Fine Asymptotics for Bergman Orthogonal Polynomials over Domains with Corners

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CMFT 2009
Ankara, June 2009
**Definition**

$\Gamma$: bounded Jordan curve, $G := \text{int}(\Gamma)$

$$\langle f, g \rangle := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle^{1/2}$$

The **Bergman polynomials** $\{p_n\}_{n=0}^{\infty}$ of $G$ are the orthonormal polynomials w.r.t. the area measure:

$$\langle p_m, p_n \rangle = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots.$$
Minimal property
\[
\frac{1}{\lambda_n} = \left\| p_n \lambda_n \right\|_{L^2(G)} = \min_{z^n+\cdots} \|z^n + \cdots\|_{L^2(G)}.
\]

The Bergman space

\[L^2_a(G) := \{f \text{ analytic in } G, \|f\|_{L^2(G)} < \infty\},\]

is a Hilbert space with reproducing kernel \(K(z, \zeta)\): For any \(\zeta \in G\),

\[f(\zeta) = \langle f, K(\cdot, \zeta) \rangle, \quad \forall \ f \in L^2_a(G).\]

Approximation Property

\(\{p_n\}_{n=0}^\infty\) is a complete ON system of \(L^2_a(G)\) and

\[K(z, \zeta) = \sum_{n=0}^\infty p_n(\zeta)p_n(z), \quad z, \zeta \in G.\]
Associated conformal maps

\[ \Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots \]

\[ \text{cap}(\Gamma) = \frac{1}{\gamma} \]

If \( \varphi_\zeta(\zeta) = 0 \) and \( \varphi'_\zeta(\zeta) > 0 \) then

\[ K(z, \zeta) = \frac{1}{\pi} \frac{\varphi'_\zeta(\zeta) \varphi'_\zeta(z)}{\varphi_\zeta(\zeta)} \]

This leads to the **Bergman kernel method** for approximating \( \varphi'_\zeta \) (and thus \( \varphi_\zeta \)) in terms of Bergman polynomials.
Weak asymptotics for $\lambda_n$ and $p_n$ in $G$

Papamichael, Saff & Gong, JCAM (1991)

If $\Gamma$ is a bounded Jordan curve then:

$$\lim_{n \to \infty} \|p_n\|_{\overline{G}}^{1/n} = 1, \quad \text{and} \quad \lim_{n \to \infty} \lambda_n^{1/n} = \gamma (= 1 / \text{cap}(\Gamma)).$$

Also, let $L_R := \{ z : |\Phi(z)| = R \}$ (\(R \geq 1\)). Then $\varphi_\zeta$ is analytic in $\text{int}(L_R)$ if and only if

$$\limsup_{n \to \infty} |p_n(\zeta)|^{1/n} = 1 / R.$$

With $\| \cdot \|_{\overline{G}}$ we denote the sup-norm on $\overline{G}$. 
Weak asymptotics for $p_n$ in $\Omega$

If $\Gamma$ is a bounded Jordan curve then:

$$\limsup_{n \to \infty} |p_n(z)|^{1/n} = |\Phi(z)|, \quad z \in \overline{\Omega} \setminus \{\infty\}.$$ 

In particular, if $z \in \Omega$ is not a limit point of zeros of $p_n$'s,

$$\lim_{n \to \infty} |p_n(z)|^{1/n} = |\Phi(z)|.$$ 

The above are based on

- Stahl & Totik, *General Orthogonal Polynomials*, CUP (1992),
Fine asymptotics when $\Gamma$ is analytic

Carleman, Ark. Mat. Astr. Fys. (1922)

If $\rho < 1$ is the smallest index for which $\Phi$ is conformal in $\text{ext}(L_\rho)$, then

$$\frac{n + 1}{\pi} \frac{\gamma^{2(n+1)}}{\chi_n^2} = 1 - \alpha_n, \quad \text{where } 0 \leq \alpha_n \leq c_1(\Gamma) \rho^{2n},$$

$$p_n(z) = \sqrt{\frac{n + 1}{\pi}} \Phi^n(z) \Phi'(z) \{1 + A_n(z)\}, \quad n \in \mathbb{N},$$

where

$$|A_n(z)| \leq c_2(\Gamma) \sqrt{n} \rho^n, \quad z \in \overline{\Omega}.$$
Fine asymptotics when $\Gamma$ is smooth

We say that $\Gamma \in C(p, \alpha)$, for some $p \in \mathbb{N}$ and $0 < \alpha < 1$, if $\Gamma$ is given by $z = g(s)$, where $s$ is the arclength, with $g^{(p)} \in \text{Lip}\alpha$. Then both $\Phi$ and $\Psi := \Phi^{-1}$ are $p$ times continuously differentiable on $\Gamma$ and $\partial D$ respectively, with $\Phi^{(p)}$ and $\Psi^{(p)} \in \text{Lip}\alpha$.


