THE SINGULAR FUNCTION BOUNDARY INTEGRAL METHOD FOR A TWO-DIMENSIONAL FRACTURE PROBLEM

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Abstract. *The Singular Function Boundary Integral Method (SFBIM) originally developed for Laplacian problems with boundary singularities is extended for solving two-dimensional fracture problems formulated as a biharmonic problem in terms of the Airy stress function. Our goal is the direct computation of the associated stress intensity factors, which appear as coefficients in the asymptotic expansion of the solution near the crack tip. In the SFBIM the leading terms of the asymptotic expansion are used to approximate the solution and to weight the governing biharmonic equation in the Galerkin sense. The discretized equations are reduced to boundary integrals by means of Green's theorem and the Dirichlet boundary conditions are weakly enforced by means of Lagrange multipliers. The numerical results on a model problem show that the method converges extremely fast and yields accurate estimates of the leading stress intensity factors.*

1 INTRODUCTION

The elastic field near the tip of a fracture in an elastic body is characterized by the *stress intensity factors* (SIFs). These are the coefficients, α_j, that appear in the asymptotic expansion of the Airy stress function u near the crack tip, which is of the general form

$$u(r,\theta) = \sum_{j=1}^{\infty} \alpha_j r^{\beta_j} f_j(\theta) , \tag{1}$$

where (r, θ) denote polar coordinates centered at the crack tip. The eigenvalues $\beta_j \in \mathbb{R}$ and the corresponding eigenfunctions $f_j(\theta)$ are known, whereas the SIFs are unknown, with the values depending on the global problem. The first SIF, α_1, plays a crucial role in the mathematical description of fracture, since

$$K = -\sqrt{2\pi}\,\alpha_1 \tag{2}$$

is the *opening mode* SIF [1].

In the last few decades there has been a plethora of work aimed at reliably computing the SIFs. The methods used include the finite element method (FEM) with post-processing [2–6], the FEM with local mesh refinement [7], enriched and generalized finite elements [8, 9], the method of fundamental solutions [10], as well as certain versions of the Trefftz method [11–14]. It should be noted that in most of the methods mentioned above, the SIFs are calculated as a post-solution operation, i.e. the solution u is approximated first and the SIFs are then calculated using the approximation to u. If the calculation of the SIFs is the main goal of the computation, then it may be beneficial to use a method in which the SIFs are calculated directly. The method of fundamental solutions, the Trefftz method and the SFBIM presented in this article fall in this category of "direct" methods.

The objective of the present work is to extend the SFBIM to two-dimensional fracture problems. The SFBIM was originally developed in [15] to solve Laplacian problems with boundary singularities aiming at resolving the convergence difficulties encountered with standard numerical methods in the vicinity of singular points. In this method the solution is approximated by the leading terms of the local asymptotic solution expansion, which are also employed to weight the governing equation in the Galerkin sense. Furthermore, the discretized equations are reduced to boundary integrals by means of the divergence theorem, and the Dirichlet boundary conditions are weakly enforced by means of Lagrange multipliers. In addition to reducing the dimension of the problem by one, another important feature of the method is that the singular coefficients α_j are

calculated directly (i.e. no post-processing is required) together with the discrete Lagrange multipliers. The SFBIM has been applied to various problems with singularities, such as the Motz problem [16], the cracked-beam problem [17], and to Laplacian problems over L-shaped domains [18, 19], exhibiting fast convergence and yielding very accurate results, especially for the leading singular coefficients. Since it yields direct estimates of the SIFs, the SFBIM appears to be an excellent candidate for solving fracture problems, which can be expressed as a biharmonic equation in terms of the Airy stress function. To illustrate the extension of the method to such problems, we have chosen a two-dimensional fracture problem, originally studied by Schiff et al. [7].

