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Characterizations, length-biasing, and infinite divisibility

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Suppose $L(X)$ is the law of a positive random variable X, and Z is positive and independent of X. Admissible solution pairs $(L(X), L(Z))$ are sought for the in-law equation $\hat{X} \cong X \circ Z$, where $L(\hat{X})$ is a weighted law constructed from $L(X)$, and $\circ$ is a binary operation which in some sense is increasing. The class of weights includes length biasing of arbitrary order. When $\circ$ is addition and the weighting is ordinary length biasing, the class of admissible $L(X)$ comprises the positive infinitely divisible laws. Examples are given subsuming all known specific cases. Some extensions for general order of length-biasing are discussed.

KEY WORDS: Characterization and structure theory; Weighted and length biased distributions; Infinite divisibility; Natural exponential families.

1. INTRODUCTION

Let X be a non-negative random variable (rv) whose law $L(X)$ is not degenerate at zero and has distribution function (DF) F. Next, let $w : \mathrm{R}_+ \mapsto \mathrm{R}_+$ be a continuous and strictly increasing weight function and $m = E(w(X)) < \infty$. The weighted law $L(\hat{X})$ formed from $L(X)$ and w has the DF $\hat{F}(x) = m^{-1} \int_0^x w(u)F(du)$. We refer to Patil & Rao (1977), and to Rao (1985), for a survey of statistical applications and to Mahfoud & Patil (1982) for some theory. The length-biased law of order r, denoted by r-LBL, corresponds to $w(x) = x^r$, and the familiar length-biased law (LBL) has $r = 1$. Our point of departure is the fact that weighting always is a stochastic order increasing operation.

Lemma 1.1. For $x \geq 0$, $P(\hat{X} > x) \geq P(X > x)$, i.e., $X \leq_{st} \hat{X}$.

Proof. When $x \geq w(1/m)$,

$$P(\hat{X} > x) = m^{-1} \int_{(x,\infty)} w(u)F(du) \geq m^{-1} w(w^{-1}(m)) \int_{(x,\infty)} F(du).$$

For $x \leq w(1/m)$ argue similarly but starting with $\hat{F}(x)$. ♣

Remark 1.1. (*i*) Essentially this result for the LBL, in terms of hazard functions for absolutely continuous F, was given by Mahfoud & Patil (1982); see p. 483. It was asserted for discrete laws by Patil, Rao & Ratnaparkhi (1985) who mention "a monotone likelihood ratio type argument between the pdf's" of X and $\hat{X}$. The above proof is simple and general.
(*ii*) The proof requires only that w is non-decreasing and has a suitable generalized inverse. If w is decreasing then weighting is a stochastically decreasing operation.

This raises the following general problem. Let $Z \geq 0$ be a rv, independent of X, and $\circ$ a binary operation on $\mathbb{R}_+ \times \mathbb{R}_+$ which is non-decreasing in each argument and which satisfies $x \circ z \geq x$, at least if z is large enough. Consider the (inflationary) stochastic equation

$$\hat{X} \cong X \circ Z, \tag{1.1}$$

where $\cong$ denotes equality in law. This equation represents X as being stochastically increased to give $L(\hat{X})$. We would like to determine the range of admissible solution pairs $(L(X), L(Z))$, and to find conditions under which $L(Z)$ uniquely determines a law $L(X)$, or a fairly restricted parametric family of laws. If $\circ$ can be chosen so $L(Z)$ is uniquely determined by $L(X)$ then we can formulate characterization theorems.

Special cases of the inflationary equation (1.1) with $\circ = +$ exist in the literature. Kirmani & Ahsanullah (1987) and Khattree (1989) have independently considered the LBL case showing that $L(X)$ comprises the inverse gaussian family $\mathrm{IG}(m, \lambda m^2)$ if and only if (iff) $Z \cong \lambda^{-1}\chi_1^2$. See Seshadri (1993), Theorem 3.12, for a more complete discussion. Khattree (1989) shows also that the gamma family $\mathrm{Gam}(m\lambda/2, 2/\lambda)$ arises iff $Z \cong \lambda^{-1}\chi_2^2$. Ahmed & Abouammoh (1993) extend both results by showing that a mixture of two χ_k^2 laws ($k = 1, 2$) gives a representation of X as a sum of two inverse gaussian, or gamma, rv's.

