Error Analysis and Numerical Performance of the BKM/AB for Conformal Mapping

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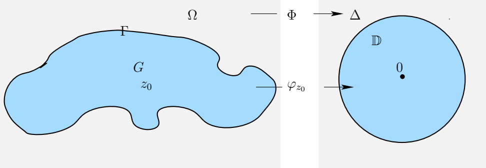
The Conformal Mapping Problem



For Γ a bounded Jordan curve, set $\boxed{G := \text{int}(\Gamma)}$ and $\boxed{\Omega := \text{ext}(\Gamma)}$.

Exterior conformal map: $\boxed{\Phi : \Omega \to \Delta}$, with $\Phi(\infty) = \infty$, $\Phi'(\infty) > 0$.

Fix $z_0 \in G$ and consider the normalized interior map: $\boxed{\varphi_{z_0} : G \to \mathbb{D}}$,
so that $\varphi_{z_0}(z_0) = 0$ and $\varphi'_{z0}(z_0) > 0$.

We want to compute the mapping $\boxed{f_0 : G \to \mathbb{D}_r}$, $r := 1/\varphi'_{z0}(z_0)$

$$f_0(z) := \frac{\varphi_{z0}(z)}{\varphi'_{z0}(z_0)}, \text{ so that } f_0(z_0) = 0 \text{ and } f_0'(z_0) = 1.$$

Note that f_0 extents homeomorphically to Γ.

The Bergman space $L^2_a(G)$

$$L^2_a(G) := \{f : f \text{ analytic in } G,\ \langle f, f\rangle < \infty\},$$

where, $\boxed{\langle f, g\rangle := \int_G f(z)\overline{g(z)}dA(z)}$ and dA denotes area measure.

$L^2_a(G)$: is a Hilbert space with corresponding norm $\|f\|_{L^2(G)} := \langle f, f\rangle^{\frac{1}{2}}$.

The Bergman polynomials $\{P_n\}$ of G

The orthonormal polynomials w.r.t. the area measure on G:

$$\langle P_m, P_n\rangle = \delta_{m,n}, \quad P_n(z) = \lambda_n z^n + \cdots,\ \lambda_n > 0,\ n = 0, 1, 2, \ldots.$$

The Bergman kernel $K(\cdot, z_0)$ of G

The reproducing kernel of $L^2_a(G)$, w.r.t. the point evaluation at z_0:

$$\langle g, K(\cdot, z_0)\rangle = g(z_0),\ \text{for all } g \in L^2_a(G).$$

Series representation for the Bergman kernel

The function $K(z, z_0)$ has the following Fourier series expansion

$$K(z, z_0) = \sum_{j=0}^{\infty} \overline{P_j(z_0)} P_j(z), \; z, z_0 \in G,$$

where, for each fixed $z_0 \in G$ the series convergence uniformly on each compact subset B of G.

Connection with the conformal mapping

The Bergman kernel $K(\cdot, z_0)$ is related to the mapping function f_0 by means of

$$f_0'(z) = \frac{K(z, z_0)}{K(z_0, z_0)}.$$

Hence

$$f_0(z) = \frac{1}{K(z_0, z_0)} \int_{z_0}^{z} K(\zeta, z_0) d\zeta.$$

The Bergman Kernel Method (BKM)

- Start with the monomials $\eta_j(z) = z^j, \ j = 0, 1, 2 \ldots,$
- Orthonormalize $\{\eta_j(z)\}$ by means of the Gram-Schmidt process to produce the orthonormal set $\{P_j(z)\}_{j=0}^{\infty}$. (Bergman polynomials)
- Approximate $K(\cdot, z_0)$ by its n-th finite Fourier expansion, with respect to $\{P_j(z)\}$: (Kernel polynomials)

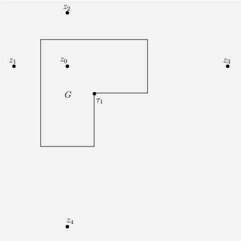
$$K_n(z, z_0) := \sum_{j=0}^{n} \overline{P_j(z_0)} P_j(z). \qquad (1)$$

- Approximate f_0 by (Bieberbach polynomials):

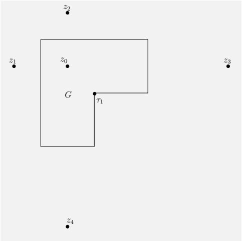
$$\pi_{n+1}(z) := \frac{1}{K_n(z_0, z_0)} \int_{z_0}^{z} K_n(t, z_0) dt. \qquad (2)$$

Application of the BKM

The above implementation of the BKM has been been suggested by pioneers of Numerical Conformal Mapping like P. Davis and D. Gaier and J. Burbea.



