Fine Asymptotics for Bergman and Szegö Polynomials over Domains with Corners

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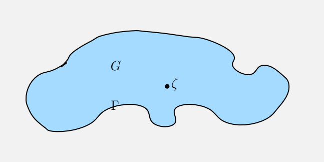
Definition: Bergman polynomials $\{p_n\}$



Γ: a Jordan curve in $\mathbb{C}$, $G := \text{int}(\Gamma)$

$$\langle f, g \rangle := \int_G f(z)\overline{g(z)}dA(z), \quad \|f\|_{L^2(G)} := \langle f, f \rangle^{1/2}$$

The Bergman polynomials $\{p_n\}_{n=0}^{\infty}$ of G are the orthonormal polynomials w.r.t. the area measure:

$$\langle p_m, p_n \rangle = \int_G p_m(z)\overline{p_n(z)}dA(z) = \delta_{m,n},$$

with

$$p_n(z) = \lambda_n z^n + \cdots, \quad \lambda_n > 0, \quad n = 0, 1, 2, \ldots.$$

Minimal property

$$\frac{1}{\lambda_n} = \left\| \frac{p_n}{\lambda_n} \right\|_{L^2(G)} = \min_{z^n+\cdots} \|z^n + \cdots\|_{L^2(G)}.$$

The Bergman space

$$L^2_a(G) := \{f \text{ analytic in } G,\ \|f\|_{L^2(G)} < \infty\},$$

is a Hilbert space with reproducing kernel $K_B(z,\zeta)$: For any $\zeta \in G$,

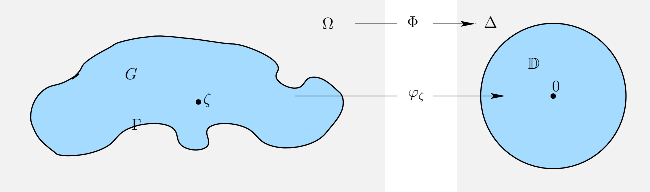
$$f(\zeta) = \langle f, K_B(\cdot,\zeta)\rangle,\ \forall\ f \in L^2_a(G).$$

Approximation Property

$\{p_n\}_{n=0}^{\infty}$ is a complete ON system of $L^2_a(G)$ and

$$K_B(z,\zeta) = \sum_{n=0}^{\infty} \overline{p_n(\zeta)} p_n(z), \quad z,\zeta \in G.$$

Associated conformal maps



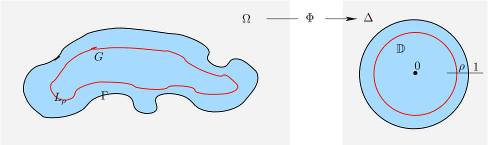
$$\Phi(z) = \gamma z + \gamma_0 + \frac{\gamma_1}{z} + \frac{\gamma_2}{z^2} + \cdots \qquad \boxed{\mathrm{cap}(\Gamma) = 1/\gamma}$$

If $\varphi_\zeta(\zeta) = 0$ and $\varphi'_\zeta(\zeta) > 0$ then

$$K_B(z,\zeta) = \frac{1}{\pi}\,\overline{\varphi'_\zeta(\zeta)}\varphi'_\zeta(z).$$

This leads to the Bergman kernel method for approximating φ'_ζ (and thus φ_ζ) in terms of Bergman polynomials.

Strong asymptotics when Γ is analytic



Carleman, Ark. Mat. Astr. Fys. (1922)

If $\rho < 1$ is the smallest index for which Φ is conformal in $\text{ext}(L_\rho)$, then for any $n \in \mathbb{N}$,

$$\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 + O(\rho^{2n}),$$

and for any $z \in \overline{\Omega}$,

$$p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\{1 + O(\sqrt{n}\,\rho^n)\}.$$

Strong asymptotics when Γ is smooth

We say that $\Gamma \in C(p,\alpha)$, for some $p \in \mathbb{N}$ and $0 < \alpha < 1$, if Γ is given by $z = g(s)$, where s is the arclength, with $g^{(p)} \in \mathrm{Lip}\alpha$. Then both Φ and $\Psi := \Phi^{-1}$ are p times continuously differentiable on Γ and $\partial\mathbb{D}$ respectively, with $\Phi^{(p)}$ and $\Psi^{(p)} \in \mathrm{Lip}\alpha$.