2   THE MODEL PROBLEM AND THE ASYMPOTIC SOLUTION

We consider here the model problem studied by Schiff et al. [7], which deals with a two-dimensional solid elastic plate containing a single edge crack, subjected to a uniform inplane load normal to the two edges parallel to the crack, while the remaining edges are stress free. Using symmetry, the problem is formulated on $\Omega = (-1, 1) \times (0, 1)$ as a biharmonic equation for of the Airy stress function $u(x, y)$ and is depicted graphically in Figure 1. For simplicity the load in the original problem from [7] has been taken to be 1.
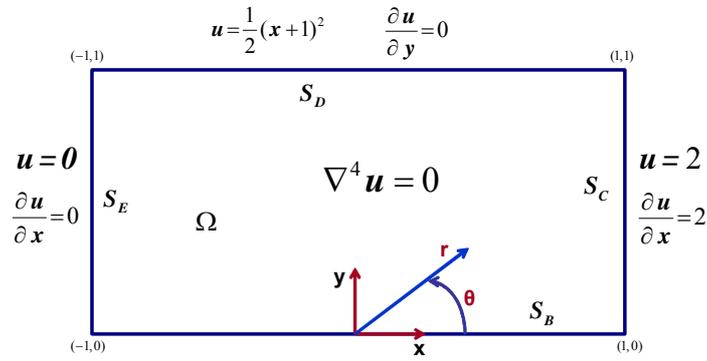
Figure 1. The model fracture problem.

The asymptotic expansion for u in the neighborhood of the singular point $(0, 0)$ can be expressed in terms of an eigenfunction expansion of the form

$$u(r,\theta) = \sum_{j=1}^{\infty} \left[c_j W_1^j(r,\theta) + d_j W_2^j(r,\theta) \right], \tag{3}$$

where (r, θ) are the polar coordinates centered at $(0, 0)$ and c_j, d_j correspond to the even and odd SIFs, respectively. Using this notation, we have $\alpha_1 = d_1$ in (2). In expansion (3) the two sets of the so-called singular functions $W_k^j, k = 1, 2$ are given by

$$W_k^j \equiv r^{\mu_j + 1} f_k(\theta, \mu_j), \quad k = 1, 2, \tag{4}$$

where

$$f_1(\theta, \mu_j) = \cos(\mu_j - 1)\theta - \cos(\mu_j + 1)\theta, \quad \mu_j = j, \quad j = 1, 2, \ldots$$
$$\tag{5}$$

and

$$f_2(\theta, \mu_j) = \cos(\mu_j - 1)\theta - \frac{\mu_j - 1}{\mu_j + 1}\cos(\mu_j + 1)\theta, \quad \mu_j = j - \frac{1}{2}, \quad j = 1, 2, \ldots \tag{6}$$

We note that the singular functions $W_k^j, k = 1, 2$ satisfy the PDE as well as the boundary conditions on S_A and S_B.

3   THE SINGULAR FUNCTION BOUNDARY INTEGRAL METHOD

In the SFBIM the solution u is approximated by the leading terms of the asymptotic expansion. By employing the first N_α terms in (3) the approximate solution $\overline{u}$, is given by

$$\overline{u} = \sum_{i=1}^{N_\alpha} \overline{c}_i W_1^i + \sum_{i=1}^{N_\alpha} \overline{d}_i W_2^i, \tag{7}$$

where $\overline{c}_i$ and $\overline{d}_i$ are the approximations to the SIFs. Obviously, the total number of singular functions involved in the approximation is $2N_\alpha$. It should be pointed out that the method is restricted to fracture problems with only one crack for which the asymptotic solution is available. Moreover, the proposed approximation (7) is valid only if the domain of the problem is a subset of the domain of convergence of the asymptotic solution. Otherwise, the

domain may be partitioned into subdomains over which separate approximations obeying appropriate compatibility conditions along the interfaces could be used.