Khattree (1989) and Ahmed & Abouammoh (1993) give their results in strictly analytical terms, not involving random variable representations of the sort (1.1). They also give what appear to be inverse versions of their results. With our notation, this amounts to solving $W \cong \hat{W} + Z$ for $L(W)$, where $L(\hat{W})$ is the LBL of $L(W)$. The solution of course is $L(W) = L(\hat{X})$. See Ahsanullah & Kirmani (1984) for an earlier formulation, and Seshadri (1993) for references and discussion. Ahmed & Abouammoh (1993) also have an error in the statement about the negative binomial family (see their eq. (3.1)). Specifically, they posit an integer valued random variable Y for which Y^{-1} has a negative binomial law – this is not valid.

Using the same underlying approach of these references, we show in the next section that for ordinary length biasing the class of admissible $L(X)$ for the additive version of (1.1) (see (2.1) below) is precisely the positive infinitely divisible laws. In its essence this is not a new result. The expression of (2.1) below in DF terms as a general criterion for infinite divisibility is due to Steutel (1971), and a lattice-law version is due to Katti (1967). However, it bears repeating in the LBL form that we give to it. We also formulate a general result relating mixture structure of $L(Z)$ to additive structure of $L(X)$. Links with self-decomposability are also mentioned. These topics comprise Section 2.

In Section 3 we present a couple of examples which subsume all those mentioned above. Specifically, we couple the Hougaard laws for X with gamma laws for Z. A discrete version is also given.

In Section 4 we discuss some extensions of Theorem 2.1 below to the case of the r-LBL operation. Gupta (1975) gave a couple of results for this case when Z in (2.1) below is a constant. We generalize one of them (Theorem 4.3 below) and give an alternative formulation of another (Theorem 4.4 below). Finally, we mention that Pakes (1994) discusses the cases where $\circ$ is maximization or multiplication, for which cases a more complete theory can be given.

2. THE ADDITIVE CASE

We consider the stochastic equation

$$\hat{X} \cong X + Z, \qquad (2.1)$$

where $L(\hat{X})$ is the LBL of $L(X)$. Let $\phi(\theta) = E(e^{-\theta X})$ be the Laplace-Stieltjes transform (LST) of $L(X)$. In this section we write this as $\phi(\theta) = \exp(-\psi(\theta))$, where ψ is the cumulant generating function (cgf) of $L(X)$. Recall that $L(X)$ is infinitely divisible (infdiv) if its cgf has the representation

$$\psi(\theta) = \int_{0-}^{\infty} (1 - e^{-\theta x})\nu(dx), \tag{2.2}$$

where ν is a (positive) measure satisfying $\int_0^1 x\nu(dx) < \infty$ and $\nu([1,\infty)) < \infty$. Often ν is called a Lévy measure and ψ is the Lévy exponent (of ϕ). See Feller (1971) and note that

$$m = \int_0^{\infty} x\nu(dx) \tag{2.3}$$

which here is finite. The following result embraces the known special solutions of (2.1).

Theorem 2.1. Equation (2.1) holds iff $L(X)$ is infdiv, with cgf (2.2), and then the LST of Z is

$$\gamma(\theta) = m^{-1} \int_0^{\infty} x e^{-\theta x} \nu(dx). \tag{2.4}$$

Conversely, if the LST of Z, γ, and the constant $m > 0$ are given, then (2.1) has a unique solution which is infdiv with cgf

$$\psi(\theta) = m \int_0^{\theta} \gamma(u) du. \tag{2.5}$$

Remark 2.1. The Lévy measure corresponding to the cgf (2.5) is

$$\nu(dx) = mG(dx)/x. \tag{2.6}$$

Proof. Note first that the LST of $\hat{X}$ is $-m^{-1}\phi'(\theta)$ and hence that (2.1) is equivalent to the LST relation

$$-m^{-1}\phi'(\theta) = \phi(\theta)\gamma(\theta). \tag{2.7}$$

Now if $L(X)$ is infdiv then we have

$$\gamma(\theta) = m^{-1}(d/d\theta)\log\phi(\theta) = m^{-1}\psi'(\theta),$$

i.e., (2.4) holds. Clearly γ is the LST of a law.