However, its application to the L-shaped domain pictured above produced the estimate $\|f_0 - \pi_n\|_{L^\infty(\overline{G})} \approx 0.19$, with $n = 24$, for the maximum BKM error.
Source: Levin, Papamichael and Sideridis, (J. Inst. Maths Applics, 1978), using double precision FORTRAN.



Note that in the case of the L-shaped domain f_0 has near τ_1 an expansion of the form

$$f_0(z) = f(\tau_1) + \sum_{j=1}^{\infty} a_j (z - \tau_1)^{2j/3}, \quad a_1 \neq 0.$$

Moreover, the extension of f_0 by reflection across the sides of G has simple poles at the reflected images z_1, z_2, z_3 and z_4 of z_0.

BKM with Augmented Basis

Based on this observation, Levin, Papamichael and Sideridis were the first to suggest, in J. Inst. Maths Applics (1978), that the error in approximating f_0 on $\overline{G}$ by polynomials of low degree will depend on both the boundary and pole singularities of f_0. Hence in order to improve the numerical performance of the Bergman Kernel Method, they proposed a modification which is based on orthonormalizing a system of functions consisting of monomials along with singular terms of the type

$$(z - z_j)^{-1}, \quad j = 1, 2, 3, 4, \quad \text{and} \quad (z - \tau_1)^{j/3}, \quad j = 1, 2, \ldots, m,$$

that reflect both pole and corner singularities of f_0. This constitutes the main idea of the modification of the BKM which is known as BKM/AB.

Application of the BKM/AB for the L-shaped

Application of the BKM/AB improves considerably the maximum error. Now:

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \approx 2.2 \times 10^{-5}, \text{ with } n = 26.$$

Source: Levin, Papamichael and Sideridis, (J. Inst. Maths Applics, 1978), using double precision FORTRAN.

Mathematical Reviews

> *... A proof of the convergence of the mathematical method given and an investigation of its convergence rate are lacking, so the results obtained are of a heuristic nature.*

(Yu. E. Khokhlov)

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Minimal properties

The polynomials $\{K_n(\cdot, z_0)\}_{n=0}^{\infty}$ provide the best $L^2(G)$-approximation to $K(\cdot, z_0)$ out of the space $\mathbb{P}_n$ of all complex polynomials of degree at most n. That is:

For any $p \in \mathbb{P}_n$

$$\|K(\cdot, z_0) - K_n(\cdot, z_0)\|_{L^2(G)} \leq \|K(\cdot, z_0) - p\|_{L^2(G)}.$$

Let $\mathbb{P}_n^* := \{p : p \in \mathbb{P}_n,\ \text{with } p(z_0) = 0 \text{ and } p'(z_0) = 1\}$. Then:

The polynomial π_n minimizes uniquely the two norms

$$\|f_0' - p'\|_{L^2(G)} \quad \text{and} \quad \|p'\|_{L^2(G)},$$

over all $p \in \mathbb{P}_n^*$.

Note

$\{P_n\}$ forms a complete ON system in $L_a^2(\Omega)$.

BKM theory of the time

Theorem (Walsh, c. 1930)

Assume that f_0 has an analytic continuation across Γ into Ω and let z_1 denote its nearest singularity. Then,

$$\|f_0 - \pi_n\|_{L^\infty(\overline{G})} = O\left(1/R^n\right), \tag{3}$$

for any $1 < R < |\Phi(z_1)|$, but for no $R > |\Phi(z_1)|$.

Recall that Ω denotes the exterior of Γ and Φ denotes the normalized exterior conformal map : $\Omega \to \Delta$.