By applying Galerkin's principle, the governing equation is weighted by the singular functions, which yields the following set of discretized equations:

$$\int_\Omega \nabla^4 \overline{u} W_k^i dV = 0 , \; i = 1, 2, ..., N_\alpha , \quad k = 1, 2. \tag{8}$$

Next, applying Green's theorem twice and taking into account that the singular functions satisfy the governing biharmonic equation, the above integrals are reduced to boundary ones:

$$\int_{\partial\Omega} \left(\frac{\partial \overline{u}}{\partial n} \nabla^2 W_k^i - \overline{u} \frac{\partial(\nabla^2 W_k^i)}{\partial n} \right) dS + \int_{\partial\Omega} \left(\frac{\partial(\nabla^2 \overline{u})}{\partial n} W_k^i - \nabla^2 \overline{u} \frac{\partial(W_k^i)}{\partial n} \right) dS = 0 , \quad i = 1, 2, ..., N_\alpha , k = 1, 2. \tag{9}$$

The Dirichlet boundary conditions are imposed by means of Lagrange multipliers. In the case of Laplacian problems, the Lagrange multipliers replace the normal derivative $\partial \overline{u} / \partial n$. In the case of biharmonic problems, another option is for the Lagrange multipliers to replace $\partial(\nabla^2 \overline{u}) / \partial n$. In the current problem, Dirichlet boundary conditions appear along the three boundary parts of interest, i.e. S_C, S_D and S_E, where the normal derivative of the solution is also specified. Therefore, Lagrange multipliers have been chosen to replace $\partial(\nabla^2 \overline{u}) / \partial n$ at boundary parts S_C, S_D and S_E. These are partitioned into three-node elements and the corresponding Lagrange multipliers, denoted respectively by λ_C, λ_D and λ_E, are expanded in terms of quadratic basis functions M^j:

$$\lambda_C = \frac{\partial(\nabla^2 \overline{u})}{\partial x} = \sum_{j=1}^{N_{\lambda_C}} \lambda_C^j M^j \quad \text{on } S_C , \tag{10}$$

$$\lambda_D = \frac{\partial(\nabla^2 \overline{u})}{\partial y} = \sum_{j=1}^{N_{\lambda_D}} \lambda_D^j M^j \quad \text{on } S_D \tag{11}$$

and

$$\lambda_E = \frac{\partial(\nabla^2 \overline{u})}{\partial x} = \sum_{j=1}^{N_{\lambda_E}} \lambda_E^j M^j \quad \text{on } S_E , \tag{12}$$

where N_{λ_C}, N_{λ_D} and N_{λ_E} are the numbers of the discrete Lagrange multipliers λ_C^j, λ_D^j and λ_E^j along the corresponding boundaries. The discrete Lagrange multipliers appear as additional unknowns in the problem. The required $N_{\lambda_C}, N_{\lambda_D}$, N_{λ_E} additional equations are obtained by weighting the Dirichlet boundary conditions along S_C, S_D and S_E by the quadratic basis functions M^i in the Galerkin sense. The following linear system of $2N_\alpha + N_{\lambda_C} + N_{\lambda_D} + N_{\lambda_E}$ discretized equations is thus obtained:

$$\int_{S_C} \left(\lambda_C W_k^i - \overline{u} \frac{\partial(\nabla^2 W_k^i)}{\partial x} - \nabla^2 \overline{u} \frac{\partial(W_k^i)}{\partial x} \right) dy + \int_{S_D} \left(\lambda_D W_k^i - \overline{u} \frac{\partial(\nabla^2 W_k^i)}{\partial y} - \nabla^2 \overline{u} \frac{\partial(W_k^i)}{\partial y} \right) dx +$$

$$+ \int_{S_E} \left(-\lambda_E W_k^i - \overline{u} \frac{\partial(\nabla^2 W_k^i)}{\partial x} + \nabla^2 \overline{u} \frac{\partial(W_k^i)}{\partial x} \right) dy = -\int_{S_C} 2\nabla^2 W_k^i dy, \quad i = 1, ..., N_\alpha , k = 1, 2 , \tag{13}$$

$$\int_{S_C} \overline{u} M^i dy = \int_{S_C} 2 M^i dy, \quad i = 1, ..., N_{\lambda_C} , \tag{14}$$

$$\int_{S_D} \overline{u} M^i dx = \int_{S_D} \left[\frac{1}{2}(x+1)^2 \right] M^i dx, \quad i = 1, ..., N_{\lambda_D} , \tag{15}$$

$$-\int_{S_E} \bar{u} M^i \, dy = 0, \quad i = 1, \dots, N_{\lambda_E}. \tag{16}$$