Conversely, suppose γ and m are given, and ψ is determined by (2.4). Then (2.2) follows by Fubini's theorem with ν given by (2.6), and which obviously is a Lévy measure. Hence $\phi = \exp(-\psi)$ is the LST of an infdiv law, $L(X)$ say, and it satisfies (2.7). ♣

Suppose for each $t \in \mathrm{R}$ that $G(x,t)$ is a DF and that H is a DF on R. Let $m(t)$ be a bounded, measurable, and non- negative function on R such that $m = \int m(t)H(dt) < \infty$. Then it is clear that

$$\nu(dx) = x^{-1} \int G(dx,t)m(t)H(dt)$$

is a Lévy measure. If ψ is its cgf then (2.1) is solved with $L(X)$ having the cgf (2.5), where $\gamma(\theta) = \int \gamma(\theta,t)m(t)H(dt)$ and $\gamma(\cdot,t)$ is the LST of $G(\cdot,t)$. This gives the most general mixture formulation of (2.1), but in this generality it has only a formal analytical significance.

Suppose now that H is a discrete DF placing mass p_j at t_j, $(j = 0,1,\cdots)$. Let $G_j(x) = G(x,t_j)$ and

$$\psi_j(\theta) = mp_j \int_0^\infty (1 - e^{-\theta x})x^{-1}G_j(dx),$$

where we have set $m_j \equiv m$. This represents no loss of generality since otherwise we just replace p_j with $p_j m_j/m$. A mixture version of the converse part of Theorem 2.1 is immediate:

Theorem 2.2. If the DF of Z has the mixture form

$$G(x) = \sum p_j G_j(x)$$

then any solution $L(X)$ of (2.1) has the representation

$$X \cong \sum X_j,$$

where the summands are independent and the cgf of X_j is $\psi_j(\theta)$. (In view of this result, the mixture theorems of Ahmed & Abouammoh (1993) are formal generalizations of single component cases.)

Theorem 2.1 places no explicit restriction on the support of the laws of X and Z. But one should note that the left extremity of $L(X)$

is, using (2.5),

$$\begin{aligned} \ell_F &= \inf supp(L(X)) = \lim_{\theta\to\infty} \theta^{-1}\psi(\theta) \\ &= \lim_{\theta\to\infty} m\gamma(\theta) = mG(0+). \end{aligned}$$

The second equality is (or should be) well-known. Since $L(X)$ is infdiv, its right extremity is infinite.

When X takes values in the set of non-negative integers $\mathbb{N}_0$ then Z is necessarily positive. It is then more convenient to replace (2.1) with

$$\hat{X} \cong X + \breve{Z} + 1 \tag{2.8}$$

where $\breve{Z} = Z - 1 \geq 0$. Let $P(s)$ be the probability generating function (pgf) of X and $\zeta(s)$ be the pgf of $\breve{Z}$. Assuming $m < \infty$, (2.8) holds iff P has the compound Poisson form

$$P(s) = \exp\left(-\rho(1 - q(s)\right), \tag{2.9}$$

where

$$\rho = mQ, \ \ Q = \int_0^1 \zeta(u)du = E((1+\breve{Z})^{-1}) \text{ and } q(s) = Q^{-1}\int_0^s \zeta(u)du. \tag{2.10}$$

The following results give characterizations of self-decomposable (SD) and of discretely SD laws based on (2.1) and (2.8), respectively.

