Fine Asymptotics for Bergman Orthogonal Polynomials over Domains with Corners

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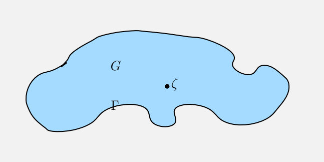
Definition



Γ: bounded Jordan curve, $G := \text{int}(\Gamma)$

$$\langle f, g\rangle := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f\rangle^{1/2}$$

The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G are the orthonormal polynomials w.r.t. the area measure:

$$\langle p_m, p_n\rangle = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots.$$

Minimal property

$$\frac{1}{\lambda_n} = \left\| \frac{p_n}{\lambda_n} \right\|_{L^2(G)} = \min_{z^n+\cdots} \|z^n + \cdots\|_{L^2(G)}.$$

The Bergman space

$$L^2_a(G) := \{f \text{ analytic in } G,\ \|f\|_{L^2(G)} < \infty\},$$

is a Hilbert space with reproducing kernel $K(z,\zeta)$: For any $\zeta \in G$,

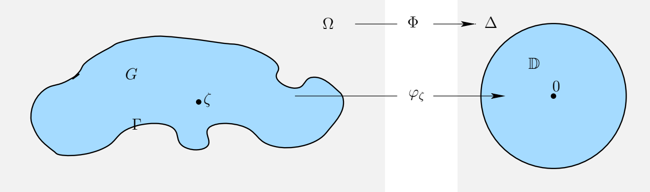
$$f(\zeta) = \langle f, K(\cdot,\zeta)\rangle,\ \forall\ f \in L^2_a(G).$$

Approximation Property

$\{p_n\}_{n=0}^{\infty}$ is a complete ON system of $L^2_a(G)$ and

$$K(z,\zeta) = \sum_{n=0}^{\infty} \overline{p_n(\zeta)} p_n(z), \quad z,\zeta \in G.$$

Associated conformal maps



$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots \qquad \boxed{\mathrm{cap}(\Gamma) = 1/\gamma}$$

If $\varphi_\zeta(\zeta) = 0$ and $\varphi'_\zeta(\zeta) > 0$ then

$$K(z,\zeta) = \frac{1}{\pi}\,\varphi'_\zeta(\zeta)\varphi'_\zeta(z).$$

This leads to the Bergman kernel method for approximating φ'_ζ (and thus φ_ζ) in terms of Bergman polynomials.

Weak asymptotics for λ_n and p_n in G

Papamichael, Saff & Gong, JCAM (1991)

If Γ is a bounded Jordan curve then:

$$\lim_{n\to\infty} \|p_n\|_{\overline{G}}^{1/n} = 1, \quad \text{and} \quad \lim_{n\to\infty} \lambda_n^{1/n} = \gamma \; (= 1/\text{cap}(\Gamma)).$$

Also, let $\boxed{L_R := \{z : |\Phi(z)| = R\}}$ $(R \geq 1)$. Then φ_ζ is analytic in $\text{int}(L_R)$ if and only if

$$\limsup_{n\to\infty} |p_n(\zeta)|^{1/n} = 1/R.$$

With $\|\cdot\|_{\overline{G}}$ we denote the sup-norm on $\overline{G}$.

Weak asymptotics for p_n in Ω

If Γ is a bounded Jordan curve then:

$$\limsup_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|, \quad z \in \overline{\Omega} \setminus \{\infty\}.$$

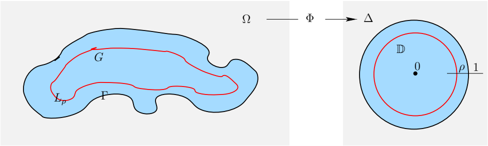
In particular, if $z \in \Omega$ is not a limit point of zeros of p_n's,

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|.$$

The above are based on

- Stahl & Totik, *General Orthogonal Polynomials*, CUP (1992),
- Ambroladze, JAT (1995).
- Saff & Totik, *Logarithmic Potentials*, Springer (1997),

Fine asymptotics when Γ is analytic



Carleman, Ark. Mat. Astr. Fys. (1922)

If $\rho < 1$ is the smallest index for which Φ is conformal in $\text{ext}(L_\rho)$, then