The above linear system can be written in block form as follows:

$$\begin{bmatrix} K & K_C & K_D & K_E \\ K_C^T & 0 & 0 & 0 \\ K_D^T & 0 & 0 & 0 \\ K_E^T & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} X_{\bar{c},\bar{d}} \\ \Lambda_C \\ \Lambda_D \\ \Lambda_E \end{bmatrix} = \begin{bmatrix} A \\ C \\ D \\ E \end{bmatrix}, \tag{17}$$

where

$$X_{\bar{c},\bar{d}} = \left[\bar{c}_1, \dots, \bar{c}_{N_\alpha}, \bar{d}_1, \dots, \bar{d}_{N_\alpha} \right]^T, \Lambda_C = \left[\lambda_C^1, \dots, \lambda_C^{N_{\lambda_C}} \right]^T, \Lambda_D = \left[\lambda_D^1, \dots, \lambda_D^{N_{\lambda_D}} \right]^T, \Lambda_E = \left[\lambda_E^1, \dots, \lambda_E^{N_{\lambda_E}} \right]^T,$$

are the vectors of unknowns, and the entries in the coefficient matrix and right hand side can be read from equations (13) – (16). It should be noted that the integrands in equations (13) – (16) are nonsingular and all integrations are carried out far from the boundaries causing the singularity. Also, the stiffness matrix in (17) is symmetric but becomes singular if $N_\lambda > 2N_\alpha$, where $N_\lambda = N_{\lambda_C} + N_{\lambda_D} + N_{\lambda_E}$. This last fact will be taken into consideration when choosing specific values for these parameters.

4   NUMERICAL EXPERIMENTS

In order to to implement the SFBIM, the boundary parts S_C, S_D and S_E, i.e. the boundary parts away from the singularity, are subdivided into quadratic elements. In particular, we use N_C elements for each of the boundaries S_C and S_E, and N_D elements for boundary S_D, which makes the total number of Lagrange multipliers $N_\lambda = N_{\lambda_C} + N_{\lambda_D} + N_{\lambda_E} = 2N_{\lambda_C} + N_{\lambda_D}$, where $N_{\lambda_C} = 2N_C + 1$ and $N_{\lambda D} = 2N_D + 1$. All integrals are calculated numerically by subdividing each quadratic element into 10 subintervals and using a 15 point Gauss-Legendre quadrature over each subinterval [18, 19].

As mentioned above, the number of the singular functions N_C should be greater than the number of Lagrange multipliers N_λ, because otherwise the stiffness matrix becomes ill-conditioned or singular. On the other hand, large values of N_C should be avoided because the contributions of the high-order singular functions become either negligible (for $r < 1$) or very large (for $r > 1$) beyond the limits double precision can handle. Since, at the moment, no a-priori way of choosing the "optimal" values for N_λ and N_C exists, we have carried out systematic runs in order to study the effects the variation of these parameters would have on the numerical results.

The effect of $2N_\alpha$ on the leading SIFs can be observed in Tables 1 and 2 which show results obtained with $N_\lambda = 39$. Initially, we observe fast convergence as $2N_\alpha$ is increased, but at very high values of the latter (i.e. above $2N_\alpha = 94$) slow divergence is observed due to the inaccuracies introduced by the high-order singular functions. Tables 3 and 4 show the effect of varying $N_\lambda = N_{\lambda_C} + N_{\lambda_D} + N_{\lambda_E}$, when $2N_\alpha = 94$. Again, fast convergence is observed initially but as N_λ approaches the value of $2N_\alpha$, the results start diverging slowly, which is attributed to the fact that the stiffness matrix becomes ill-conditioned.