Theorem 2.3. Let X and Z be independent and non-negative random variables, and let $E(X) < \infty$. Then any two of the following assertions implies the third:
(a) $L(X)$ is SD;
(b) (2.1) holds;
(c) $L(Z)$ is absolutely continuous in $(0, \infty)$ with a non-increasing density.

Proof. One simply observes that Biggins & Shanbhag (1981) show that $L(X)$ is SD iff its LST ϕ satisfies the relation (2.7) with

$$m\gamma(\theta) = \delta + \int_0^\infty e^{-\theta x} w(x)dx,$$

where $\delta \geq 0$ and w is non-increasing. ♣

By appealing to a criterion for discrete SD of Steutel and van Harn (1979) (see their (2.7)) we obtain the following discrete analogue of Theorem 2.3.

Theorem 2.4. Let X and $\breve{Z}$ be independent and $\mathbb{N}_0$ – valued random variables, and let $E(X) < \infty$. Then any two of the following assertions implies the third:
(a) $L(X)$ is discretely SD;
(b) (2.8) holds;
(c) $P(\breve{Z} = j) \propto P(U > j)$ for some $\mathbb{N}_0$ – valued random variable U.

3. EXAMPLES

Our first example subsumes the cases examined by Kirmani & Ahsanullah (1987), Khattree (1989) and Ahmed & Abouammoh (1993). Suppose $L(Z) = \text{Gam}(a, \lambda)$, i.e.,

$$\gamma(\theta) = \left(\frac{\lambda}{\lambda + \theta}\right)^a,$$

where $a, \lambda > 0$. Then (2.5) yields

$$\psi(\theta) = \begin{cases} m\lambda \log(1 + \theta/\lambda), & \text{if } a = 1; \\ \frac{m\lambda^a}{1-a}\left[(\lambda + \theta)^{1-a} - \lambda^{1-a}\right], & \text{if } a \neq 1. \end{cases}$$

When $a = 1$ we find that $L(X) = \text{Gam}(m\lambda, \lambda)$ which, in essence, is Khattree's (1989) second characterization. When $a \neq 1$ it is convenient to set $\alpha = 1 - a$, so

$$\psi(\theta) = \frac{\delta}{\alpha}\left[(\lambda + \theta)^\alpha - \lambda^\alpha\right],$$

where

$$\delta = m\lambda^{1-\alpha} \quad \text{and} \quad -\infty < \alpha < 1,\ \alpha \neq 0.$$

We write $\text{H}(\alpha, \delta, \lambda)$ to denote this family of laws. When $0 < \alpha < 1$ this family was determined by Hougaard (1986) as the natural exponential family (NEF) generated from a positive stable law with index α. This determination entails the above restriction on α, but our genesis shows the parameter space can be enlarged. The puncture at $\alpha = 0$ is filled by the above $\text{Gam}(m\lambda, \lambda)$ laws. The case $a = \alpha = 1/2$, discussed by Kirmani & Ahsanullah (1987) and Khattree (1989), gives the inverse gaussian law $\text{IG}(m, 2\delta^2)$, using Seshadri's (1993) notation. When $\alpha < 0$ we have the following (and obvious) compound-Poisson representation

$$X \cong \sum_{j=1}^{\mathcal{N}} \Gamma_j,$$

where $\mathcal{N}$ and the Γ_j's are mutually independent, $L(\mathcal{N}) = \text{Poi}(-\delta\lambda^\alpha/\alpha)$, and $L(\Gamma_j) = \text{Gam}(\lambda, -\alpha)$. Hence $L(X)$ assigns positive mass to the origin iff $\alpha < 0$.

We summarise the main points of this example in the following theorem which generalizes earlier results, cited above, even when $a = 1/2$ and $a = 1$.

Theorem 3.1. Let $a = 1 - \alpha$ and λ be positive constants. Any pair of the following implies the third:
(a) (2.1) holds;
(b) $L(Z) = \text{Gam}(a, \lambda)$;
(c) For some $m > 0$, $L(X) = \text{H}(\alpha, \delta, \lambda)$ where $\delta = m\lambda^a$.

