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CHARACTERIZATIONS OF SOME WELL-KNOWN DISCRETE DISTRIBUTIONS BASED ON VARIANTS OF THE RAO-RUBIN CONDITION

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SUMMARY. The present note investigates characterizations of some well-known discrete distributions based on variants of the Rao-Rubin condition. Shanbhag and Clark (1972), and Patil and Ratnaparkhi (1977), have given characterizations for the Poisson, and for the binomial and negative binomial distributions, respectively, replacing the Rao-Rubin condition by conditional expectations of the distributions of the r.v.'s X and Y. Our work focuses on unification and generalization as well as some extensions of these results. A result relating to the Srivastava and Singh (1975) conjecture and modified versions of the results of Shanbhag (1977) and Alzaid (1986) are also given.

1. Introduction

A damage model may be described by a random vector (X, Y) of non-negative, integer-valued components with $Y \leqslant X$ so that the joint probability law for X and Y is of the form

$$P(X=n,\ Y=r) = S(r \mid n)g_n, \quad r = 0, 1, \ldots, n\ ; n = 0, 1, \ldots,$$

where $\{S(r \mid n)\}$ is a probability function for each fixed n and $\{g_n\}$ is the (marginal) probability law of X. (The conditional probability function of Y given X here is termed the survival distribution.)

It is customary to call Y the undamaged part of X and $X-Y$ the damaged part. The concept of damage models in discrete probability theory was introduced by Rao (1963). Rao and Rubin (1964) initiated research in this area and formulated a characterization theorem for the Poisson distribution based on conditional and marginal distributions of the r.v. Y. Later authors have considered conditions that involve only conditional and marginal expectations of Y instead of the corresponding distributions.

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*We are sorry to report that this author expired in a tragic accident in 1988.

In order to state these results we need some notation.

(1) $X \sim PSD(\theta)$ refers to a power series distribution with parameter θ of the form

$$P(X = n) = a_n \theta^n / A(\theta),\ n = 0, 1, 2 \ldots$$

(2) For a, c real numbers and r an integer,

$$a^{(r)} = a(a-1)(a-2) \ldots (a-r+1)\ ;\ a^{(0)} = 1,$$

$$a_{(r)} = a(a+1)(a+2) \ldots (a+r-1)\ ;\ a_{(0)} = 1,$$

$$a_{(r,c)} = a(a+c)(a+2c) \ldots (a+(r-1)c)\ ;\ a_{(0,c)} = 1.$$

Theorem 1.1 (Shanbhag and Clark (1972)). *Let $X \sim PSD(\theta)$ and the survival distribution $\{S(r \mid n)\}$ have $S(n \mid n)$ to be independent of θ and its mean and variance as $n\alpha$ and $nx\ (1-\alpha)$ with α independent of θ. Then, provided $P(X = Y) > 0$,*

$$E(Y) = E(Y \mid X = Y) \text{ and } \operatorname{var}(Y) = \operatorname{var}(Y \mid X = Y)$$

iff $X \sim$ Poisson and $S(n \mid n) = \alpha^n$.

Patil and Ratnaparkhi (1977) have provided a slightly different formulation of the above theorem characterizing the Poisson distribution and then have stated and proved a series of results that lead additionally to the characterizations of the binomial and negative binomial distributions. They have also given some interesting interpretations of the underlying assumptions that can be of some use in statistical ecology. Specically, these authors have proved :

Theorem 1.2 (Patil and Ratnaparkhi (1977)). *Let $X \sim PSD(\theta)$ and the survival distribution $\{S(r \mid n)\}$ have $S(n \mid n)$ to be independent of θ. Then, provided $P(X = Y) > 0$, the following assertions hold :*

(a) *If the first two factorial moments of the survival distribution are given by*

$$E(Y^{(i)} \mid n) = n^{(i)} \frac{m^{(i)}}{N^{(i)}},\ i = 1, 2,$$

where $0 < m \leqslant N$, then

$$E(Y^{(i)}) = E(Y^{(i)} \mid X = Y),\ i = 1, 2 \qquad \ldots \quad (1.1)$$

iff $X \sim$ Binomial $(N, .)$ and $S(n \mid n) = \dfrac{m^{(n)}}{N^{(n)}}$.

(b) *If the first two factorial moments of the survival distribution are given by*

$$E(Y^{(i)} \mid n) = n^{(i)} \frac{m_{(i)}}{N_{(i)}},\quad i = 1, 2,$$

where $0 < m \leqslant N$, *then* (1.1) *is valid* iff $X \sim$ *Negative Binomial* $(N, .)$ *and* $S(n \mid n) = \frac{m_{(n)}}{N_{(n)}}$.

