequence of the form

$$p_n(\mu, z) = \gamma_n(\mu)z^n + \cdots, \quad \gamma_n(\mu) > 0, \quad n = 0, 1, 2, \ldots,$$

satisfying $\langle p_m(\mu, \cdot), p_n(\mu, \cdot)\rangle_\mu = \delta_{m,n}$.

# Distribution of zeros: The tools

For any polynomial $q_n(z)$, of degree $n$, we denote by $\nu_{q_n}$ the normalized counting measure for the zeros of $q_n(z)$; that is,

$$\nu_{q_n} := \frac{1}{n} \sum_{q_n(z)=0} \delta_z,$$

where $\delta_z$ is the unit point mass (Dirac delta) at the point $z$.
For any measure $\mu$ with compact support in $\mathbb{C}$,

$$U^{\mu}(z) := \int \log \frac{1}{|z-t|} d\mu(t), \quad z \in \mathbb{C}.$$

denotes the logarithmic potential on $\mu$. In particular, if $q_n$ is monic, then

$$U^{\nu_{q_n}}(z) = \frac{1}{n} \log \frac{1}{|q_n(z)|}, \quad z \in \mathbb{C}.$$

With $\mu_K$ we denote the equilibrium measure of a compact set $K$ of positive logarithmic capacity.

# Potential Theory: Five theorems

**Theorem (Generalized Minimum Principle)**

*Let $G \in \overline{\mathbb{C}}$ be a domain and $h$ a superharmonic function on $G$ that is bounded from below and for which*

$$\limsup_{z\to\zeta, z\in G} h(z) \geq m,$$

*is satisfied for quasi-every $\zeta \in \partial G$. Then,*

$$h(z) > m, \quad z \in G,$$

*unless $h$ is constant.*

Saff & Totik, Logarithmic Potentials, Springer, 1997.

# Potential Theory: Five theorems

**Theorem (Principle of Descent)**

*Let $\mu_n$, $n = 1, 2, \ldots$ , be probability measures, supported on the same compact subset of $\mathbb{C}$, such that*

$$\mu_n \xrightarrow{*} \mu.$$

*Suppose that for each n, a point $z_n$ is given so that $z_n \to z$, for some $z \in \mathbb{C}$. Then,*

$$U^{\mu}(z) \leq \liminf_{n\to\infty} U^{\mu_n}(z_n).$$

We say that $\mu_n \xrightarrow{*} \mu$, if

$$\int f\, d\mu_n \to \int f\, d\mu, \quad n \to \infty,$$

for every function $f$ continuous on $\overline{\mathbb{C}}$.

# Potential Theory: Five theorems

**Theorem (Lower Envelope Theorem)**

*Let $\mu_n$, n = 1, 2, ... , be a sequence of positive unit Borel measures, supported on the same compact subset of $\mathbb{C}$, such that*

$$\mu_n \xrightarrow{*} \mu.$$

*Then,*

$$\liminf_{n\to\infty} U^{\mu_n}(z) = U^{\mu}(z),$$

*for quasi-every $z \in \mathbb{C}$.*

# Potential Theory: Five theorems

**Theorem (Unicity Theorem)**

*Suppose that the positive measures $\mu$ and $\nu$ have compact support and in a region $D \subset \mathbb{C}$ the potentials $U^\nu$ and $U^\mu$ satisfy*

$$U^\mu(z) = U^\nu(z) + u(z),$$

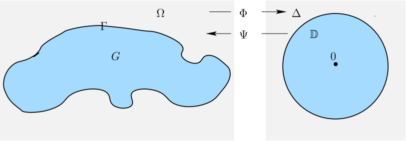
*almost everywhere with respect to two-dimensional Lebesgue measure, where the function $u$ is harmonic in $D$. Then, in $D$ the measures $\mu$ and $\nu$ coincide.*

# Potential Theory: Five theorems

**Theorem (Carleson's Unicity Theorem)**

*Let $K$ be a compact set of positive capacity, and let $\Omega$ denote the unbounded component of $\overline{\mathbb{C}} \setminus K$. If $\mu$ and $\nu$ are two unit measures supported on $\partial\Omega$, and if the potentials $U^{\mu}$ and $U^{\nu}$ coincide in $\Omega$, then $\mu = \nu$.*