P.K. Suetin, Proc. Steklov Inst. Math. AMS (1974)

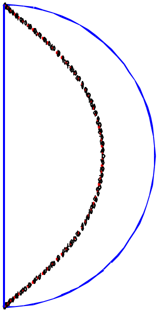
Assume that $\Gamma \in C(p+1,\alpha)$, with $p+\alpha > 1/2$. Then, then for any $n \in \mathbb{N}$,

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 + O(\frac{1}{n^{2(p+\alpha)}}),}$$

and for any $z \in \overline{\Omega}$,

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\{1 + O(\frac{\log n}{n^{p+\alpha}})\}.}$$

Strong asymptotics for Γ non-smooth: An example



$$\gamma = \frac{1}{\text{cap}(\Gamma)} = \frac{3\sqrt{3}}{4}$$

We compute, by using the Gram-Schmidt process (in finite precision), the Bergman polynomials $p_n(z)$ for the unit half-disk, for n up to 60 and test the hypothesis

$$\alpha_n := 1 - \frac{n+1}{\pi} \frac{\gamma^{2(n+1)}}{\lambda_n^2} \approx C \frac{1}{n^s}.$$

Strong asymptotics for Γ non-smooth: Numerical data

n	α_n	s
51	0.003 263 458 678	-
52	0.003 200 769 764	0.998 887
53	0.003 140 444 435	0.998 899
54	0.003 082 351 464	0.998 911
55	0.003 026 369 160	0.998 923
56	0.002 972 384 524	0.998 934
57	0.002 920 292 482	0.998 946
58	0.002 869 952 027	0.998 957
59	0.002 821 401 485	0.998 968
60	0.002 774 426 207	0.998 979

The numbers indicate clearly that $\alpha_n \approx C\frac{1}{n}$. Accordingly, we have made conjectures regarding fine asymptotics in Oberwolfach Reports (2004) and ETNA (2006).

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The numbers indicate clearly that $\alpha_n \approx C\,\frac{1}{n}$. Accordingly, we have made conjectures regarding fine asymptotics in Oberwolfach Reports (2004) and ETNA (2006).

Strong asymptotics for the leading coefficient

Theorem (I)

Assume that Γ is piecewise analytic without cusps, then

$$\boxed{\frac{n+1}{\pi}\frac{\gamma^{2(n+1)}}{\lambda_n^2} = 1 - \alpha_n},$$

where

$$0 \leq \alpha_n \leq c(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}$$

and $C(\Gamma)$ depends on Γ only.

Fine asymptotics for p_n in Ω

Theorem (II)

Assume that Γ is piecewise analytic w/o cusps. Then, for any $z \in \Omega$,

$$\boxed{p_n(z) = \sqrt{\frac{n+1}{\pi}}\Phi^n(z)\Phi'(z)\left\{1 + A_n(z)\right\}},$$

where

$$|A_n(z)| \leq \frac{c_1(\Gamma)}{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}$$

A lower bound for α_n - Coefficient estimates

Let Ψ denote the inverse conformal map $\Phi^{-1} : \{w : |w| > 1\} \to \Omega$. Then

$$\Psi(w) = bw + b_0 + \frac{b_1}{w} + \frac{b_2}{w^2} + \cdots, \quad |w| > 1.$$

Theorem (III)

Assume that Γ is quasiconformal and rectifiable. Then,

$$\alpha_n \geq \frac{\pi(1-k^2)}{A(G)}(n+1)|b_{n+1}|^2.$$

The above provides a connection with the well-studied problem of estimating coefficients of univalent functions.

Quasiconformal curves

In Theorem (II), $k := \dfrac{K-1}{K+1} < 1$, where $K \geq 1$, is the characteristic constant of the quasiconformal reflection defined by Γ.

Definition

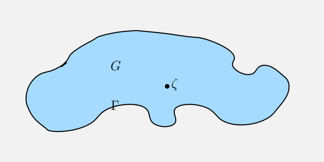
A Jordan curve Γ is quasiconformal if there exists a constant $M > 0$, such that

$$\operatorname{diam}\Gamma(a,b) \leq M\,|a-b|, \text{ for all } a,b \in \Gamma,$$

where $\Gamma(a,b)$ is the arc (of smaller diameter) of Γ between a and b.

Note: A piecewise analytic Jordan curve is quasiconformal if and only if has no cusps (0 and 2π angles).