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n,} \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\rho^{2n},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\{1 + A_n(z)\},} \quad n \in \mathbb{N},$$

where

$$|A_n(z)| \le c_2(\Gamma)\sqrt{n}\,\rho^n, \quad z \in \overline{\Omega}.$$

Fine asymptotics when Γ is smooth

We say that $\Gamma \in C(p,\alpha)$, for some $p \in \mathbb{N}$ and $0<\alpha<1$, if Γ is given by $z=g(s)$, where s is the arclength, with $g^{(p)} \in \mathrm{Lip}\alpha$. Then both Φ and $\Psi := \Phi^{-1}$ are p times continuously differentiable on Γ and $\partial\mathbb{D}$ respectively, with $\Phi^{(p)}$ and $\Psi^{(p)} \in \mathrm{Lip}\alpha$.

P.K. Suetin, Proc. Steklov Inst. Math. AMS (1974)

Assume that $\Gamma \in C(p+1,\alpha)$, with $p+\alpha>1/2$. Then

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1-\alpha_n,} \quad \text{where } 0 \le \alpha_n \le c_1(\Gamma)\,\frac{1}{n^{2(p+\alpha)}},$$

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\,\Phi^n(z)\Phi'(z)\{1+A_n(z)\},} \quad n \in \mathbb{N},$$

where

$$|A_n(z)| \le c_2(\Gamma)\,\frac{\log n}{n^{p+\alpha}}, \quad z \in \overline{\Omega}.$$

Fine asymptotics for Γ non-smooth ?

Does it hold $\lim_{n\to\infty} \alpha_n = 0$?

We are not aware of a single case of non-smooth Γ for which the leading coefficients λ_n, $n = 0, 1, \ldots$, are known explicitly.

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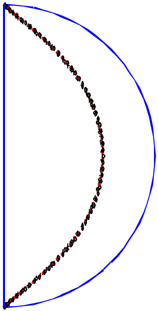
Fine asymptotics for Γ non-smooth: Numerical data



$$\gamma = \frac{1}{\mathrm{cap}(\Gamma)} = \frac{3\sqrt{3}}{4}$$

We compute, by using the Gram-Schmidt process (in finite precision), the Bergman polynomials $p_n(z)$ for the unit half-disk, for n up to 60 and test the hypothesis

$$\alpha_n := 1 - \frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} \approx C\,\frac{1}{n^s}.$$

Fine asymptotics for Γ non-smooth: Numerical data

n	α_n	s
51	0.003 263 458 678	-
52	0.003 200 769 764	0.998 887
53	0.003 140 444 435	0.998 899
54	0.003 082 351 464	0.998 911
55	0.003 026 369 160	0.998 923
56	0.002 972 384 524	0.998 934
57	0.002 920 292 482	0.998 946
58	0.002 869 952 027	0.998 957
59	0.002 821 401 485	0.998 968
60	0.002 774 426 207	0.998 979

The numbers indicate clearly that $\alpha_n \approx C\frac{1}{n}$. Accordingly, we have made conjectures regarding fine asymptotics in Oberwolfach Reports (2004) and ETNA (2006).

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Main actors

Recall: $\Phi(z)=\gamma z+\gamma_0+\frac{\gamma_1}{z}+\frac{\gamma_2}{z^2}+\cdots$ and let

$$\boxed{\Phi^n(z)=F_n(z)-E_n(z)} \quad \text{and} \quad \boxed{\Phi^n(z)\Phi'(z)=G_n(z)-H_n(z)}$$

where

- $F_n(z)=\gamma^n z^n+\cdots\in\mathbb{P}_n,$ is the Faber poly of G,
- $E_n(z)=\frac{c_1}{z}+\frac{c_2}{z^2}+\frac{c_3}{z^3}+\cdots,$ is the singular part of Φ^n,
- $G_n(z)=\gamma^{n+1}z^n+\cdots\in\mathbb{P}_n$, is the Faber poly of the 2nd kind of G,
- $H_n(z)=\frac{d_2}{z^2}+\frac{d_3}{z^3}+\frac{d_4}{z^4}+\cdots,$ is the singular part of $\Phi^n\Phi'$.

Note:

$$G_n(z)=\frac{F'_{n+1}(z)}{n+1} \quad \text{and} \quad H_n(z)=\frac{E'_{n+1}(z)}{n+1}.$$

Fine asymptotics for λ_n

Theorem (I)

Assume that Γ is piecewise analytic without cusps, then

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n},$$

where

$$0 \le \alpha_n \le c(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}$$

and $C(\Gamma)$ depends on Γ only.