$2N_\alpha$	d_1	d_2	d_3	d_4	d_5	d_{10}
70	2.12751291	− 1.03669169	0.0371710	0.1177493	− 0.1227288	− 0.01108
80	2.12751343	− 1.03669221	0.0371701	0.1177510	− 0.1227319	− 0.01103
88	2.12751347	− 1.03669218	0.0371701	0.1177511	− 0.1227313	− 0.01103
90	2.12751342	− 1.03669217	0.0371701	0.1177510	− 0.1227316	− 0.01103
92	2.12751342	− 1.03669217	0.0371701	0.1177509	− 0.1227316	− 0.01103
94	**2.12751343**	**− 1.03669217**	**0.0371702**	**0.1177509**	**− 0.1227315**	**− 0.01103**
96	2.12751343	− 1.03669217	0.0371702	0.1177509	− 0.1227314	− 0.01103
100	2.12751343	− 1.03669219	0.0371702	0.1177509	− 0.1227315	− 0.01103
110	2.12751347	− 1.03669237	0.0371705	0.1177508	− 0.1227315	− 0.01102
120	2.12751343	− 1.03669229	0.0371705	0.1177508	− 0.1227314	− 0.01103

Table 1 : Convergence of the leading *odd* SIFs d_i with $2N_\alpha$; $N_\lambda = 39$.

These computations suggest that the "optimal" values for the numbers of singular functions and Lagrange multipliers are $2N_\alpha = 94$ and $N_\lambda = 39$, respectively. For higher values of $2N_\alpha$ (e.g., $2N_\alpha = 120$) satisfactory values of the SIFs are still obtained, but the quality of the global solution is not very good. When comparing the

performance of the method with that in the case of Laplacian problems [16–19], we note that convergence is slower in the case of the biharmonic equation, which is reasonable since the latter is more complicated than the Laplace equation. If the smoothness of the calculated Lagrange multiplier functions is used as an indication of the quality of the solution, then for the combination $2N_\alpha = 94$ and $N_\lambda = 39$, the calculated Lagrange multiplier functions along boundary parts S_C, S_D and S_E are the smoothest possible (see Figure 2). We note that for a slightly different value of N_λ the estimated values of the SIFs are essentially unaffected, while the calculated Lagrange multipliers exhibit oscillations.

$2N_\alpha$	c_1	c_2	c_3	c_4	c_5	c_{10}
70	0.16676222	0.0624426	− 0.1324729	− 0.0102230	0.1058502	0.004334
80	0.16676181	0.0624440	− 0.1324747	− 0.0102203	0.1058466	0.004262
88	0.16676181	0.0624439	− 0.1324746	− 0.0102211	0.1057471	0.004263
90	0.16676182	0.0624439	− 0.1324745	− 0.0102208	0.1057474	0.004264
92	0.16676184	0.0624439	− 0.1324745	− 0.0102208	0.1057474	0.004264
94	**0.16676184**	**0.0624439**	**− 0.1324745**	**− 0.0102209**	**0.1057472**	**0.004264**
96	0.16676184	0.0624439	− 0.1324745	− 0.0102208	0.1057471	0.004264
100	0.16676184	0.0624439	− 0.1324745	− 0.0102207	0.1057470	0.004264
110	0.16676179	0.0624441	− 0.1324753	− 0.0102196	0.1057450	0.004262
120	0.16676181	0.0624440	− 0.1324751	− 0.0102200	0.1057457	0.004265

Table 2 : Convergence of the leading *even* SIFs c_i with $2N_\alpha$; $N_\lambda = 39$.

N_λ	d_1	d_2	d_3	d_4	d_5	d_{10}
7+13+7	2.12751309	− 1.03669185	0.0371707	0.1177516	− 0.1227324	− 0.01103
7+17+7	2.12751334	− 1.03669214	0.0371702	0.1177509	− 0.1227318	− 0.01103
7+21+7	2.12751338	− 1.03669217	0.0371702	0.1177507	− 0.1227319	− 0.01103
7+23+7	2.12751343	− 1.03669217	0.0371702	0.1177510	− 0.1227314	− 0.01103
7+25+7	**2.12751343**	**− 1.03669217**	**0.0371702**	**0.1177509**	**− 0.1227315**	**− 0.01103**
7+27+7	2.12751342	− 1.03669217	0.0371702	0.1177509	− 0.1227315	− 0.01103
7+29+7	2.12751347	− 1.03669213	0.0371703	0.1177513	− 0.1227309	− 0.01103
7+31+7	2.12751346	− 1.03669213	0.0371703	0.1177513	− 0.1227310	− 0.01103
7+33+7	2.12751335	− 1.03669221	0.0371701	0.1177506	− 0.1227324	− 0.01103

Table 3 : Convergence of the leading *odd* SIFs d_i with N_λ ; $2N_\alpha = 94$.

