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Characterizations of probability distributions based on discrete p-monotonicity

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Abstract

In this paper, a general characterization of a class of discrete self-decomposable distributions based on discrete p-monotonicity is obtained. This result can be thought as the discrete version of an earlier result due to Shanbhag (1972), Artikis (1982) and Alamatsaz (1985). Applying this result, discrete versions of some recent characterizations obtained by Kotz and Steutel (1988), Huang and Chen (1989) and Alamatsaz (1993) are given. Also, as a by-product of another related result, an extension of Forst's (1979) characterization of discrete self-decomposable Poisson mixtures is arrived at.

Keywords: Characterization; Probability generating function; Discrete p-monotonicity; Discrete self-decomposability; Discrete stable distribution; Mixtures of distributions

1. Introduction

The notions of unimodality exist for both non-lattice as well as lattice distributions each one having its own interpretation. Khintchine (1938) established a characterization for unimodal distributions by proving that a real-valued random variable X has a unimodal distribution with mode at 0 if and only if $X \overset{d}{=} UZ$, where U is uniformly distributed on $(0, 1)$ and independent of some random variable Z. Olshen and Savage (1970), general ized, amongst other things, the concept of unimodality to p-unimodality and proved that a one-dimensional random variable X is p-unimodal if and only if

$$X \overset{d}{=} U^{1/p}Z, \tag{1}$$

where $p > 0$ and U is uniform in $(0, 1)$. In terms of characteristic functions (ch.f.'s), (1) is equivalent to stating that

$$\phi(t) = p \int_0^1 \psi(ut)u^{p-1}\,\mathrm{d}u, \quad -\infty < t < \infty, \tag{2}$$

where ϕ is the ch.f. of the random variable X and ψ is some ch.f. Shanbhag (1972) characterized ϕ for which ψ is a certain power of ϕ itself when $p = 1$. By following Shanbhag's exact line of proof, Artikis (1982) (with an incomplete proof) and Alamatsaz (1985) generalized this result for $p \geqslant 1$. Recently, Kotz and Steutel (1988), Huang and Chen (1989) and Yeo and Milne (1991) obtained interesting

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characterizations based on the relation (and some of its extensions)

$$X \stackrel{d}{=} U(X_1 + X_2), \tag{3}$$

where X_1 and X_2 are independent random variables with distribution the same as that of X. More recently, Alamatsaz (1993) obtained, as a corollary to his earlier result (Alamatsaz, 1985), the solution of the equation

$$X_1 + \cdots + X_K \stackrel{d}{=} U^{1/p}(X_{K+1} + \cdots + X_{K+M}), \tag{4}$$

where M, K $(M > K)$ are integers, $p > 0$, U is uniformaly distributed on $(0, 1)$, X_i's are identical, $X_1, \ldots, X_K$ are independent, and U and $X_{K+1}, \ldots, X_{K+M}$ are also independent. He also revealed that the results obtained by Kotz and Steutel (1988), Huang and Chen (1989), Devroye (1990) and Yeo and Milne (1991) follow as special cases of his results.

Unimodal discrete[1] distributions have a different structure from that of generalized unimodal distributions. Steutel (1988), gave a definition analogous to (1), relative to discrete distributions. Restrictions to distributions on $N_0 = \{0, 1, 2, \ldots\}$, he defined that a random variable X is p-monotone, where $p > 0$ if

$$X \stackrel{d}{=} U^{1/p} \oplus Z, \tag{5}$$

where U is uniformly distributed on $(0, 1)$ and independent of the non-negative integer-valued r.v. Z; the multiplication $u \oplus Z$ is defined (see Steutel and van-Harn, 1979) by $\sum_{i=0}^{X} Z_i$, where $\{Z_i\}$ is a sequence of independent Bernoulli random variables with success probability u and X is a non-negative integer-valued random variable independent of $\{Z_i\}$. By considering a discrete version of (3), that is,

$$X \stackrel{d}{=} U^{1/p} \oplus (X_1 + X_2), \tag{6}$$

where X, X_1 and X_2 are non-negative integer-valued random variables that are indentically distributed, U is uniform in $(0, 1)$ and independent of X_1 and X_2, Alzaid and Al-Osh (1990) obtained, amongst other things, a characterization of a class of infinitely divisible distributions which parallels the result of Kotz and Steutel (1988) for the continuous case.