Remark 3.1. (*i*) An alternative expression of this result is that the $\text{H}(\alpha, \delta, \lambda)$ family members are the only laws whose LBL's are a convolution of themselves and a gamma law. This is well known for the inverse gaussian laws; see Seshadri (1993), p. 52.
(*ii*) When $0 < a \leq 1$, Theorem 2.3 shows that the Hougaard laws are SD.

Now consider (2.8). Suppose $0 < p < 1$ and let

$$\zeta(s) = \left(\frac{q}{1 - ps}\right)^a, \tag{3.1}$$

where $q = 1-p$. This is the pgf of the negative binomial law $\text{NBin}(p, a)$.

With $\alpha = 1 - a$, integration in (2.10) leads to the pgf: If $a \neq 1$,

$$P(s) = \exp[-(d/\alpha)((1-ps)^\alpha - q^\alpha)], \quad d = mq^{1-\alpha}/p, \tag{3.2}$$

and if $a = 1$ then

$$P(s) = \left(\frac{1-p}{1-ps}\right)^d .$$

Denote this family by $\mathrm{DH}(\alpha, d, p)$.

When $0 < \alpha < 1$ the laws represented by (3.2) comprise the NEF generated from the discrete stable law of index α. For this see Pakes (1995) and the references therein. Its pgf is

$$\sum \sigma_j s^j = \exp[-(d/\alpha)(1-s)^\alpha],$$

and from this we see that the pgf (3.2) has masses $\sigma_j p^j \exp((d/\alpha)q^\alpha)$. This is an exact analogue of Hougaard's (1986) continuous laws. But note here that when $\alpha = 1$ we get a Poisson family; the NEF operation only changes the rate parameter. When $\alpha < 0$ we have a compound Poisson representation

$$X \cong \sum_{j=1}^{\mathcal{N}} \beta_j,$$

where $L(\mathcal{N}) = \mathrm{Poi}(dq^\alpha/p)$ and $L(\beta_j) = \mathrm{NBin}(p, \alpha)$. The following result is a consequence of the above.

Theorem 3.2. Let $a = \alpha - 1 > 0$ and $0 < p < 1$. Any pair of the following implies the third:
(a) (2.8) holds;
(b) $L(\breve{Z}) = \mathrm{NBin}(p, a)$;
(c) For any $m > 0$, $L(X) = \mathrm{DH}(\alpha, d, p)$, where $d = mq^a/p$.

Remark 3.2. (*i*) The case $a = 1$ is a characterization of the $\mathrm{NBin}(p, d)$ family ($d > 0$), giving a precise version of Theorem 3.1 of Ahmed & Abouammoh (1993).
(*ii*) It follows from Theorem 2.4 that $\mathrm{DH}(\alpha, d, p)$ is discretely SD when $0 \leq \alpha \leq 1$. This can still be true when $\alpha < 0$ and p is sufficiently small.

When $L(\breve{Z})$ is degenerate at zero we have $a = 0$ in (3.1), giving the Poisson law $\mathrm{Poi}(m)$ for $L(X)$. This was noted by Gupta (1975), p. 763.

The following assertion, a corollary of Theorem 2.1, extends this a little.

Theorem 3.3. Let $0 < c < \infty$ be fixed. Each of the following implies the other:
(a) $\hat{X} \cong X + c$;
(b) For any $m > 0$, $L(X/c) = \text{Poi}(m)$.

We end this section by mentioning two more examples for (2.8). If $L(\breve{Z}) = \text{Poi}(\lambda)$ then the allowable laws $L(X)$ comprise the Poisson-Poisson laws having $q(s) = \exp(-\lambda(1-s))$ in (2.9). If $L(Z) = \text{Bin}(n,p)$ then $q(s)$ is the pgf of $\text{Bin}(n+1,p)$, i.e., $L(X)$ is a family of Poisson-binomial compounds. The particular case $n = 1$ gives the Hermite laws, $L(X) = \text{Herm}(\sqrt{\rho p}, (1-p)\sqrt{\rho/p})$, where we have used the parametrization of Johnson, Kotz & Kemp (1993), p. 357.