$2N_\alpha$	c_1	c_2	c_3	c_4	c_5	c_{10}
7+13+7	0.16676176	0.0624436	− 0.1324753	− 0.0102209	0.1058486	0.004267
7+17+7	0.16676184	0.0624439	− 0.1324745	− 0.0102207	0.1058479	0.004264
7+21+7	0.16676185	0.0624439	− 0.1324744	− 0.0102204	0.1057480	0.004266
7+23+7	0.16676185	0.0624439	− 0.1324745	− 0.0102209	0.1057471	0.004264
7+25+7	**0.16676184**	**0.0624439**	**− 0.1324745**	**− 0.0102209**	**0.1057472**	**0.004264**
7+27+7	0.16676184	0.0624439	− 0.1324745	− 0.0102208	0.1057470	0.004263
7+29+7	0.16676180	0.0624438	− 0.1324748	− 0.0102213	0.1057461	0.004263
7+31+7	0.16676181	0.0624438	− 0.1324748	− 0.0102212	0.1057462	0.004262
7+33+7	0.16676187	0.0624440	− 0.1324743	− 0.0102201	0.1057485	0.004262

Table 4 : Convergence of the leading *even* SIFs c_i with N_λ ; $2N_\alpha = 94$.

In Table 5 the converged values of coefficients d_i and c_i, $i = 1, \ldots, 10$ obtained with the SFBIM are compared with the most accurate values obtained by the collocation Trefftz method of Li et al. [14], who reported that the leading SIF d_1 is converged up to the seventh significant digit. The SFBIM appears to be more accurate as it achieves convergence up to the eighth significant digit. Since Li et al. [14] do not provide information about the convergence of the other SIFs, in Table 5 we tabulate their computed values with