ON A DOMAIN DECOMPOSITION METHOD FOR THE COMPUTATION OF CONFORMAL MODULES

N. PAPAMICHAEL and S. N. STYLIANOPOULOS
Department of Mathematics and Statistics, Brunel University

Abstract. We consider a domain decomposition method for approximating the conformal modules of a certain class of long quadrilaterals.

1. INTRODUCTION

Let $\Omega$ be a Jordan domain in the complex $z$-plane ($z = x + iy$), and consider a system consisting of $\Omega$ and four points $z_j; j = 1, 2, 3, 4$, in counter-clockwise order on its boundary $\partial \Omega$. Such a system is said to be a quadrilateral $Q := \{\Omega; z_1, z_2, z_3, z_4\}$.

The conformal module $m(Q)$ of $Q$ is defined as follows.

Let $R_H$ denote a rectangle of the form

$$R_H = \{(\xi, \eta) : 0 < \xi < 1, 0 < \eta < H\},$$

in the $w$-plane ($w = \xi + i\eta$). Then, $m(Q)$ is the unique value of $H$ for which $Q$ is conformally equivalent to $R_H$. That is, for $H = m(Q)$ and for this value only there exists a unique conformal map

$$f : \Omega \to R_H = R_{m(Q)},$$

which takes the four points $z_j; j = 1, 2, 3, 4$, respectively onto the four corners of the rectangle, i.e. $f$ is such that

$$f(z_1) = 0, f(z_2) = 1, f(z_3) = 1 + \text{im}(Q) \text{ and } f(z_4) = \text{im}(Q).$$

The conformal map (1.2) has many practical applications, and in these the value of $m(Q)$ is often of special significance; see e.g. [2], [4:§ 16.111] and [7].

Now let $\Omega$ be a domain bounded by the two straight lines $x = 0$ and $x = 1$, and two Jordan arcs with cartesian equations $y = -r_1(x)$ and $y = r_2(x)$, where $r_j; j = 1, 2$, are positive in $[0,1]$, and let $z_j; 1, 2, 3, 4$, be the four corners of $\Omega$. That is, let

$$Q = \{\Omega; z_1, z_2, z_3, z_4\},$$

where

$$\Omega = \{(x, y) : 0 < x < 1, -r_1(x) < y < r_2(x)\},$$

and

$$z_1 = -ir_1(0), z_2 = 1 - ir_1(1), z_3 = 1 + ir_2(1), z_4 = ir_2(0).$$

Also, let

$$\Omega_1 = \{(x, y) : 0 < x < 1, -r_1(x) < y < 0\},$$

$$\Omega_2 = \{(x, y) : 0 < x < 1, r_2(x) < y < 0\}.$$
and
\[ \Omega_2 = \{(x, y) : 0 < x < 1, \ 0 < y < \tau_2(x)\}, \tag{1.4b} \]
so that \( \bar{\Omega} = \bar{\Omega}_1 \cup \bar{\Omega}_2 \), and let
\[ Q_1 = \{\Omega_1; x_1, x_2, 1, 0\} \quad \text{and} \quad Q_2 = \{\Omega_2; 0, 1, x_3, x_4\} \tag{1.4c} \]
Then, as is shown in [4: p. 437], \( m(Q) \geq m(Q_1) + m(Q_2) \), and equality occurs only in the two trivial cases where: (a) \( \Omega \) is a rectangle, and (b) \( \tau_1(x) = \tau_2(x), x \in [0,1] \).
The purpose of this note is to consider the problem of approximating \( m(Q) \) by the sum \( m(Q_1) + m(Q_2) \) and, in particular, to give an estimate of \( m(Q) - (m(Q_1) + m(Q_2)) \). In other words, we are concerned with a method for approximating \( m(Q) \), based on decomposing the original quadrilateral (1.3) into the two quadrilaterals given by (1.4). Such a method is of practical interest, mainly, because it can reduce considerably the "crowding" difficulties associated with the computation of the conformal modules of "long" quadrilaterals of the form (1.3); see [5].