4. SOME EXTENSIONS

We note first a minor extension of Theorem 2.1. Fix a constant $c > 0$ and let $L(X)$ be as above. Seshadri (1993) considers the affine weight $w(x) = c + x$. It is easily shown that the the main assertions of Theorem 2.1 continue to hold with (2.5) replaced by

$$\psi(\theta) = -c\theta + (c+m)\int_0^\theta \gamma(u)du,$$

where we still have $m = E(X)$. Note that now the left extremity is

$$\ell_F = -c + (c+m)G(0),$$

which may be negative.

The case of rth order length biasing in (2.1), that is $L(\hat{X})$ is now the r-LBL of $L(X)$, appears to be much harder. See Gupta (1975) in relation to (2.8) when r is an integer. In general (2.7) can be replaced by a differential equation of possibly fractional order, a formal move that seems of little help. But if $L(Z) = \text{Exp}(\lambda)$ then $L(X)$ has a density function f, and the convolution equation for f obtained from (2.1) can easily be solved to give

$$f(x) = Cx^{-r}\exp\left[-\lambda x\left(1 + \frac{m}{r-1}x^{-r}\right)\right],$$

where C is a normalizing constant. (When $r = 2$ this gives Halphen's harmonic laws and $C^{-1} = 2K_0(2\lambda m^{1/2})$; see Seshadri (1993), Section 1.9.) The full family can be regarded as the r-LBL obtained from the NEF generated by a Type II extreme-value law. Hence this NEF is characterized by a reciprocal relation of the type mentioned in Section 1.

If (2.1) is satisfied by (X, Z) and $L(Z)$ is infdiv, then $L(\hat{X})$ is too. So if $m_2 = E(X^2) < \infty$, then $\hat{X}$ has a 2-LBL, $L(\hat{X}^{(2)})$, and Theorem 2.1 gives the representation

$$\hat{X}^{(2)} \cong \hat{X} + Z_2 \cong X + Z + Z_2,$$

where Z_2 is independent of Z and its law can be deduced from Theorem 2.1. Continuing in this way we obtain the following somewhat weak extension of this theorem. When $m_n = E(X^n) < \infty$, let $\hat{X}^{(n)}$ denote the n-LBL of $L(X)$.

Theorem 4.1. For integer $r \geq 2$ let $m_r < \infty$ and suppose for $n = 0, \cdots, r-1$ that $\phi_n(\theta) = (-1)^n \phi^{(n)}(\theta)/m_n$, the LST of $\hat{X}^{(n)}$, is infdiv. Then $L(\hat{X}^{(r)})$ can be represented by

$$\hat{X}^{(r)} \cong X + S_r,$$

where $S_r = \sum_{n=1}^{r} Z_n$, the Z_n are independent with LST

$$\gamma_n(\theta) = -\frac{m_{n-1}}{m_n} \cdot \frac{\phi^{(n)}(\theta)}{\phi^{(n-1)}(\theta)}.$$

The cgf of $L(\hat{X}^{(n)})$ is

$$\psi_n(\theta) = \frac{m_{n+1}}{m_n} \int_0^\theta \gamma_{n+1}(u)du.$$

Example 4.1. Starting with $L(X) = \mathrm{Gam}(a, \lambda)$ we obtain the very well-known decomposition, where $L(Z_n) = \mathrm{Exp}(\lambda)$. Taking $r = 2$ and $L(X) = \mathrm{H}(\alpha, \delta, \lambda)$ some algebra gives the above decomposition with $L(Z_1) = \mathrm{Gam}(a, \lambda)$ and $L(Z_2)$ is the mixture law $p\mathrm{Exp}(\lambda) + (1 -$

$p)\mathrm{Gam}(a, \lambda)$, where $p = \frac{a}{a+\delta/\lambda}$. A little suprisingly, this representation does not extend in all cases when $r = 3$; the 2-LBL's are infdiv only when $\alpha \geq 0$. We see this from the following general considerations.