# Bergman polynomials $\{p_n\}$ on an Jordan domain $G$




$\Gamma := \partial G$ $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$

$$\langle f, g \rangle := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle^{1/2}.$$

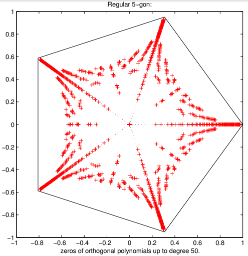
The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of $G$ are the orthonormal polynomials w.r.t. the area measure on $G$:

$$\langle p_m, p_n \rangle = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots.$$

# Example: $G$ is the canonical pentagon




## Theorem (Levin, Saff & St., Constr. Approx. 2003)

*Let $\varphi$ be a conformal map of $G$ onto the unit disk $\mathbb{D}$. Then, there is a subsequence $\mathcal{N}$ of $\mathbb{N}$ such that*

$$\nu_{p_n} \xrightarrow{*} \mu_\Gamma, \quad n \to \infty, \quad n \in \mathcal{N},$$

*if and only if $\varphi$ cannot be analytically continued to some open set containing $\overline{G}$.*

# Key results

The above theorem is based on the following facts:

- The area measure on $G$ belongs to the class $\texttt{Reg}$, that is,

$$\lim_{n\to\infty} \|p_n\|_G^{1/n} = 1.$$

- The kernel $K(z,\zeta)$, of the Bergman space $L^2_a(G)$ satisfies,

$$K(z,\zeta) = \sum_{n=0}^{\infty} \overline{p_n(\zeta)} p_n(z), \quad z,\zeta \in G,$$

  and is is related to a normalized conformal map $\varphi_\zeta : G \to \mathbb{D}$, $\varphi_\zeta(\zeta) = 0$, $\zeta \in G$, by

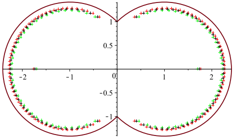
$$K(z,\zeta) = \frac{1}{\pi} \overline{\varphi'_\zeta(\zeta)} \varphi'_\zeta(z).$$

An application of Walsh's maximal convergence then yields

$$\limsup_{n\to\infty} |p_n(\zeta)|^{1/n} = 1, \quad n \in \mathcal{N},$$

and the result then follows from Theorem III.4.1 in Saff and Totik.

# The two intersecting circles




Zeros of $p_n(z)$, with $n = 80, 100, 120$.

**Theorem (Saff & St, JAT 2015)**

*If the boundary $\Gamma$ of $G$ contains an inward corner point, then*

$$\nu_{p_n} \xrightarrow{*} \mu_\Gamma, \quad n \to \infty, \quad n \in \mathbb{N},$$

*where $\mu_\Gamma$ denotes the equilibrium measure on $\Gamma$.*

Based on Gardiner and Pommerenke, Constr, Approx, 2002.
The reluctance of the zeros to approach the points $\pm i$, is due to the fact that $\boxed{d\mu_\Gamma(z) = |\Phi'(z)|ds}$, where $s$ denotes the arclength on $\Gamma$.

# The circular sector

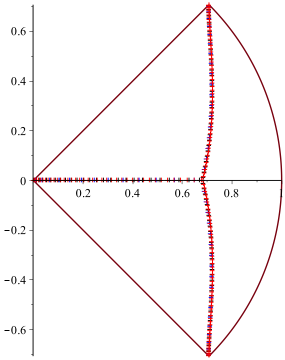
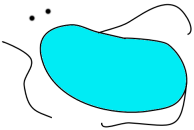



Figure: Zeros of $p_n$, $n = 50, 100, 150$, for the circular sector with opening angle $\pi/2$.

# Explanation



**Theorem (Mina-Diaz, Saff & St., CMFT 2005)**

*Let $E \neq \emptyset$ be a compact subset of $\mathbb{C}$ such that both $\overline{\mathbb{C}} \setminus E$ and $\mathring{E} := \mathrm{int}(E)$ are connected. Let $g : \overline{\mathbb{C}} \setminus \mathring{E} \to \overline{\mathbb{C}}$ be such that $g$ is analytic in $\mathbb{C} \setminus E$, $|g|$ is continuous and never zero in $\overline{\mathbb{C}} \setminus \mathring{E}$, $g(\infty) = \infty$ and $g'(\infty) = 1$. Let $\{q_n\}_{n=1}^{\infty}$ be a sequence of monic polynomials of respective degrees $n = 1, 2, \ldots$, such that $\infty$ is not an accumulation point of the set of zeros of the $q_n$'s. Further, assume that*

$$\limsup_{n\to\infty} |q_n(z)|^{1/n} \leq |g(z)| \qquad q.e. \quad z \in \partial E\,.$$