Theorem (Simonenko, 1979)

Assume that Γ is piecewise analytic. Then, for some $\gamma > 0$,

$$\|f_0 - \pi_n\|_{L^\infty(\overline{G})} \leq c\frac{1}{n^\gamma} \quad n \in \mathbb{N},$$

where $c > 0$ does not depend on n.

BKM theory today: f_0 singular on Γ

Theorem (D. Gaier, Arch. Math., 1992)

Assume that the boundary Γ is piecewise analytic without cusps. Then,

$$\|f_0 - \pi_n\|_{L^\infty(\overline{G})} = O(\log n)\frac{1}{n^s},$$

where $s := \lambda/(2-\lambda)$ and $\lambda\pi$ $(0 < \lambda < 2)$ denotes the smallest exterior angle where two analytic arcs of Γ meet.

Theorem (Maymeskul, Saff and St., Numer. Math., 2002)

If, in addition the interior angle related to λ is NOT of the form $1/m$, $m \in \mathbb{N}$, then

$$c_1\frac{1}{n^s} \le \|f_0 - \pi_n\|_{L^\infty(\overline{G})} \le c_2\sqrt{\log n}\,\frac{1}{n^s}, \quad n \ge 2.$$

BKM/AB with corners singularities: Assumptions

- The boundary Γ of G consist of N analytic arcs that meet at corner points τ_k, $k = 1, 2, \dots, N$.
- $\alpha_k \pi$: denotes the interior angle at τ_k, $k = 1, 2, \dots, N$ $(0 < \alpha_k < 2)$.
- $\lambda_k \pi$: denotes the exterior angle at τ_k $(\lambda_k = 2 - \alpha_k)$.
- NO logarithmic terms occur in the expansions of f_0 near τ_k, $k = 1, 2, \dots, N$. Then, for any $p_k \in \mathbb{N}_0$,

$$f_0(z) = \sum_{j=0}^{p_k} a_j^{(k)} (z - \tau_k)^{\gamma_j^{(k)}} + O\left((z - \tau_k)^{\gamma_{p_k+1}^{(k)}} \right),$$

  where $a_1^{(k)} \neq 0$, $\gamma_0^k := 0$, $\gamma_j^{(k)} = p + q/\alpha_k$, $p \in \mathbb{N}_0$, $q \in \mathbb{N}$.
- M $(M \geq 1)$: denotes the number of corners of Γ for which α_k is NOT of the special form $1/m$, $m \in \mathbb{N}$. (If $N > M$, then f_0 has an analytic continuation in some neighborhood of the corner τ_N.)

BKM/AB with corners singularities: Implementation

- Start with the augmented system $\{\eta_j\}$:

$$\eta_j(z) = [(z-\tau_k)^{\gamma_j^{(k)}}]', \quad j = 1, 2, \ldots p_k, \quad k = 1, 2, \ldots, M,$$

$$\eta_{r_M+j}(z) = (z^j)', \quad j = 1, 2, \ldots, n, \quad r_M := \sum_{k=1}^{M} p_k.$$

- Orthonormalize $\{\eta_j(z)\}$ by means of the Gram-Schmidt process to produce the orthonormal set $\{\widetilde{P}_j(z)\}$.
- Approximate $K(\cdot, z_0)$ and f_0 respectively by,

$$\widetilde{K}_n(z, z_0) := \sum_{j=1}^{r_M+n} \overline{\widetilde{P}_j(z_0)}\widetilde{P}_j(z),$$

$$\widetilde{\pi}_{n+1}(z) := \frac{1}{\widetilde{K}_n(z_0, z_0)} \int_{z_0}^{z} \widetilde{K}_n(t, z_0)dt.$$

BKM/AB with corner singularities: Theory

- $\mathbb{P}_n^{A_2}$: denotes the space of augmented polynomials:

$$\mathbb{P}_n^{A_2} := \{p : p(z) = \sum_{j=1}^{r_M+n} t_j \eta_j(z),\ t_j \in \mathbb{C}\},$$

- $\widetilde{\pi}_n$: denotes the BKM/AB approximation resulting from $\mathbb{P}_n^{A_2}$ to f_0.