Let μ be a (positive) measure supported in $[0, \infty)$ and suppose

$$\chi = \inf\left\{\eta : M(\eta) \equiv \int e^{-\eta x}\mu(dx)\right\} < \infty.$$

The NEF generated by μ, and denoted by $NEF(\mu)$, is the family of DF's $(F_\eta : \eta > \chi)$, where $F_\eta(dx) = e^{-\eta x}\mu(dx)/M(\eta)$. For any $r > 0$, the $r-$LBL exists for each F_η and together they comprise the NEF with generator $x^r\mu(dx)$. Infinite divisibility for general NEF's receive detailed discussion in Seshadri's (1993) monograph, but the following result gives a much easier treatment, and in more familiar terms, for positive laws. In our more restricted context, it also gives a somewhat stronger assertion than the structural results discussed by Seshadri (1993).

Theorem 4.2. If the members of $NEF(\mu)$ are infdiv then

$$\tau(\eta) = -M'(\eta)/M(\eta)$$

is completely monotone (CM): For $\eta > \chi$,

$$\tau(\eta) = \int_0^\infty e^{-\eta x}\zeta(dx). \tag{4.1}$$

The Lévy measure of F_η is

$$\nu_\eta(dx) = e^{-\eta x}x^{-1}\zeta(dx).$$

Conversely, if $M(\cdot)$ is a positive function defined and differentiable in (χ, ∞) such that (4.1) holds, then (4.3) below defines a family of infdiv laws which comprise a NEF whose generator μ is uniquely defined up to a multiplicative constant by

$$M(\eta) = \exp\left[-\int_{\eta'}^{\eta}\tau(u)du\right], \tag{4.2}$$

where $\eta' \in (\chi, \infty)$ is an arbitrary constant.

Proof. Observe first that the LST of F_η is $\phi_\eta(\theta) = M(\eta+\theta)/M(\eta)$. If F_η is infdiv then, with $\alpha(\eta) = \log M(\eta)$,

$$\alpha(\eta+\theta) - \alpha(\eta) = -\int_0^\infty (1-e^{-\theta x})\nu_\eta(dx),$$

where ν_η is a Lévy measure. Differentiation with respect to θ yields

$$-\alpha'(\eta+\theta) = \int_0^\infty e^{-\theta x} x \nu_\eta(dx).$$

Further differentiation, each time setting $\theta = 0$, shows that $-\alpha'(\eta) = \tau(\eta)$ is CM in (χ, ∞) and hence (4.1) holds for some measure ζ. In particular the right-hand side of the above equation is now seen to be $\int_0^\infty e^{-x(\theta+\eta)}\zeta(dx)$, and hence integration yields

$$\phi_\eta(\theta) = \exp\left[-\int_0^\infty (1-e^{-\theta x})\nu_\eta(dx)\right]. \tag{4.3}$$

For the converse, we note that ν_η as defined above is indeed a Lévy measure and hence (4.3) defines a family (F_η) of infdiv laws. Clearly $\phi_\eta(\theta) = M(\eta+\theta)/M(\eta)$, where M is given by (4.2). But (4.2) expresses M as the composition of a CM function with one whose derivative is CM. Hence (Feller (1971)) M is CM with generating measure μ, say, and it follows that $(F_\eta) = NEF(\mu)$. ♣

If r is a positive integer and $F_\eta \in NEF(\mu)$ then the DF of the r − LBL, F_η, has the LST $M^{(r)}(\eta+\theta)/M^{(r)}(\eta)$. Consequently the conditions of Theorem 4.1 are satisfied if $\tau_n(\eta) = -M^{(n+1)}(\eta)/M^{(n)}(\eta)$ is CM for $n = 1, \ldots, r-1$. The family Gam(a,η), $\eta > 0$, is a NEF and $\tau_n(\eta) = (a+n-1)/\eta$, which is CM for each n. We anticipate this from Example 4.1.