Clearly (6) is a special case of (5) which, in terms of probability generating functions (p.g.f.'s), the latter is equivalent to stating that

$$P(t) = p \int_0^1 G(1 - u(1 - t))u^{p-1} \, \mathrm{d}u, \quad t \in [-1, 1], \tag{7}$$

where P is the p.g.f. of the random variable X and G is some p.g.f.

In the present paper, we shall investigate characterizations of a class of discrete self-decomposable distributions based on the relation (7). Specifically, using (7), we shall characterize P for which G is a certain power of P itself. Furthermore, we shall prove that P (and hence G) in this case is discrete self-decomposable and therefore unimodal for all $p > 0$. In connection with the above, an extension of a result due to Forst (1979) concerning the discrete self-decomposability of Poisson mixtures is obtained. Finally, as applications of the above results, we shall arrive at characterizations of the negative binomial (and hence for the geometric) distribution which can be thought as the discrete versions of the recent results of Kotz and Steutel (1988), Huang and Chen (1989) and Alamatsaz (1993).

2. The characterizations

Theorem 1. *Let P in* (7) *be non-degenerate. Then*

$$G(t) = (P(t))^{(p+r-1)/p}, \quad \textit{for all } t \in [-1, 1],$$

if and only if

$$P(t) = (1 + c(1 - t)^{\gamma})^{-p/\gamma}, \quad t \in [-1, 1], \tag{8}$$

where $\gamma = r - 1$, $1 < r \leqslant 2$ and $c > 0$.

Proof. Eq. (7) is equivalent to

$$P(t) = \frac{p}{(1 - t)^p} \int_0^{1-t} G(1 - s)s^{p-1} \, \mathrm{d}s, \quad t \in [-1, 1], \tag{9}$$

[1] A discrete distribution $\{p_n\}$ with support as a subset of the lattice of integers is said to be unimodal if there exists at least one integer n_0 such that $p_n \geqslant p_{n-1}$ for all $n \leqslant n_0$ and $p_{n+1} \leqslant p_n$ for all $n \geqslant n_0$; the point n_0 is called the vertex (or the mode) of $\{p_n\}$.

which, in view of the assumption that $G(t) = (P(t))^{(p+r-1)/p}$, is equivalent to

$$\frac{d}{dt}\{P(t)(1-t)^p\}\{P(t)(1-t)^p\}^{-(p+r-1)/p}$$
$$= -p(1-t)^{-r}, \quad t \in (-1,1). \tag{10}$$

From this equation, we get $P(t) = 1$ if $r = 1$; as P is non-degenerate, we have to exclude then this case. Integrating on both sides of (10), and appealing to the continuity of P, we get

$$P(t) = (1 + c(1-t)^{r-1})^{-p/(r-1)}, \quad t \in [-1,1],$$

where c is a non-zero constant with respect to t. Now, define

$$P_m(t) = [P(1-(1-t)/m^{1/(r-1)})]^m, \quad m = 1,2,\ldots.$$

Hence, $\lim_{m\to\infty} P_m(t) = P^*(t)$, where

$$P^*(t) = \exp\left\{-\frac{pc}{r-1}(1-t)^{r-1}\right\}, \quad t \in [-1,1]. \tag{11}$$

From the continuity theorem for p.g.f.'s, it follows that P^* is a p.g.f. Clearly, P^* is now the p.g.f. of a discrete stable distribution. From what is observed by Steutel and van-Harn (1979), it is immediate that we need r here to be such that $1 < r \leqslant 2$ and $c > 0$; in that case, the P^* can indeed be seen to be discrete stable with exponent $\gamma = r - 1$. Hence, it is clear that P is given by (8). The converse is immediate by the fact that

$$\frac{d}{dt}P(t) = cp(1-t)^{\gamma-1}(1+c(1-t)^{\gamma})^{-1}P(t)$$

and then

$$G(t) = P(t) - \frac{(1-t)}{p}\frac{d}{dt}\{P(t)\} = (P(t))^{(p+\gamma)/p}.$$

This completes the proof of the theorem. □

Obviously, P (and hence G) given by (8) is the p.g.f. of a discrete infinitely divisible random variable. In the sequel, we show that P (and hence G) is also discrete self-decomposable and hence unimodal for all $p > 0$. Before that, we prove a general theorem concerning discrete (infinitely divisible) self-decomposable p.g.f.'s.

Theorem 2. *Let P^* be the p.g.f. of a discrete (infinitely divisible) self-decomposable distribution on N_0. Then, for every $0 < \gamma \leqslant 1$, the function G^* given by*

$$G^*(t) = P^*(1-(1-t)^{\gamma}), \quad t \in [-1,1] \tag{12}$$

is the p.g.f. of a discrete (infinitely divisible) self-decomposable distribution on N_0.