Fine asymptotics for p_n in Ω

Theorem (II)

Assume that Γ *is piecewise analytic w/o cusps. Then, for any* $z \in \Omega$*,*

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\,\Phi^n(z)\Phi'(z)\left\{1 + A_n(z)\right\}},$$

where

$$|A_n(z)| \le \frac{c(\Gamma)}{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}\,\frac{1}{\sqrt{n}}, \quad n \in \mathbb{N}$$

A lower bound for α_n - Coefficient estimates

Let Ψ denote the inverse conformal map $\Phi^{-1} : \{w : |w| > 1\} \to \Omega$. Then

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots, \quad |w| > 1.$$

Theorem (III)

Assume that Γ is quasiconformal and rectifiable. Then,

$$\alpha_n \geq \frac{\pi(1-k^2)}{A(G)} (n+1) |b_{n+1}|^2.$$

The above provides a connection with the well-studied problem of estimating coefficients of univalent functions.

Quasiconformal curves

In Theorem (II), $k := \dfrac{K-1}{K+1} < 1$, where $K \geq 1$, is the characteristic constant of the quasiconformal reflection defined by Γ.

Definition

A Jordan curve Γ is quasiconformal if there exists a constant $M > 0$, such that

$$\operatorname{diam}\Gamma(a,b) \leq M\,|a-b|, \text{ for all } a,b \in \Gamma,$$

where $\Gamma(a,b)$ is the arc (of smaller diameter) of Γ between a and b.

Note: A piecewise analytic Jordan curve is quasiconformal if and only if has no cusps (0 and 2π angles).

A Bernstein-Walsh type lemma

Recall:

Lemma (Bernstein-Walsh)

For any $P \in \mathbb{P}_n$,

$$|P(z)| \leq \|P\|_{\overline{G}} \, |\Phi(z)|^n, \quad z \in \Omega.$$

We can replace $\|P\|_{\overline{G}}$ by $\|P\|_{L^2(G)}$:

Lemma (I)

Assume that Γ is quasiconformal and rectifiable. Then, for any $P \in \mathbb{P}_n$,

$$|P(z)| \leq \frac{c(\Gamma)}{\operatorname{dist}(z,\Gamma)} \sqrt{n} \, \|P\|_{L^2(G)} \, |\Phi(z)|^{n+1}, \quad z \in \Omega.$$

Decay of Faber polynomials in G

Recall: $\{F_n\}$ are the Faber polynomials of G.

Theorem (Gaier, Analysis, 2001)

Assume that Γ is piecewise analytic w/o cusps and let $\lambda\pi$ $(0<\lambda<2)$ be the smallest exterior angle of Γ. Then, for any $z\in G$,

$$|F_n(z)| \leq \frac{c(\Gamma)}{\operatorname{dist}(z,\Gamma)}\,\frac{1}{n^{\lambda}}, \quad n\in\mathbb{N}.$$

For the Faber polynomials of the 2nd kind $\{G_n\}$ we have:

Theorem (IV)

Assume that Γ is piecewise analytic w/o cusps. Then, for any $z\in G$,

$$|G_n(z)| \leq \frac{c(\Gamma)}{\operatorname{dist}(z,\Gamma)}\,\frac{1}{n}, \quad n\in\mathbb{N}.$$

Ratio asymptotics

From Thm (I) we have immediately:

Corollary (Ratio asymptotics for λ_n)

$$\boxed{\sqrt{\frac{n+1}{n+2}}\,\frac{\lambda_{n+1}}{\lambda_n} = \gamma + \xi_n},$$

where

$$|\xi_n| \le c(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}.$$

We note however that numerical evidence suggests that $|\xi_n| \approx C\dfrac{1}{n^2}$.

Since $\boxed{\mathrm{cap}(\Gamma) = 1/\gamma}$, the above relation provides the means for computing approximations to the capacity of Γ, by using only the leading coefficients of the associated orthonormal polynomials.