Assume that $\Gamma \in C(p + 1, \alpha)$, with $p + \alpha > 1/2$. Then

\[
\frac{n + 1}{\pi} \frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n, \quad \text{where} \quad 0 \leq \alpha_n \leq c_1(\Gamma) \frac{1}{n^{2(p+\alpha)}},
\]

\[
\rho_n(z) = \sqrt{\frac{n + 1}{\pi}} \Phi^n(z)\Phi'(z)\{1 + A_n(z)\}, \quad n \in \mathbb{N},
\]

where

\[
|A_n(z)| \leq c_2(\Gamma) \frac{\log n}{n^{p+\alpha}}, \quad z \in \overline{\Omega}.
\]
Does it hold \( \lim_{n \to \infty} \alpha_n = 0 \) ?

We are not aware of a single case of non-smooth \( \Gamma \) for which the leading coefficients \( \lambda_n, n = 0, 1, \ldots \), are known explicitly.

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Luckily, we have plenty of ...
We compute, by using the Gram-Schmidt process (in finite precision), the Bergman polynomials $p_n(z)$ for the unit half-disk, for $n$ up to 60 and test the hypothesis

$$\alpha_n := 1 - \frac{n + 1}{\pi} \frac{\gamma^{2(n+1)}}{\chi_n^2} \approx C \frac{1}{n^s}.$$
The numbers indicate clearly that $\alpha_n \approx C \frac{1}{n}$. Accordingly, we have made conjectures regarding fine asymptotics in Oberwolfach Reports (2004) and ETNA (2006).

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<tr>
<th>$n$</th>
<th>$\alpha_n$</th>
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<td>51</td>
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<td>0.002 774 426 207</td>
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Fine asymptotics for \( \Gamma \) non-smooth: Numerical data

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Main actors

Recall: $\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots$ and let

$$\Phi^n(z) = F_n(z) - E_n(z)$$

and

$$\Phi^n(z)\Phi'(z) = G_n(z) - H_n(z)$$

where

- $F_n(z) = \gamma^n z^n + \cdots \in \mathbb{P}_n$, is the Faber poly of $G$,
- $E_n(z) = \frac{c_1}{z} + \frac{c_2}{z^2} + \frac{c_3}{z^3} + \cdots$, is the singular part of $\Phi^n$,
- $G_n(z) = \gamma^{n+1} z^n + \cdots \in \mathbb{P}_n$, is the Faber poly of the 2nd kind of $G$,
- $H_n(z) = \frac{d_2}{z^2} + \frac{d_3}{z^3} + \frac{d_4}{z^4} + \cdots$, is the singular part of $\Phi^n\Phi'$.

Note:

$$G_n(z) = \frac{F'_{n+1}(z)}{n+1}$$

and

$$H_n(z) = \frac{E'_{n+1}(z)}{n+1}.$$
Fine asymptotics for $\lambda_n$

**Theorem (I)**

Assume that $\Gamma$ is piecewise analytic without cusps, then

$$\frac{n + 1}{\pi} \frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n,$$

where

$$0 \leq \alpha_n \leq c(\Gamma) \frac{1}{n}, \quad n \in \mathbb{N}$$

and $C(\Gamma)$ depends on $\Gamma$ only.
Fine asymptotics for $p_n$ in $\Omega$

**Theorem (II)**

Assume that $\Gamma$ is piecewise analytic w/o cusps. Then, for any $z \in \Omega$,

$$p_n(z) = \sqrt{\frac{n + 1}{\pi}} \Phi^n(z) \Phi'(z) \{1 + A_n(z)\},$$

where

$$|A_n(z)| \leq \frac{c(\Gamma)}{\text{dist}(z, \Gamma) |\Phi'(z)|} \frac{1}{\sqrt{n}}, \quad n \in \mathbb{N}.$$
Let $\Psi$ denote the inverse conformal map $\Phi^{-1} : \{ w : |w| > 1 \} \rightarrow \Omega$. Then