Proof. The infinitely divisibility part is obvious. To prove the other part, set $h(t) = 1-(1-t)^{\gamma}$ and note that

$$\frac{G^*(t)}{G^*(1-c(1-t))} = \frac{P^*(h(t))}{P^*(1-(1-(1-c(1-t)))^{\gamma})}$$
$$= \frac{P^*(h(t))}{P^*(1-c^{\gamma}+c^{\gamma}h(t))}$$
$$= P^*_{c^{\gamma}}(h(t)), \quad t \in [-1,1]$$

is a compound p.g.f. for each $c \in (0,1)$. Hence, the result follows. □

Remark 1. (i) By taking P^* in the above theorem as the p.g.f. of a negative binomial distribution with index p/γ and parameter $1/(1+c)$, it is easily seen that the p.g.f. G^* given by (12) coincides with P given by (8). This implies that P (and hence G) in question is discrete (infinitely divisible) self-decomposable and hence, in view of a result due to Steutel and van Harn (1979), unimodal for all $p > 0$.

(ii) It is also worth pointing out here that P given by (8) can be thought as the p.g.f. of a more general form of the discrete Linnik distribution studied recently by Devroye (1993).

The following corollary of Theorem 2 is the discrete version of Theorem 2 of Alamatsaz (1985) concerning mixtures of stable distributions, and itself is an extension of a result due to Forst (1979).

Corollary 1. *Let G_1 be the p.g.f. of a discrete stable distribution with exponent γ given by*

$$G_1(t) = \exp\{-\lambda(1-t)^{\gamma}\}, \quad t \in [-1,1],$$

where $\lambda > 0$ and $\gamma \in (0,1]$. Then

$$P_1(t) = \int_0^{\infty} (G_1(t))^x \, dF(x), \tag{13}$$

with F as some self-decomposable distribution function on $\mathbb{R}_+ = [0, \infty)$, *is the p.g.f. of a discrete self-decomposable distribution.*

Proof. Let X be the random variable corresponding to the self-decomposable distribution function F on $\mathbb{R}_+$. Then, in view of the result that Poisson (θ) mixtures are discrete self-decomposable when the mixing random variable θ is self-decomposable (see Forst, 1979),

$$P^*(t) = E(\mathrm{e}^{-\lambda X(1-t)}), \quad t \in [-1, 1]$$

is the p.g.f. of a discrete self-decomposable distribution for some $\lambda > 0$. Moreover,

$$G^*(t) = P^*(1 - (1-t)^\gamma) = E(\mathrm{e}^{-\lambda X(1-t)^\gamma}) = P_1(t), \quad t \in [-1, 1].$$

This implies that P_1 in question is discrete self-decomposable. □

Remark 2. (i) It is worth mentioning here that by taking F in the above corollary as a gamma distribution with index p/γ and scale parameter 1, the p.g.f. P_1 given by (13) also coincides with P given by (8).

(ii) It is also worth pointing out here that mixtures of discrete self-decomposable distributions are not necessarily discrete self-decomposable even if the mixing distribution is also self-decomposable with support on $[0, \infty)$ (see, for example, Alamatsaz, 1983). It follows, however, from Corollary 1 that: "Power mixtures of all discrete stable p.g.f.'s are discrete self-decomposable when the mixing distribution is self-decomposable with support on $[0, \infty)$". On taking $\gamma = 1$, Corollary 1 gives Forst's (1979) result.

In the sequel, we reveal, amongst other things, how discrete versions of some recent results in the literature are connected with Theorem 1. (Our results are analogous to that of Alamatsaz (1993) for the continuous case.) To achieve our goal, Theorem 1 is interpreted in terms of random variables as follows.

Theorem 3. *Let U be a uniformly distributed random variable on* $(0, 1)$ *independent of* $X_1, \ldots, X_K$, *where* X_i*'s are independent non-negative integer-valued random variables and identical to a random variable X with p.g.f. P. Then*

$$X \stackrel{\mathrm{d}}{=} U^{1/p} \oplus (X_1 + \cdots + X_K), \tag{14}$$

if and only if

$$P(t) = (1 + c(1-t)^\gamma)^{-p/\gamma}, \quad t \in [-1, 1],$$

where $\gamma = p(K-1)$, $1 < K \leqslant (1/p + 1)$ *and* $c > 0$.