When $\alpha < 1$, then for $\mathrm{H}(\alpha, \delta, \eta)$ we have $M(\eta) = \exp(-(\delta/\alpha)\eta^\alpha)$, and hence

$$\tau_1(\eta) = \delta\eta^{\alpha-1} \quad \text{and} \quad \tau_2(\eta) = \delta\eta^{-a} + a\eta^{-1}.$$

These are CM, as we expect from Example (4.1). Further algebra yields

$$\tau_3(\eta) = 2a\eta^{-1} + \delta\eta^{-a} + a(a-1)\eta^{-1}(a+\delta\eta^{1-a})^{-1}.$$

The first two terms are CM, but the third is CM iff $a \le 1$. When $a > 1$ the last term is negative near the origin and it dominates the whole.

Hence Theorem 4.1 is applicable to the 3-LBL only for Hougaard's parameter range, $0 \leq \alpha < 1$, and *not* to the above compound Poisson laws.

To extend Theorem 3.3 consider

$$\hat{X}^{(r)} \cong X + c \tag{4.4}$$

for fixed positive r and c. The DF version is the integral equation

$$\int_0^x u^r dF(u) = mF(x-c), \qquad x \geq 0.$$

Clearly, $P(0 < X < c) = 0$ whence $L(X)$ is a lattice law with span c. If $P(X = 0) = 0$ then $F(X) \equiv 0$. When $P(X = 0) > 0$ the integral equation determines all the weights $P(X = cj)$ as follows:

Theorem 4.3. Let r and c be positive constants. The following assertions are equivalent:
(a) (4.4) holds;
(b) For any $m > 0$,

$$P(X = cj) = p_0 \frac{(mc^{-r})^j}{(j!)^r}, \quad j = 0, 1, \cdots,$$

where p_0 is chosen so $P(X < \infty) = 1$. (The case $r = 1$ is just Theorem 3.3, and when $r = 2$ we have $p_0^{-1} = I_0(2\sqrt{m}/c)$, where I_0 is a zero-order modified Bessel function.)

Let $r \in N$ and $L(X_f^{(r)})$ denote the weighted law induced by the rth order factorial moment:

$$P(X_f^{(r)} = j) = m^{-1} j^{(r)} P(X = j),$$

where $m = E(X_f^{(r)}) < \infty$ and $j^{(r)} = j!/(j-r)!$. Gupta (1975) solved the stochastic relation

$$X_f^{(r)} \cong X + r \tag{4.5}$$

by solving a linear differential equation of order r satisfied by the pgf of $L(X)$. His solution depends on r (not quite) arbitrary constants (and it contains an error arising by conflating notation for a summation index and the imaginary number).

Let $p_j = P(X = j)$. The mass function version of (4.5) is

$$j^{(r)} p_j = m p_{j-r}.$$

For each $\ell = 0, 1, \cdots, r-1$ this equation determines $p_{\ell+kr}$ $(k \in \mathbb{N})$ in terms of p_ℓ only:

$$p_{\ell+kr} = p_\ell \prod_{\nu=1}^{k} \frac{m}{(\ell+\nu r)^{(r)}}.$$

Choose p_ℓ so that these weights determine a law on $\ell + r\,\mathbb{N}_0$, which we denote by $\Lambda(m, r, \ell)$. Let $\mathbf{a} = (a_1, \cdots, a_r)$ denote an element of the r – dimensional simplex $\mathcal{S}_r$. The following result is a constructive rendition of Theorem 1 in Gupta (1975).

Theorem 4.4. Let $r \in \mathbb{N}$ be fixed and $L(X)$ be a discrete law with $E(X^r) < \infty$. The following assertions are equivalent:
(a) $L(X)$ satisfies (4.5);
(b) For each $m > 0$ and $\mathbf{a} \in \mathcal{S}_r$,

$$L(X) = \sum_{\ell=0}^{r-1} a_{\ell+1} \Lambda(m, r, \ell).$$

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