## Theorem (Mina-Diaz, Saff & St., CMFT 2005, cont.)

*Then, any measure $\sigma$ that is a weak*-limit point of the sequence $\{\nu_{q_n}\}_{n=1}^{\infty}$ is supported on $E$ and*

$$U^{\sigma}(z) = \log|g(z)|^{-1} \qquad \forall z \in \mathbb{C} \setminus \overset{\circ}{E}\,. \tag{1}$$

*Moreover, there is a unique measure $\mu_g$ supported on $\partial E$ such that (1) holds with $\sigma = \mu_g$. For such a measure, we have*

**(a)** *if $\overset{\circ}{E} = \emptyset$, then $\nu_{q_n} \xrightarrow{*} \mu_g$ as $n \to \infty$;*

**(b)** *if $\overset{\circ}{E} \neq \emptyset$ and for some $z_0 \in \overset{\circ}{E}$ and a subsequence $\mathcal{N} \subset \mathbb{N}$*

$$\lim_{\substack{n\to\infty \\ n\in\mathcal{N}}} |q_n(z_0)|^{1/n} = e^{-U^{\mu_g}(z_0)},$$

*then*

$$\nu_{q_n} \xrightarrow{*} \mu_g \quad \text{as} \quad n \to \infty, \quad n \in \mathcal{N}\,.$$

# Used in the proof

Observe that the assumption of the theorem is equivalent to

$$\liminf_{n\to\infty} U^{\nu_{q_n}}(z) \geq \log|g(z)|^{-1} \qquad q.e. \quad z \in \partial E\,. \tag{2}$$

Let $\sigma$ be a weak*-limit point of the sequence $\{\nu_{q_n}\}_{n=1}^{\infty}$, so that for some subsequence $\mathcal{N} \subset \mathbb{N}$

$$\nu_{q_n} \xrightarrow{*} \sigma \quad \text{as} \quad n \to \infty\,, \quad n \in \mathcal{N}\,.$$

Then $\sigma$ is a probability measure and by (2) and the Lower Envelope Theorem, we have for q.e. $z \in \partial E$,

$$U^{\sigma}(z) = \liminf_{\substack{n\to\infty \\ n\in\mathcal{N}}} U^{\nu_{q_n}}(z) \geq \liminf_{n\to\infty} U^{\nu_{q_n}}(z) \geq \log|g(z)|^{-1}. \tag{3}$$

# Used in the proof

By the assumptions on $g$, the function

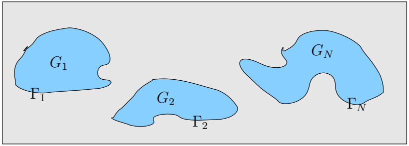
$$F^{\sigma}(z) := U^{\sigma}(z) - \log|g(z)|^{-1}, \quad z \in \mathbb{C} \setminus E,$$

is superharmonic and lower bounded in $\mathbb{C} \setminus E$, harmonic and equal to zero at $\infty$, and in view of (3) and the lower semicontinuity of $U^{\sigma}$, it also satisfies for *quasi-every* $z' \in \partial E$

$$\liminf_{\substack{z\to z'\\ z\in\mathbb{C}\setminus E}} F^{\sigma}(z) \geq \liminf_{z\to z'} U^{\sigma}(z) - \lim_{\substack{z\to z'\\ z\in\mathbb{C}\setminus E}} \log|g(z)|^{-1} \geq U^{\sigma}(z') - \log|g(z')|^{-1} \geq 0\,.$$

Then, by the generalized minimum principle for superharmonic functions we conclude that $F^{\sigma} \equiv 0$, which implies that (1) holds in $\mathbb{C} \setminus E$. It also implies that $U^{\sigma}$ is harmonic in $\mathbb{C} \setminus E$ and therefore, in view of the Unicity Theorem $\operatorname{supp}(\sigma)$ must be contained in $E$.
It is a direct consequence of Carleson's Unicity Theorem that there can be at most one measure $\mu_g$ supported on $\partial E$ that satisfies (1) with $\sigma = \mu_g$.