Then we have the following:

Theorem (Maymeskul, Saff & St., Numer. Math., 2002)

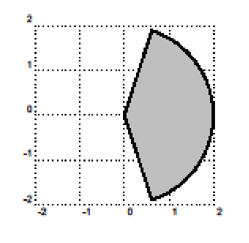
Assume that Γ *is piecewise analytic without cusps and set* $s^\star := \min\{(2-\alpha_k)\gamma_{\nu_k}^{(k)} :\ 1 \le k \le M\}$. *Then,*

$$\frac{1}{n^{s^\star}} \preceq \|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \preceq \sqrt{\log n}\,\frac{1}{n^{s^\star}}, \quad n \ge 2,$$

where $\nu_k := \min\{j > p_k :\ \gamma_j^{(k)} \notin \mathbb{N},\ a_j^{(k)} \ne 0\}$.

BKM/AB with corner singularities: Illustration

Let G denote the circular sector
$G_\alpha := \{z : |z| < 2, -\alpha\pi/2 < \arg z < \alpha\pi/2\}$, with $\alpha = 4/5$.



We fix $z_0 = 1$ and consider the BKM/AB reflecting the corner singularity $(z - \tau_1)^{5/4}$ of f_0 at $\tau_1 = 0$.
Note that f_0 is known exactly and let $E_{n,\infty}(f_0, G) := \|f_0 - \pi_n\|_{L^\infty(\overline{G})}$ and $\widetilde{E}_{n,\infty}(f_0, G) := \|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})}$ denote respectively the uniform BKM and BKM/AB error.

Numerical Example: Circular sector

We test numerically the two hypotheses:

- $E_{n,\infty} \approx c\sqrt{\log n}\frac{1}{n^s}$, $\boxed{s = (2-\alpha)(1/\alpha) = 1.5}$ in BKM.
- $\widetilde{E}_{n,\infty} \approx c\sqrt{\log n}\frac{1}{n^{s^\star}}$, $\boxed{s^\star = (2-\alpha)(2/\alpha) = 3}$ in BKM/AB.

n	$E_{n,\infty}$	s_n	$\widetilde{E}_{n,\infty}$	$s_n^\star$
20	1.2e-02	-	3.6e-04	-
30	6.4e-03	1.55	9.9e-05	3.20
40	4.1e-03	1.54	4.1e-05	3.10
50	2.9e-03	1.52	2.0e-05	3.12
60	2.2e-03	1.51	1.2e-05	3.09
70	1.8e-03	1.51	7.2e-06	3.07

The computations were carried out in Maple 11, using `Digits:=64`.

BKM theory: A Refinement

- L_R: The level curve $\{z : |\Phi(z)| = R, \;\; R \geq 1\}$.
- G_R: The interior of L_R, i.e., $G_R := \text{int}(L_R)$.
- For quantities $A > 0$, $B > 0$ we use $A \preceq B$ if $A \leq cB$, where c is a constant independent of n.

The next theorem complements the classical result (3) of Walsh, in the sense that that it provides a lower estimate and uses the precise ϱ in the denominator.

Theorem (Lytrides & St., CMFT, to appear)

Assume that Γ is piecewise analytic without cusps. Assume further that the conformal map f_0 has an analytic continuation across Γ, such that f_0 is analytic on $\overline{G_\varrho}$, for some $\varrho > 1$, apart from a finite number of poles on L_ϱ. Let m denote the highest order of the poles of f_0 on L_ϱ. Then,

$$c_1 \frac{n^{m-1}}{\varrho^n} \leq \|f_0 - \pi_n\|_{L^\infty(\overline{G})} \leq c_2 \frac{n^m \sqrt{\log n}}{\varrho^n}, \quad n \geq 2.$$

The above result is based on the following lemma, which is an easy consequence of an earlier result of E.B. Saff in Trans. Amer. Math. Soc.

Lemma (Lytrides & St., CMFT, to appear)

Assume that the boundary Γ of G is piecewise Dini-smooth and consider a function f which is analytic on $\overline{G_\varrho}$, for some $\varrho > 1$, apart from a finite number of poles on L_ϱ. Let m denote the highest order of the poles of f on L_ϱ. Then,

$$\inf_{p\in\mathbb{P}_n} \|f - p\|_{L^\infty(\overline{G})} \asymp \frac{n^{m-1}}{\varrho^n}.$$

The expression $A \asymp B$ means that $A \preceq B$ and $B \preceq A$ simultaneously.