From the above, when $p = 1$ it is seen that a solution to (14) exists only for $K = 2$. In this case, we obtain a characterization of the geometric distribution analogous to that of Kotz and Steutel (1988) for the exponential distribution.

Corollary 2. *Under the notation of Theorem* 3,

$$X \stackrel{\mathrm{d}}{=} U \oplus (X_1 + X_2)$$

if and only if

$$P(t) = (1 + c(1-t))^{-1}, \quad t \in [-1, 1],$$

for some $c > 0$; *i.e.* $X \sim$ *geometric.*

By taking $K = 2$, Theorem 3 of Alzaid and Al-Osh (1990) is obtained.

Corollary 3. *Under the notation of Theorem* 3,

$$X \stackrel{\mathrm{d}}{=} U^{1/p} \oplus (X_1 + X_2)$$

if and only if

$$0 < p \leqslant 1 \quad \text{and} \quad P(t) = (1 + c(1-t)^p)^{-1}, \quad t \in [-1, 1],$$

for some $c > 0$.

Remark 3. In proving their result, Alzaid and Al-Osh (1990) preassumed that $0 < p \leqslant 1$. They also showed that p.g.f.'s of the form given in Corollary 3 are infinitely divisible. Obviously, in view of Remark 1, these p.g.f.'s are more specifically discrete self-decomposable and hence unimodal. Thus, Corollary 3 is an improved version of Alzaid and Al-Osh result.

By choosing different values of p ($p \leqslant 1$), Theorem 3 results in several other characterizations of e.g. negative binomial distributions. For instance, if we take $p = \frac{1}{2}$, a solution to (14) exists only for $K = 2$ or 3. For example, in the case of $K = 3$, we obtain the following corollary.

Corollary 4. *Under the notation of Theorem* 1,

$$X \stackrel{d}{=} U^2 \oplus (X_1 + X_2 + X_3)$$

if and only if

$$P(t) = (1 + c(1 - t))^{-1/2}, \quad t \in [-1, 1],$$

for some $c > 0$; *i.e.* $X \sim$ *negative binomial with index* $\frac{1}{2}$.

By considering another interpretation of Theorem 1 in terms of random variables, we could obtain the solution of (15) given below. This equation can be thought as the discrete version of (4).

Theorem 4. *Let* M *and* K $(M > K)$ *be positive integers,* $X_1, \ldots, X_{K+M}$ *be non-negative integer-valued random variables and identical with a common p.g.f.* P *such that* $X_1, \ldots, X_K$ *are independent* U, $X_{K+1}, \ldots, X_{K+M}$ *are independent and* U *are uniformly distributed on* $(0, 1)$. *Then*

$$X_1 + \cdots + X_K \stackrel{d}{=} U^{1/p} \oplus (X_{K+1} + \cdots + X_{K+M}) \tag{15}$$

if and only if

$$P(t) = (1 + c(1 - t)^{\gamma})^{-1/(M-K)}, \quad t \in [-1, 1],$$

where $\gamma = (M - K)p/K$, $0 < \gamma \leqslant 1$ *and* $c > 0$.

As an application of Theorem 4, a characterization of the negative binomial distribution analogous to that given by Huang and Chen (1989) and Alamatsaz (1993) for the gamma distribution is obtained.

Corollary 5. *Under the notation of Theorem* 4, *relation* (15) *holds with* $p = K/(M - K)$ *if and only if*

$$P(t) = (1 + c(1 - t))^{-1/(M-K)}, \quad t \in [-1, 1],$$

for some $c > 0$; *i.e.* $X \sim$ *negative binomial with index* $1/(M - K)$.

In particular, when $M = K + 1$, *then*

$$X_1 + \cdots + X_K \stackrel{d}{=} U^{1/K} \oplus (X_{K+1} + \cdots + X_{2K+1}) \tag{16}$$

if and only if

$$P(t) = (1 + c(1 - t))^{-1}, \quad t \in [-1, 1],$$

for some $c > 0$; *i.e.* $X \sim$ *geometric.*

In concluding, we mention that the forthcoming monograph of C.R. Rao and D.N. Shanbhag ("*Choquet-Deny type functional equations with applications to stochastic models*", 1994) will address multivariate extensions of some of the preceding results.

Errata

Note added in proof: Independently to our work, van Harn and Steutel (1993) have obtained characterizations related to our Theorem 1 by considering a generalized multiplication operation in which subcritical branching processes, both with discrete and continuous state space, play an important role. Also, Pakes (1994) has studied the problem in the general case where U is any random variable with support on $(0, 1)$.

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