# Bergman polynomials on an archipelago




$\Gamma_j$, $j = 1, \ldots, N$, a system of disjoint and mutually exterior Jordan curves in $\mathbb{C}$, $\boxed{G_j := \text{int}(\Gamma_j)}$, $\boxed{\Gamma := \cup_{j=1}^N \Gamma_j}$, $\boxed{G := \cup_{j=1}^N G_j}$.

$$\langle f, g \rangle_G := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle_G^{1/2}$$

The Bergman polynomials $\{p_n\}_{n=0}^\infty$ of $G$ are the unique orthonormal polynomials w.r.t. the area measure on $G$:

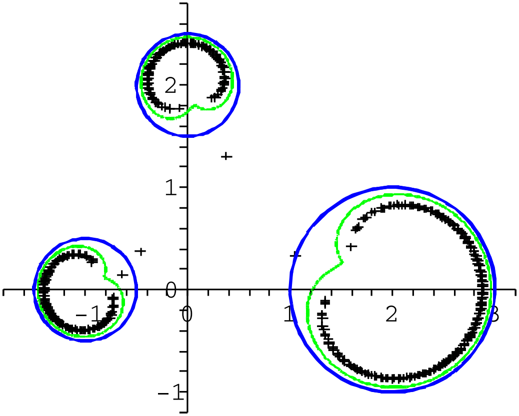
$$\langle p_m, p_n \rangle_G = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots.$$

# Three-disks

Zeros of the Bergman polynomials $p_{140}$, $p_{150}$ and $p_{160}$.




Theory in: Gustafsson, Putinar, Saff & St, Adv. Math., 2009.

# The basic tool for the distribution of zeros

- $\Omega := \overline{\mathbb{C}} \setminus \overline{G}$.
- $K(z,\zeta)$: the Bergman (reproducing) kernel function of $L_a^2(G)$.
- $L_R := \{z : g_\Omega(z,\infty) = \log R\}$ the level lines of the Green function.
- $\varrho(\zeta) := \sup\{R : K(z,\zeta)$ *has an analytic continuation inside* $L_R\}$.
- $$h(z) := \begin{cases} g_\Omega(z,\infty), & z \in \overline{\Omega}, \\ -\log \varrho(z), & z \in G, \end{cases}$$
- $\boxed{\beta := \frac{1}{2\pi}\Delta h}$, in the sense of distributions.
- $\nu_{p_n}$: the *normalized counting measure of zeros* of $p_n$.
- $\mathcal{C}$: the set of *weak-star cluster points* of the counting measures $\{\nu_{p_n}\}_{n=1}^{\infty}$, i.e., the set of measures $\sigma$ for which there exists a subsequence $\mathcal{N}_\sigma \subset \mathbb{N}$ such that $\nu_{p_n} \xrightarrow{*} \sigma$, as $n \to \infty$, $n \in \mathcal{N}_\sigma$.
- $\mu_\Gamma$: the *equilibrium measure* on the boundary $\Gamma$.

# The basic result for the distribution of zeros

**Theorem (Gustafsson, Putinar, Saff & St, Advances in Math, 2009)**

(i) $\beta$ *is a positive unit measure with support contained in* $\overline{G}$.

(ii) *The balayage of* $\beta$ *onto* $\Gamma$ *gives the equilibrium measure* $\mu_\Gamma$*:*

$$\begin{cases} U^\beta \geq U^{\mu_\Gamma} & \text{in } \mathbb{C}, \\ U^\beta = U^{\mu_\Gamma} & \text{in } \Omega. \end{cases}$$

(iii) $\mathcal{C}$ *is nonempty, and for any* $\sigma \in \mathcal{C}$,

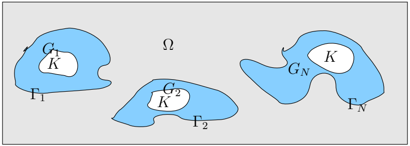
$$\begin{cases} U^\sigma \geq U^\beta & \text{in } \mathbb{C}, \\ U^\sigma = U^\beta & \text{in the unbounded component of } \overline{\mathbb{C}} \setminus \operatorname{supp}\beta. \end{cases}$$

(iv) *The measure* $\beta$ *is the lower envelope of* $\mathcal{C}$*:* $U^\beta = \operatorname{lsc}(\inf_{\sigma\in\mathcal{C}} U^\sigma)$.