Definition: Szegö polynomials $\{P_n\}$



Γ: rectifiable Jordan curve.

$$\langle f, g \rangle_\Gamma := \frac{1}{2\pi}\int_\Gamma f(z)\overline{g(z)}|dz|, \quad \|f\|_{L^2(\Gamma)} := \langle f, f \rangle_\Gamma^{1/2}$$

The Szegö polynomials $\{P_n\}_{n=0}^{\infty}$ of Γ are the orthonormal polynomials w.r.t. the normalized arc length measure measure:

$$\langle P_m, P_n \rangle_\Gamma = \frac{1}{2\pi}\int_\Gamma P_m(z)\overline{P_n(z)}|dz| = \delta_{m,n},$$

with

$$P_n(z) = \mu_n z^n + \cdots, \quad \mu_n > 0, \quad n = 0, 1, 2, \ldots.$$

Minimal property

$$\frac{1}{\mu_n} = \left\| \frac{P_n}{\mu_n} \right\|_{L^2(\Gamma)} = \min_{z^n+\cdots} \|z^n + \cdots\|_{L^2(\Gamma)}.$$

The Smirnov space

$$E^2(G) := \{f \text{ analytic in } G, \ \|f\|_{L^2(\Gamma)} < \infty\},$$

is a Hilbert space with reproducing kernel $K_S(z,\zeta)$: For any $\zeta \in G$,

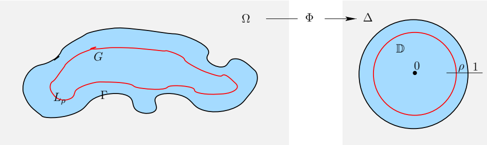
$$f(\zeta) = \langle f, K_S(\cdot,\zeta)\rangle, \ \forall \ f \in E^2(G).$$

Approximation Property

If G is a Smirnov domain then the $\{P_n\}_{n=0}^{\infty}$ is a complete ON system of $E^2(G)$ and

$$K_S(z,\zeta) = \sum_{n=0}^{\infty} \overline{P_n(\zeta)} P_n(z), \quad z,\zeta \in G.$$

Strong asymptotics when Γ is analytic



G. Szegö, Math. Z. (1921)

If $\rho < 1$ is the smallest index for which Φ is conformal in $\text{ext}(L_\rho)$, then for any $n \in \mathbb{N}$,

$$\frac{\gamma^{2n+1}}{\mu_n^2} = 1 + O(\rho^{2n}),$$

and for any $z \in \overline{\Omega}$,

$$P_n(z) = \Phi^n(z)\sqrt{\Phi'(z)}\{1 + O(\sqrt{n}\,\rho^n)\}.$$

Strong asymptotics when Γ is smooth

P.K. Suetin, (1964)

Assume that $\Gamma \in C(p+1,\alpha)$, with $0<\alpha<1$. Then, for any $n \in \mathbb{N}$,

$$\frac{\gamma^{2n+1}}{\mu_n^2} = 1 + O\left(\frac{1}{n^{2(p+\alpha)}}\right),$$

and for any $z \in \overline{\Omega}$,

$$\boxed{P_n(z) = \Phi^n(z)\sqrt{\Phi'(z)}\{1 + O\left(\frac{\log n}{n^{p+\alpha}}\right)\}.}$$

Strong asymptotics for the leading coefficient

Theorem (IV)

Assume that Γ is piecewise analytic without cusps, then

$$\boxed{\frac{\gamma^{2n+1}}{\mu_n^2} = 1 + \alpha_n},$$

where

$$0 \leq \alpha_n \leq c(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}$$

and $C(\Gamma)$ depends on Γ only.

Fine asymptotics for P_n in Ω

Theorem (V)

Assume that Γ is piecewise analytic w/o cusps. Then, for any $z \in \Omega$,

$$\boxed{P_n(z) = \Phi^n(z)\sqrt{\Phi'(z)}\,\{1 + A_n(z)\}},$$

where

$$|A_n(z)| \leq \frac{c_1(\Gamma)}{\sqrt{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}$$

Sharp estimates for $\|p_n\|_{\overline{G}}$ and $\|P_n\|_{\overline{G}}$

Theorem (VI)

Assume that Γ is piecewise analytic w/o cusps and let $\lambda\pi$ denote the largest exterior angle of Γ $(1 \le \lambda \le 2)$. Then

$$\|p_n\|_{\overline{G}} \le c(\Gamma)\, n^{\lambda-1/2}, \quad n \in \mathbb{N}. \tag{1}$$

and

$$\|P_n\|_{\overline{G}} \le c(\Gamma)\, n^{\lambda/2-1/2}, \quad n \in \mathbb{N}. \tag{2}$$

Note:

- The order $\lambda - 1/2$ in (1) is sharp and $\lambda/2 - 1/2$ in (2) is sharp, both for Γ smooth (hence $\lambda = 1$). This follows immediately from the fine asymptotic formula of Suetin.