BKM/AB with pole singularities: Assumptions

- f_0 has an analytic continuation across Γ in Ω.
- The nearest singularities of f_0 in Ω are poles at points z_j, $j = 1, 2, \ldots, \kappa$, of the form $(z - z_j)^{-k_j}$, $k_j \in \mathbb{N}$, where $|\Phi(z_1)| \leq |\Phi(z_2)| \leq \ldots \leq |\Phi(z_\kappa)|$.
- The other singularities of f_0 in Ω occur at points $z_{\kappa+1}$, $z_{\kappa+2}, \ldots$, where $|\Phi(z_k)| < |\Phi(z_{\kappa+1})| \leq |\Phi(z_{\kappa+2})| \leq \ldots$.

Therefore, it is natural to expect an improvement in the convergence rate of the BKM error, if we “remove" the singularities at z_j of f_0 in Ω. This would be possible by introducing in the basis set $\{\eta_j(z)\}_{j=1}^{\infty}$, functions of the form

$$(z - z_j)^{-k_j}, \quad k_j \in \mathbb{N},\ j = 1, 2, \ldots \kappa,$$

which reflect the singularities of f_0. (That was the original idea of Levin, Papamichael and Sideridis in J. Inst. Math. Appl., 1978.)

BKM/AB with pole singularities: Implementation

- Start with the augmented system $\{\eta_j\}$ consisting of:

$$\eta_j(z) = \left[(z - z_j)^{-k_j}\right]', \; j = 1, 2 \ldots \kappa,$$
$$\eta_{\kappa+j}(z) = (z^j)', \quad j = 1, 2, \ldots, n.$$

- Orthonormalize $\{\eta_j\}$ by means of the Gram-Schmidt process to produce the orthonormal set $\{\widetilde{P}_j\}$,
- Approximate $K(\cdot, z_0)$ by its n-th finite Fourier expansion:

$$\widetilde{K}_n(z, z_0) := \sum_{j=1}^{\kappa+n} \overline{\widetilde{P}_j(z_0)}\widetilde{P}_j(z).$$

- Approximate f_0 by $\widetilde{\pi}_{n+1}(z) := \dfrac{1}{\widetilde{K}_n(z_0, z_0)} \displaystyle\int_{z_0}^{z} \widetilde{K}_n(t, z_0)dt.$

BKM/AB with pole singularities: Theory

- Let $\mathbb{P}_n^{A_1}$ denotes the space of augmented polynomials: $\mathbb{P}_n^{A_1} := \{p : p(z) = \sum_{j=1}^{\kappa+n} t_j\eta_j(z),\ t_j \in \mathbb{C}\}$.
- Let $\widetilde{\pi}_n$ denotes the BKM/AB approximation to f_0 resulting from $\mathbb{P}_n^{A_1}$. Then we have the followings:

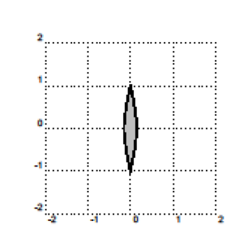
Theorem (Lytrides & St, CMFT, to appear)

Assume that Γ is piecewise analytic without cusps and set $\varrho := |\Phi(z_{\kappa+1})|$. Assume, in addition that f_0 has a finite number of poles and no other singularities on L_ϱ and let m denote their highest order. Then,

$$\frac{n^{m-1}}{\varrho^n} \preceq \|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \preceq \frac{n^m\sqrt{\log n}}{\varrho^n}, \quad n \geq 2.$$

Numerical Example: Lens-domain

Let G denote the symmetric lens domain formed by two intersecting circles that meet at -i and i and form equal angles $\pi/13$ with the imaginary axis.



We fix $z_0 = 0$ and note that f_0 is known exactly. Since, from BKM theory above,

$$E_{n,\infty}(f_0, G) := \|f_0 - \pi_n\|_{L^\infty(\overline{G})} \approx n^{-s} \quad \text{with} \quad s = 12,$$

we consider the classical BKM (and expect to do just GREAT!).