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots , \quad |w| > 1.$$ 

**Theorem (III)**

Assume that $\Gamma$ is quasiconformal and rectifiable. Then,

$$\alpha_n \geq \frac{\pi (1 - k^2)}{A(G)} (n + 1) |b_{n+1}|^2.$$ 

The above provides a connection with the well-studied problem of estimating coefficients of univalent functions.
Quasiconformal curves

In Theorem (II), \( k := \frac{K - 1}{K + 1} < 1 \), where \( K \geq 1 \), is the characteristic constant of the quasiconformal reflection defined by \( \Gamma \).

**Definition**

A Jordan curve \( \Gamma \) is quasiconformal if there exists a constant \( M > 0 \), such that

\[
\text{diam} \, \Gamma(a, b) \leq M |a - b|, \quad \text{for all} \quad a, b \in \Gamma,
\]

where \( \Gamma(a, b) \) is the arc (of smaller diameter) of \( \Gamma \) between \( a \) and \( b \).

Note: A piecewise analytic Jordan curve is quasiconformal if and only if has no cusps (0 and \( 2\pi \) angles).
Recall:

**Lemma (Bernstein-Walsh)**

*For any* $P \in \mathbb{P}_n$,

$$|P(z)| \leq \|P\|_{\overline{G}} \left|\Phi(z)\right|^n, \quad z \in \Omega.$$  

We can replace $\|P\|_{\overline{G}}$ by $\|P\|_{L^2(G)}$:

**Lemma (I)**

Assume that $\Gamma$ is quasiconformal and rectifiable. Then, for any $P \in \mathbb{P}_n$,

$$|P(z)| \leq \frac{c(\Gamma)}{\text{dist}(z, \Gamma)} \sqrt{n} \|P\|_{L^2(G)} \left|\Phi(z)\right|^{n+1}, \quad z \in \Omega.$$  

Basic  Asymptotics  Estimates  Applications
Decay of Faber polynomials in $G$

Recall: $\{F_n\}$ are the Faber polynomials of $G$.

**Theorem (Gaier, Analysis, 2001)**

Assume that $\Gamma$ is piecewise analytic w/o cusps and let $\lambda \pi$ ($0 < \lambda < 2$) be the smallest exterior angle of $\Gamma$. Then, for any $z \in G$,

$$|F_n(z)| \leq \frac{c(\Gamma)}{\text{dist}(z, \Gamma)} \frac{1}{n^\lambda}, \quad n \in \mathbb{N}.$$ 

For the Faber polynomials of the 2nd kind $\{G_n\}$ we have:

**Theorem (IV)**

Assume that $\Gamma$ is piecewise analytic w/o cusps. Then, for any $z \in G$,

$$|G_n(z)| \leq \frac{c(\Gamma)}{\text{dist}(z, \Gamma)} \frac{1}{n}, \quad n \in \mathbb{N}.$$
From Thm (I) we have immediately:

**Corollary (Ratio asymptotics for $\lambda_n$)**

$$\sqrt{n+1} \frac{\lambda_{n+1}}{n+2} \frac{\lambda_n}{\lambda_n} = \gamma + \xi_n,$$

where

$$|\xi_n| \leq c(\Gamma) \frac{1}{n}, \quad n \in \mathbb{N}.$$  

We note however that numerical evidence suggests that $|\xi_n| \approx C \frac{1}{n^2}$. Since $\text{cap}(\Gamma) = 1/\gamma$, the above relation provides the means for computing approximations to the capacity of $\Gamma$, by using only the leading coefficients of the associated orthonormal polynomials.
Similarly, from Thm (II) we have:

**Corollary (Ratio asymptotics for \( p_n \))**

\[
\sqrt{\frac{n+1}{n+2}} \frac{p_{n+1}(z)}{p_n(z)} = \Phi(z) + B_n(z), \quad z \in \Omega,
\]

where

\[
|B_n(z)| \leq \frac{c(\Gamma)}{\text{dist}(z, \Gamma)|\Phi'(z)|} \frac{1}{\sqrt{n}}, \quad n \in \mathbb{N}
\]