A result about the zeros of p_n and $\|P_n\|_{\overline{G}}$

Since for any $z \in \Omega$, $|\Phi(z)| > 1$ and $|\Phi'(z)| \neq 0$, Thms II and V yield:

Theorem (VII)

Assume that Γ is piecewise analytic w/o cusps. Then for any closed set $E \subset \Omega$, there exists $n_0 \in \mathbb{N}$, such that for $n \geq n_0$, $p_n(z)$ has no zeros on E. The same holds true for $P_n(z)$.

This leads at once to the refinement:

Corollary

Assume that Γ is piecewise analytic w/o cusps. Then

$$\lim_{n\to\infty} |p_n(z)|^{1/n} = |\Phi(z)|, \quad z \in \Omega \setminus \{\infty\},$$

and

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = |\Phi(z)|, \quad z \in \Omega \setminus \{\infty\}.$$

Ratio asymptotics

From Thm (I) we have immediately:

Corollary (Ratio asymptotics for λ_n)

$$\boxed{\sqrt{\frac{n+1}{n+2}}\,\frac{\lambda_{n+1}}{\lambda_n} = \gamma + \xi_n},$$

where

$$|\xi_n| \le c(\Gamma)\,\frac{1}{n}, \quad n \in \mathbb{N}.$$

We note however that numerical evidence suggests that $|\xi_n| \approx C\frac{1}{n^2}$.

Since $\boxed{\operatorname{cap}(\Gamma) = 1/\gamma}$, the above relation provides the means for computing approximations to the capacity of Γ, by using only the leading coefficients of the associated orthonormal polynomials.

Ratio asymptotics

Similarly, from Thm (II) we have:

Corollary (Ratio asymptotics for p_n)

$$\boxed{\sqrt{\frac{n+1}{n+2}}\,\frac{p_{n+1}(z)}{p_n(z)} = \Phi(z)\{1+B_n(z)\}}, \quad z\in\Omega,$$

where

$$|B_n(z)| \le \frac{c_1(\Gamma)}{\sqrt{\operatorname{dist}(z,\Gamma)\,|\Phi'(z)|}}\,\frac{1}{\sqrt{n}} + c_2(\Gamma)\,\frac{1}{n}, \quad n\in\mathbb{N}.$$

The above relation provides the means for computing approximations to the conformal map Φ in Ω, by simply taking the ratio of two consequent orthonormal polynomials. This leads to an efficient algorithm for recovering the shape of G, from a finite collection of its power moments $\langle z^m, z^n\rangle$, $m,n=0,1,\ldots,N$.

Only ellipses carry finite-term recurrences for p_n

Definition

We say that the polynomials $\{p_n\}_{n=0}^{\infty}$ satisfy a $(N+1)$-term recurrence relation, if for any $n \geq N-1$,

$$zp_n(z) = a_{n+1,n}p_{n+1}(z) + a_{n,n}p_n(z) + \ldots + a_{n-N+1,n}p_{n-N+1}(z).$$

Theorem (Putinar & St. CAOT, 2007)

Assume that:

- $\Gamma = \partial G$, *where* G *is a Caratheodory domain;*
- *the Bergman polynomials* $\{p_n\}_{n=0}^{\infty}$ *satisfy a* $(N+1)$*-term recurrence relation, with some* $N \geq 2$*;*
- $\Gamma \subset B := \{(x,y) \in \mathbb{R}^2 : \psi(x,y) = 0\}$, *where* B *is bounded.*

Then $N = 2$ *and* Γ *is an ellipse.*

An application of the Suetin's asymptotics for p_n leads to:

Theorem (Khavinson & St., 2010)

Assume that:

- $\Gamma = \partial G$ *is a* C^2*-smooth Jordan curve;*
- *the Bergman polynomials* $\{p_n\}_{n=0}^{\infty}$ *satisfy a* $(N+1)$*-term recurrence relation, with some* $N \geq 2$*.*

Then $N = 2$ *and* Γ *is an ellipse.*

However, by using the ratio asymptotics corollary above:

Theorem (VIII)

Assume that:

- $\Gamma = \partial G$ *is piecewise analytic without cusps;*
- *the Bergman polynomials* $\{p_n\}_{n=0}^{\infty}$ *satisfy a* $(N+1)$*-term recurrence relation, with some* $N \geq 2$*.*

Then $N = 2$ *and* Γ *is an ellipse.*