(v) *If* $\mathcal{C}$ *has only one element, then this is* $\beta$ *and*

$$\nu_{p_n} \xrightarrow{*} \beta, \quad n \to \infty, \ n \in \mathbb{N}.$$

# Bergman polynomials on archipelago with lakes




With $K$ is a compact subset of $G$, set $\boxed{G^* := G \setminus K}$ and consider

$$\langle f, g \rangle_{G^*} := \int_{G^*} f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G^*)} := \langle f, f \rangle_{G^*}^{1/2}.$$

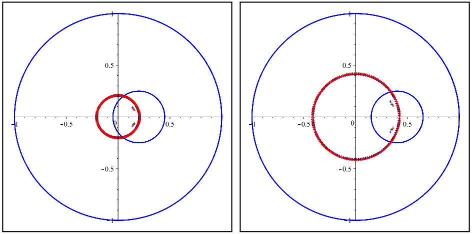
The Bergman polynomials $\{p_n^*\}_{n=0}^{\infty}$ of $G^*$ are the unique orthonormal polynomials w.r.t. the area measure on $G^*$:

$$\langle p_m^*, p_n^* \rangle_{G^*} = \int_{G^*} p_m^*(z)\overline{p_n^*(z)}dA(z) = \delta_{m,n},$$

with

$$p_n^*(z) = \gamma_n^* z^n + \cdots, \quad \gamma_n^* > 0, \quad n = 0, 1, 2, \ldots.$$

# The annular case




Plots of the zeros of $p_n^*(z)$, for $n = 120, 140$ and $160$.

Let $G = \mathbb{D}$, $\mathcal{K} := \{z : |z - a| \leq \varrho\}$, $|a| + \varrho < 1$, $\varrho > 0$, $G^* = \mathbb{D} \setminus \mathcal{K}$,
We recall that there exists a unique pair of points $z_1$ and $z_2$ that are mutually inverse points with respect to the two circles $\mathbb{T} := \partial\mathbb{D}$ and $\{z : |z - a| = \varrho\}$, that is

$$z_1 \overline{z_2} = 1 \quad \text{and} \quad (z_1 - a)(\overline{z_2 - a}) = \varrho^2.$$

Let $z_1$ denote the point that lies in $\mathcal{K}$ ($z_2$ will then lie outside $\mathbb{D}$).

# The annular case: Explanation

**Proposition (Saff & St, Mat. Sbornik, 2018)**

*With the above notation, there exists a subsequence $\mathcal{N} \subset \mathbb{N}$ such that the normalized zero counting measures for $p_n^*(z)$ satisfy*

$$\nu_{p_n^*} \stackrel{*}{\longrightarrow} \mu_{|z_1|}, \quad n \to \infty, \quad n \in \mathcal{N},$$

*where $\mu_{|z_1|}$ denotes the normalized arclength measure on the circle $|z| = |z_1|$.*

Thus, no matter what the relative position of $\mathcal{K}$, a weak limit of $\nu_n$ will invariably be the arclength measure on a specific circle in $\mathbb{D}$, always centered at the origin.

# Shift Operator on $L^2(\mu)$

Let $N_z$ denote the shift operator on $L^2(\mu)$. That is,

$$N_z : L^2(\mu) \to L^2(\mu) \quad \text{with} \quad N_z f = zf.$$

$N_z$ defines a normal operator on $L^2(\mu)$. Furthermore,

$$p_n(\mu, z) = \lambda_n(\mu) \det(z - \pi_n N_z \pi_n),$$

where $\pi_n$ is the projection onto the $n$-dimensional subspace onto $\mathbb{P}_{n-1}$.

**Theorem (B. Simon, Duke Math. J., 2009)**

*Let*

$$N(\mu) := \sup\{|z| : z \in S_\mu\}.$$

*Then, for any $k \in \mathbb{N}$,*

$$\pi_n N_z^k \pi_n - (\pi_n N_z \pi_n)^k,$$

*is an operator of rank at most $k$ and norm at most $2N(\mu)^k$.*

# Shift Operator on $L^2(\mu)$

Let $\mu_n$ denote the unit measures $d\mu_n = \frac{1}{n}\sum_{j=0}^{n-1} |p_n(\mu, z)|^2 d\mu(z)$.