The above relation provides the means for computing approximations to the conformal map \( \Phi \) in \( \Omega \), by simply taking the ratio of two consequent orthonormal polynomials. This leads to an efficient algorithm for **recovering the shape** of \( G \), from a finite collection of its power moments \( \langle z^m, z^n \rangle, m, n = 0, 1, \ldots, N \).
Only ellipses carry finite-term recurrences for $p_n$

**Definition**
We say that the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a $(N + 1)$-term recurrence relation, if for any $n \geq N - 1$,

$$zp_n(z) = a_{n+1,n}p_{n+1}(z) + a_{n,n}p_n(z) + \ldots + a_{n-N+1,n}p_{n-N+1}(z).$$

**Theorem (Putinar & St. CAOT, 2007)**
Assume that:

- $\Gamma = \partial G$, where $G$ is a Caratheodory domain;
- the Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a $(N + 1)$-term recurrence relation, with some $N \geq 2$;
- $\Gamma \subset B := \{(x, y) \in \mathbb{R}^2 : \psi(x, y) = 0\}$, where $B$ is bounded.

Then $N = 2$ and $\Gamma$ is an ellipse.
An application of the Suetin’s asymptotics for \( p_n \) leads to:

**Theorem (Khavinson & St., 2009)**

Assume that:
- \( \Gamma = \partial G \) is a \( C^2 \)-smooth Jordan curve;
- the Bergman polynomials \( \{ p_n \}_{n=0}^{\infty} \) satisfy a \( (N+1) \)-term recurrence relation, with some \( N \geq 2 \).

Then \( N = 2 \) and \( \Gamma \) is an **ellipse**.

However, by using the ratio asymptotics corollary above:

**Theorem (V)**

Assume that:
- \( \Gamma = \partial G \) is piecewise analytic without cusps;
- the Bergman polynomials \( \{ p_n \}_{n=0}^{\infty} \) satisfy a \( (N+1) \)-term recurrence relation, with some \( N \geq 2 \).

Then \( N = 2 \) and \( \Gamma \) is an **ellipse**.
Where are the zeros of $p_n$?

Fejer
All the zeros of $p_n$ lie in the convex hull of $\overline{G}$.

Saff
All the zeros of $p_n$ lie in the interior convex hull of $\overline{G}$.

Widom
For any $n \in \mathbb{N}$, $p_n$ has at most a bounded number of zeros (independent of $n$) on any closed set $E \subset \Omega$. 
A result about the zeros of $p_n$

Since for any $z \in \Omega$, $|\Phi(z)| > 1$ and $|\Phi'(z)| \neq 0$, Thm II yields:

**Theorem (VI)**

Assume that $\Gamma$ is piecewise analytic w/o cusps. Then for any closed set $E \subset \Omega$, there exists $n_0 \in \mathbb{N}$, such that for $n \geq n_0$, $p_n(z)$ has no zeros on $E$.

This leads at once to the refinement:

**Corollary**

Assume that $\Gamma$ is piecewise analytic w/o cusps. Then

$$\lim_{n \to \infty} |p_n(z)|^{1/n} = |\Phi(z)|, \quad z \in \Omega \setminus \{\infty\}.$$
A sharp estimate for $\|p_n\|_G$

**Theorem (VII)**

Assume that $\Gamma$ is piecewise analytic w/o cusps and let $\lambda\pi$ denote the largest exterior angle of $\Gamma$ ($1 \leq \lambda \leq 2$). Then

$$\|p_n\|_G \leq c(\Gamma) n^{\lambda - 1/2}, \quad n \in \mathbb{N}.$$ 

**Note:**

- The order $\lambda - 1/2$ is sharp for $\Gamma$ smooth (hence $\lambda = 1$). This follows immediately from the fine asymptotic formula of Suetin.
- The above should be compared with the “norm comparison” estimate (holding for any $P \in \mathbb{P}_n$)

$$\|P\|_G \leq c(\Gamma) n^{\lambda} \|P\|_{L^2(G)}, \quad n \in \mathbb{N},$$

A sharp estimate for $\| p_n \|_G$

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