2. ESTIMATE OF \( m(Q) - (m(Q_1) + m(Q_2)) \)
Let \( Q \) and \( Q_j; j = 1,2 \), denote the quadrilaterals defined by (1.3)- (1.4), and assume that the functions \( \tau_j; j = 1,2, \) in (1.3b) are absolutely continuous in \([0,1] \) and
\[ d_j := \sup_{0 \leq x \leq 1} |\tau_j'(x)| < \infty; \ j = 1,2. \tag{2.1} \]
Also, let
\[ m_j := \max_{0 \leq x \leq 1} \{\exp(-\pi \tau_j(x))\}; \ j = 1,2, \tag{2.2} \]
and
\[ d = \max(d_1, d_2), \ m = \max(m_1, m_2), \ h = \min \{m(Q_1), m(Q_2)\}. \tag{2.3} \]
Then, our main result can be stated as follows.

**THEOREM 2.1.** If
\[ \epsilon := d \{(1 + m^2)/(1 - m^2)\} < 1, \tag{2.4} \]
then
\[ m(Q) - (m(Q_1) + m(Q_2)) < A(\epsilon)e^{-2\pi h}(2 + e^{-2\pi h}), \tag{2.5a} \]
where
\[ A(\epsilon) = \left(\frac{8}{3}\right)^{1/2}\epsilon^2/\{(1 - \epsilon)(1 - \epsilon^2)^{1/2}\}. \tag{2.5b} \]
2.1. Our method of proof makes extensive use of the theory of the Theodersen-Garrick method, and involves expressing the problem $f: \Omega \rightarrow \mathbb{H}$ as an equivalent problem for the conformal map of a doubly-connected domain onto a circular annulus; see [1: pp. 194-206] and [3]. The detailed proof will be given in a subsequent paper [6], together with the analysis of the decomposition method for determining the full conformal map $f$.

2.2. The theorem shows that for “large” $h$,

$$m(Q) - (m(Q_1) + m(Q_2)) \sim e^{-2\pi h}. \quad (2.6)$$

That is, if $Q$ is a “long” quadrilateral then $m(Q)$ can be approximated closely by $m(Q_1) + m(Q_2)$.

2.3. Although the bound (2.5) is in terms of $h = \min\{m(Q_1), m(Q_2)\}$, it is in general very easy to obtain crude estimates of $m(Q_j); j = 1, 2$. For example,

$$\min_{0 \leq x \leq 1} \tau_j(x) < m(Q_j) < \max_{0 \leq x \leq 1} \tau_j(x); j = 1, 2.$$  \hspace{1cm} (2.7)

2.4. The condition (2.4) is needed for our method of proof. However, the results of several numerical experiments suggest that (2.6) holds even when the condition (2.4) is violated; see [6].

2.5. Considerable simplifications occur in the case where $\tau_1(x) = c > 0, x \in [0,1]$, i.e. where $Q_1$ is a rectangle of height $c$. In this special case corresponding to Theorem 2.1 we have the following:

**Theorem 2.2.** Let $\tau_1(x) = c > 0, x \in [0,1]$, so that $m(Q_1) = c$, and let $d_2$ and $m_2$ be defined by (2.1) and (2.2). If

$$\epsilon := d_2 \left\{(1 + m_2^2)/(1 - m_2^2)\right\} < 1, \quad (2.7)$$

then

$$(M/Q) - (c + m(Q_2)) < A(\epsilon)e^{-2\pi m(Q_3)}(1 + e^{-2\pi c}); \quad (2.8a)$$

where

$$A(\epsilon) = \left(\frac{2}{3}\right)^{1/2} e^{2/\{(1 - \epsilon)(1 - \epsilon^2)^{1/2}\}}. \quad (2.8b)$$

3. Example

Let $Q$ and $Q_j; j = 1, 2,$ be defined by (1.3)-(1.4) with

$$\tau_1(x) = 1 + 0.2 \text{sech}^2(2.5x) \text{ and } \tau_2(x) = 1 + 0.5x.$$  \hspace{1cm} (2.9)

Then, by using the algorithms of [3] we find that

$$m(Q) = 2.25194, m(Q_1) = 1.06549 \text{ and } m(Q_2) = 1.18628.$$  \hspace{1cm} (2.10)

Thus,

$$m(Q) - (m(Q_1) + m(Q_2)) = 1.7 \times 10^{-4},$$

whilst (2.5) with $h = m(Q_1)$ gives
m(Q) - (m(Q_1) + m(Q_2)) < 2.4 \times 10^{-3}.

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Department of Mathematics and Statistics, Brunel University, Uxbridge, Middx UB8 3PH, U. K.