**Theorem (B. Simon, Duke Math. J., 2009)**

$$\frac{1}{n}\mathrm{Tr}(\pi_n N_z \pi_n)^k = \int z^k d\nu_{p_n}.$$

$$\frac{1}{n}\mathrm{Tr}(\pi_n N_z^k \pi_n) = \int z^k d\mu_n.$$

*Thus, from the previous theorem, for any $k = 0, 1, 2, \ldots$,*

$$\left| \int z^k d\nu_{p_n} - \int z^k d\mu_n \right| \leq \frac{2kN^k(\mu)}{n}.$$

*Furthermore, if $K$ is a compact set containing the supports of all $\nu_n$ and $\mu$, such that $\{z_k\}_{k=0}^{\infty} \cup \{\overline{z}_k\}_{k=0}^{\infty}$ are $\|\cdot\|_\infty$-total in $\mathcal{C}(K)$, then for any subsequence $\{n_j\}$, $\boxed{\mu_{n_j} \xrightarrow{*} \nu}$ if and only if $\boxed{\mu_{n_j} \xrightarrow{*} \nu}$.*

# Krylov subspaces

Let $A \in \mathcal{L}(H)$ be a linear bounded operator acting on the complex Hilbert space $H$ and let $\xi \in H$ be a non-zero vector. We denote $H_n(A, \xi)$ the linear span of the vectors $\xi, A\xi, ..., A^{n-1}\xi$ and let $\pi_n$ be the orthogonal projection of $H$ onto $H_n(A, \xi)$. Let $a_n$ denote the counting measures of the spectra of the *finite central truncations* $A_n = \pi_n A \pi_n$. Note that for any complex polynomial $p(z)$ it holds that

$$\int p(z) da_n(z) = \frac{\operatorname{tr} p(A_n)}{n}.$$

The orthogonal monic polynomials $P_n$ in this case are defined as minimizers of the functional (semi-norm):

$$\|q\|_{A,\xi}^2 = \|q(A)\xi\|^2, \quad q \in \mathbb{C}[z],$$

and the zeros of $P_n$ (whenever $P_n$ exists) coincide with the spectrum of $A_n$.

**Theorem (Gustafsson & Putinar, Springer 2017)**

*Let $A, B \in \mathcal{L}(H)$ with $A - B$ of finite trace: $A - B \in \mathcal{C}_1(H)$. Then for every polynomial $p \in \mathbb{C}[z]$ we have*

$$\lim_{n\to\infty} \frac{\mathrm{Tr}(p(A_n)) - \mathrm{Tr}(p(B_n))}{n} = 0.$$

**Corollary**

*Let $a_n, b_n$ denote the counting measures of the spectra of $A_n$ and $B_n$, respectively. Then,*

$$\lim_{n\to\infty} [\int \frac{da_n(\zeta)}{\zeta - z} - \int \frac{db_n(\zeta)}{\zeta - z}] = 0,$$

*uniformly on compact subsets which are disjoint of the convex hull of $\sigma(A) \cup \sigma(B)$.*

# Conclusion

All the results in this section yield information for the analytic moments:

$$\lim_{n\to\infty} \int z^k d\nu_n = \int z^k d\nu, \quad k = 0, 1, 2, \ldots,$$

where $\nu$ is a known positive measure and $\{\nu_n\}$ are a sequence of positive measures (all supported on the same compact set $K$ in the complex plane) we want to describe its weak limit points. Note that the measures being positive implies the same information for the anti-analytic moments:

$$\lim_{n\to\infty} \int \overline{z}^k d\nu_n = \int \overline{z}^k d\nu, \quad k = 1, 2, \ldots.$$

# Conclusion

However, according to the complex Stone-Weierstrass theorem, in order to establish

$$\nu_n \xrightarrow{*} \nu,$$

we need the limits of all the complex moments

$$\lim_{n\to\infty} \int z^k \overline{z}^j d\nu_n = \int z^k \overline{z}^j d\nu, \quad k, j = 0, 1, 2, \ldots,$$

unless $K$ is of a special form (Mergelyan, Walsh), where the analytic moments constitute sufficient information.