Numerical Example: Lens-domain

Far from doing so...

n	$E_{n,\infty}$
8	0.3817
12	0.2044
16	0.1180
20	0.0702
24	0.0424
28	0.0259
32	0.0160

Note: $1/32^{-12} \approx 10^{-19}(!)$ However, f_0 has two simple poles at $z_1 = \tan(\pi/13)$ and $z_2 = -z1$, with $|\Phi(z_1)| \approx 1.119$ and the next pole is at a point z_3 with $|\Phi(z_3)| \approx 2.055$. So, we consider the BKM/AB, reflecting the simple poles at z_1 and z_2.

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Numerical Example: The influence of poles

We test numerically the two hypotheses:

- $E_{n,\infty} \approx c\frac{1}{\varrho^n}$, $\boxed{\varrho = |\Phi(z_1)| \approx 1.119}$ in BKM.
- $\widetilde{E}_{n,\infty} \approx c\frac{1}{\widetilde{\varrho}^n}$, $\boxed{\widetilde{\varrho} = |\Phi(z_3)| \approx 2.055}$ in BKM/AB.

n	$E_{n,\infty}$	ϱ_n	$\widetilde{E}_{n,\infty}$	$\widetilde{\varrho}_n$
8	0.3817	-	6.1e-04	-
12	0.2044	1.170	3.4e-05	2.053
16	0.1180	1.147	2.0e-06	2.029
20	0.0702	1.139	1.2e-07	2.028
24	0.0424	1.135	7.0e-09	2.028
28	0.0259	1.131	4.1e-10	2.028
32	0.0160	1.128	2.5e-11	2.028

BKM/AB with Pole and Corner Singularities

- Start with the augmented system $\{\eta_j\}$:

$$\eta_j(z) = \left[(z - z_j)^{-k_j}\right]', \; j = 1, 2 \ldots \kappa,$$
$$\eta_{\kappa+j}(z) = [(z - \tau_k)^{\gamma_j^{(k)}}]', \quad j = 1, 2, \ldots p_k, \quad k = 1, 2, \ldots, M,$$
$$\eta_{\kappa+r_M+j}(z) = (z^j)', \quad j = 1, 2, \ldots, n.$$

- Orthonormalize $\{\eta_j(z)\}$ by means of the Gram-Schmidt process to produce the orthonormal set $\{\widetilde{P}_j(z)\}$,
- Approximate $K(\cdot, z_0)$ and f_0 respectively by,

$$\widetilde{K}_n(z, z_0) := \sum_{j=1}^{\kappa+r_M+n} \overline{\widetilde{P}_j(z_0)}\widetilde{P}_j(z),$$
$$\widetilde{\pi}_{n+1}(z) := \frac{1}{\widetilde{K}_n(z_0, z_0)} \int_{z_0}^{z} \widetilde{K}_n(t, z_0) dt.$$

BKM/AB with corners and poles: Theory

- Let $\mathbb{P}_n^{A_3}$ denote the spaces of augmented polynomials: $\mathbb{P}_n^{A_3} := \{p : p(z) = \sum_{j=1}^{\kappa+r_M+n} t_j \eta_j(z),\ t_j \in \mathbb{C}\}$.
- Let $\widetilde{\pi}_n$ denote the BKM/AB approximation to f_0 resulting from $\mathbb{P}_n^{A_3}$.

Theorem (Lytrides & St, CMFT, to appear)

Assume that Γ *is piecewise analytic without cusps and set* $\varrho := |\Phi(z_{\kappa+1})|$ *and* $s^\star := \min\{(2-\alpha_k)\gamma_{\nu_k}^{(k)} :\ 1 \le k \le M\}$. *Then,*

$$\|f_0 - \widetilde{\pi}_n\|_{L^\infty(\overline{G})} \le c_1 \sqrt{\log n}\frac{1}{n^{s^\star}} + c_2 \frac{1}{R^n}, \quad n \ge 2, \tag{4}$$

for any R, $1 < R < \varrho$.

This result should lead to the optimal choice of monomial, corner and pole singular basis functions.