one additional digit than the converged values of the SFBIM. Nevertheless, there is excellent agreement between the results of the two methods. Finally, Figures 3 and 4 show the surface plots of the approximate solution $\bar{u}$ and its partial derivatives $\bar{u}_x, \bar{u}_y, \bar{u}_{xx}$ and $\bar{u}_{yy}$. The effect of the singularity at (0, 0) is clearly visible in these profiles.
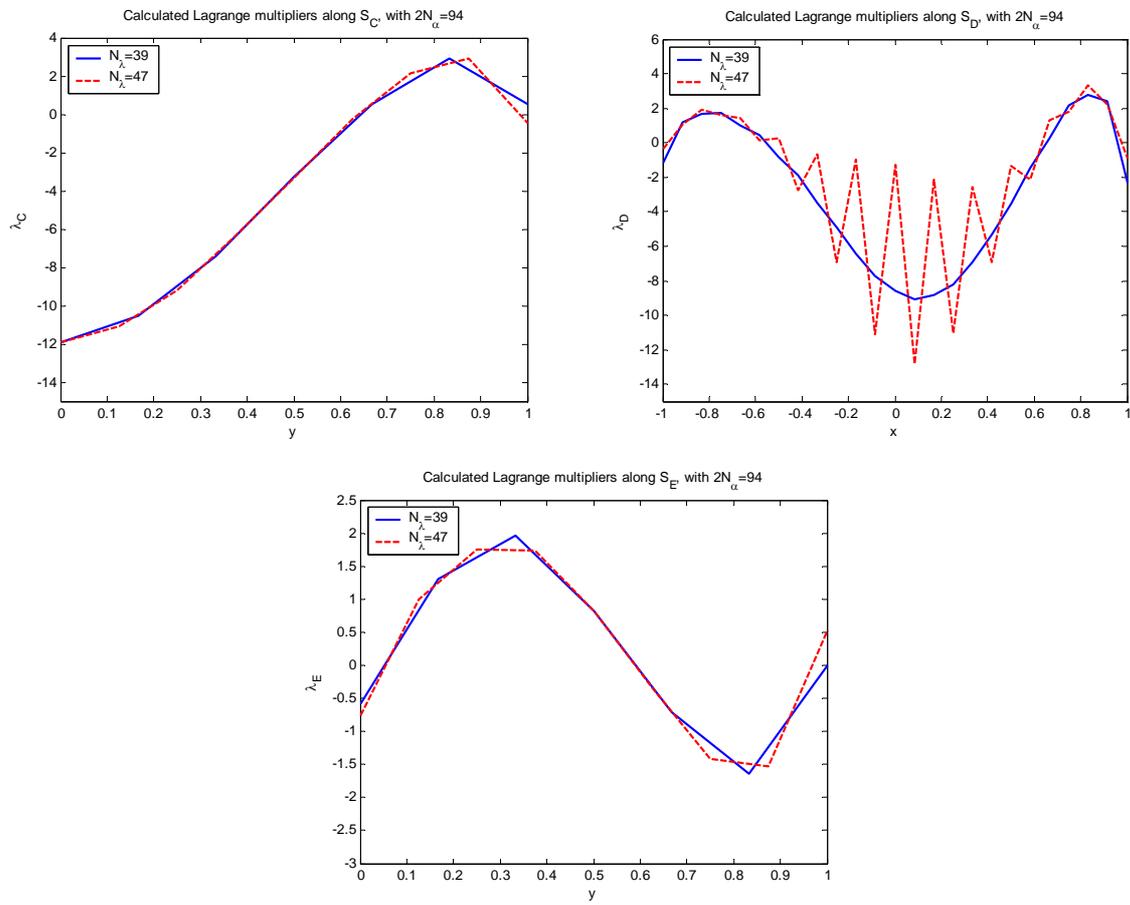
Figure 2. Calculated Lagrange multipliers along S_C, S_D, S_E with $2N_\alpha = 94$; $N_\lambda = 39$ (solid, $N_{\lambda_C} = N_{\lambda_E} = 7$, $N_{\lambda_D} = 25$) and $N_\lambda = 43$ (dashed, $N_{\lambda_C} = N_{\lambda_E} = 9$, $N_{\lambda_D} = 25$).
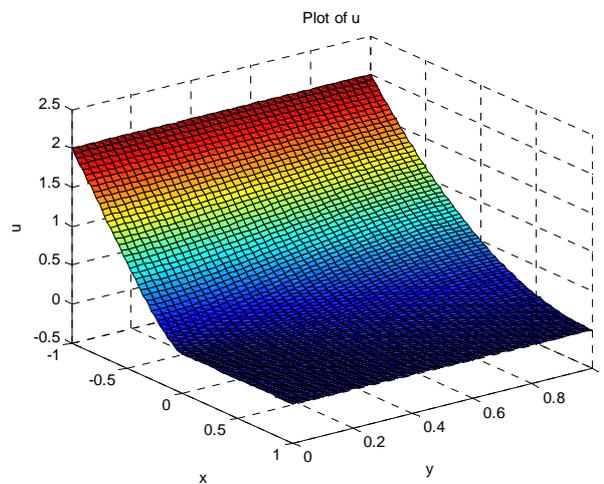
Figure 3. Plot of the converged solution $\bar{u}$.
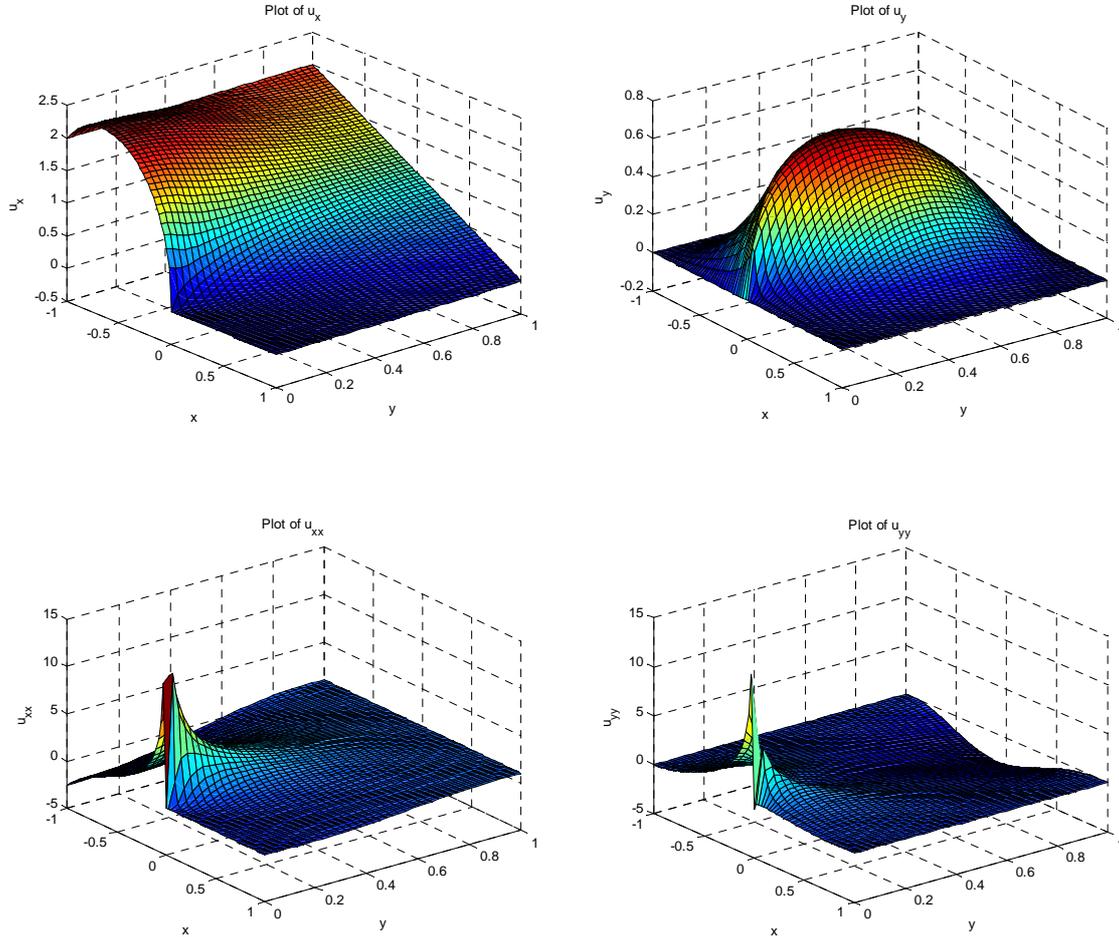
Figure 4. Plots of the first derivatives $\bar{u}_x$ and $\bar{u}_y$ (top) and second derivatives $\bar{u}_{xx}$ and $\bar{u}_{yy}$ (bottom).

5   CONCLUSIONS

The singular function boundary integral method (SFBIM) has been developed for solving two-dimensional fracture problems in terms of the Airy stress function. In this method the leading terms of the asymptotic solution are used to approximate the solution and thus the SIFs are calculated directly (i.e. no post-processing of the numerical solution is required). The governing biharmonic equation is weighted by the singular functions in the Galerkin sense, and the discretized equations are then reduced to boundary integrals by means of a double application of the divergence theorem, which leads to a significant reduction in the computational cost. Another attractive feature of the method is that integration is necessary only along boundary parts that are away from the crack tip. The Dirichlet boundary conditions are weakly enforced by means of Lagrange multipliers which, depending on the type of the boundary conditions, may replace either $\partial u / \partial n$ or $\partial \left(\nabla^2 u \right) / \partial n$ in the integrands of the discretized equations. The Lagrange multipliers are calculated together with the SIFs. The SFBIM has been applied to a model problem proposed by Schiff et al. [7]. The numerical calculations showed that the method converges very fast with the number of singular functions and the number of Lagrange multipliers, and yields accurate estimates of the leading SIFs. The value of the leading SIF, in particular, is converged up to

eight significant digits. Our results agree well with the values obtained by Li et al. [14] using the collocation Trefftz method.

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