Ratio asymptotics

Similarly, from Thm (II) we have:

Corollary (Ratio asymptotics for p_n)

$$\boxed{\sqrt{\frac{n+1}{n+2}}\,\frac{p_{n+1}(z)}{p_n(z)} = \Phi(z) + B_n(z)}, \quad z \in \Omega,$$

where

$$|B_n(z)| \leq \frac{c(\Gamma)}{\text{dist}(z,\Gamma)|\Phi'(z)|}\,\frac{1}{\sqrt{n}}, \quad n \in \mathbb{N}$$

The above relation provides the means for computing approximations to the conformal map Φ in Ω, by simply taking the ratio of two consequent orthonormal polynomials. This leads to an efficient algorithm for recovering the shape of G, from a finite collection of its power moments $\langle z^m, z^n \rangle$, $m, n = 0, 1, \ldots, N$.

Only ellipses carry finite-term recurrences for p_n

Definition

We say that the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a $(N+1)$-term recurrence relation, if for any $n \geq N-1$,

$$zp_n(z) = a_{n+1,n}p_{n+1}(z) + a_{n,n}p_n(z) + \ldots + a_{n-N+1,n}p_{n-N+1}(z).$$

Theorem (Putinar & St. CAOT, 2007)

Assume that:

- $\Gamma = \partial G$, *where* G *is a Caratheodory domain;*
- *the Bergman polynomials* $\{p_n\}_{n=0}^{\infty}$ *satisfy a* $(N+1)$*-term recurrence relation, with some* $N \geq 2$*;*
- $\Gamma \subset B := \{(x,y) \in \mathbb{R}^2 : \psi(x,y) = 0\}$, *where* B *is bounded.*

Then $N = 2$ *and* Γ *is an ellipse.*

An application of the Suetin's asymptotics for p_n leads to:

Theorem (Khavinson & St., 2009)

Assume that:

- $\Gamma = \partial G$ *is a* C^2*-smooth Jordan curve;*
- *the Bergman polynomials* $\{p_n\}_{n=0}^{\infty}$ *satisfy a* $(N+1)$*-term recurrence relation, with some* $N \geq 2$*.*

Then $N = 2$ *and* Γ *is an ellipse.*

However, by using the ratio asymptotics corollary above:

Theorem (V)

Assume that:

- $\Gamma = \partial G$ *is piecewise analytic without cusps;*
- *the Bergman polynomials* $\{p_n\}_{n=0}^{\infty}$ *satisfy a* $(N+1)$*-term recurrence relation, with some* $N \geq 2$*.*

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Where are the zeros of p_n?

Fejer

All the zeros of p_n lie in the convex hull of $\overline{G}$.

Saff

All the zeros of p_n lie in the interior convex hull of $\overline{G}$.

Widom

For any $n \in \mathbb{N}$, p_n has at most a bounded number of zeros (independent of n) on any closed set $E \subset \Omega$.

A result about the zeros of p_n

Since for any $z \in \Omega$, $|\Phi(z)| > 1$ and $|\Phi'(z)| \neq 0$, Thm II yields:

Theorem (VI)

Assume that Γ is piecewise analytic w/o cusps. Then for any closed set $E \subset \Omega$, there exists $n_0 \in \mathbb{N}$, such that for $n \geq n_0$, $p_n(z)$ has no zeros on E.

This leads at once to the refinement:

Corollary

Assume that Γ is piecewise analytic w/o cusps. Then

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|, \quad z \in \Omega \setminus \{\infty\}.$$

A sharp estimate for $\|p_n\|_{\overline{G}}$

Theorem (VII)

Assume that Γ is piecewise analytic w/o cusps and let $\lambda\pi$ denote the largest exterior angle of Γ $(1 \le \lambda \le 2)$. Then

$$\|p_n\|_{\overline{G}} \le c(\Gamma)\, n^{\lambda - 1/2}, \quad n \in \mathbb{N}.$$

Note:

- The order $\lambda - 1/2$ is sharp for Γ smooth (hence $\lambda = 1$). This follows immediately from the fine asymptotic formula of Suetin.
- The above should be compared with the “norm comparison” estimate (holding for any $P \in \mathbb{P}_n$)

$$\|P\|_{\overline{G}} \le c(\Gamma)\, n^{\lambda}\, \|P\|_{L^2(G)}, \quad n \in \mathbb{N},$$

of Pritsker, J. Math. Anal. Appl. (1997) and Abdulayev, Ukrain. Math. J. (2000).

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