In Section 2, we unify these theorems by characterizing the Polya-Eggenberger distribution based on the first two factorial moments of Y given $X = n$. A generalization of that result is also obtained. Furthermore, we obtain extensions of these results based on any three consecutive factorial moments of the conditional distribution of Y given $X = n$.

In Section 3, we give a result relating to the Srivastava-Singh (1975) conjecture. Applying this result, modified versions of Shanbhag (1977) and Alzaid (1986) results are obtained.

2. A UNIFICATION AND EXTENSIONS OF SHANBHAG AND CLARK (1972), AND PATIL AND RATNAPARKHI (1977) RESULTS

Definition 2.1. A non-negative, integer-valued r.v. X is said to have a Markov-Polya Distribution i.e., $X \sim MPD(n, a, b, c)$ if

$$P(X = r) = \binom{n}{r} \frac{a_{(r,c)}\, b_{(n-r,c)}}{(a+b)_{(n,c)}},\ r = 0, 1, \ldots, n\ ;\ n = 0, 1, \ldots,$$

where $a, b > 0$ and c is a real number such that the distribution is well defined. (Note that this has binomial, hypergeometric and negative hypergeometric as special cases.)

Definition 2.2. A non-negative, integer-valued r.v. X is said to have a Polya-Eggenberger Distribution i.e., $X \sim PED(h, c, \lambda)$ if

$$P(X = r) = K(h, c, \lambda)\, h_{(r,c)} \frac{\lambda^r}{r\,!},\ r = 0, 1 \ldots,$$

for some $\lambda > 0$, $h > 0$, and a real uumber c such that the distribution is well defined with $K(h, c, \lambda)$ as the normalizing constant. (Note that this has Poisson, binomial and negative binomial as special cases).

Theorem 2.1. *Let* $X \sim PSD(\theta)$ *and the survival distribution* $\{S(r \mid n)\}$ *have* $S(n \mid n)$ *to be independent of* θ *and its two first factorial moments to be equal to the corresponding moments of an* $MPD(n, a, b, c)$, *i.e. given respectively by* $n \frac{a}{(a+b)}$ *and* $n(n-1) \frac{a(a+c)}{(a+b)(a+b+c)}$. *Then* (1.1) *is valid iff* $X \sim PED(a+b, c, .)$ *and* $n(n-1) \frac{a(a+c)}{(a+b)(a+b+c)}$.

Proof. The necessary part of the assertion is trivial. To prove sufficiency, setting $i = 1$ in (1.1) giving

$$\frac{a}{(a+b)} \sum_{n=0}^{\infty} \frac{na_n \theta^n}{A(\theta)} = \frac{\sum_{n=0}^{\infty} na_n \theta^n S(n \mid n)}{\sum_{n=0}^{\infty} a_n \theta^n S(n \mid n)}.$$

By setting $B(\theta) = \sum_{n=0}^{\infty} a_n \theta^n S(n \mid n)$, the above equation can be written as

$$\frac{a}{(a+b)} \frac{A'(\theta)}{A(\theta)} = \frac{B'(\theta)}{B(\theta)},$$

which leads to

$$B(\theta) = K\,[A(\theta)]^{a/(a+b)} \quad \ldots \quad (2.1)$$

where $K > 0$ is an arbitrary constant. Similarly, $i = 2$ in (1.1) yields

$$\frac{a(a+c)}{(a+b)\,(a+b+c)} \frac{A''(\theta)}{A(\theta)} = \frac{B''(\theta)}{B(\theta)}. \quad \ldots \quad (2.2)$$

Substituting for $B(\theta)$ from (2.1) we get

$$\frac{d}{d\theta}\left\{\frac{A'(\theta)}{A(\theta)}\right\} = \frac{c}{(a+b)} \left\{\frac{A'(\theta)}{A(\theta)}\right\}^2,$$

which implies that

$$A(\theta) = \begin{cases} d\left\{1 - \dfrac{\theta c}{\gamma(a+b)}\right\}^{-\frac{(a+b)}{c}} & \text{if } c \neq 0 \\[2ex] d \exp\left\{\dfrac{\theta}{\gamma}\right\} & \text{if } c = 0 \end{cases} \quad \ldots \quad (2.3)$$

where d and γ are positive constants. It further follows that $S(n \mid n) = \dfrac{a_{(n,c)}}{(a+b)_{(n,c)}}$. This completes the proof of the theorem.

Corollary 2.1. *Set $c = 0$ in (2.3). Then $A(\theta) = d\, exp\left\{\dfrac{\theta}{\gamma}\right\}$, which implies that $X \sim$ Poisson and moreover that $S(n \mid n) = \left(\dfrac{a}{a+b}\right)^n$. This is Theorem* 1.1.

Corollary 2.2. *Set $c = -1$ in (2.3). Then $A(\theta) = d\left\{1 + \dfrac{\theta}{\gamma(a+b)}\right\}^{(a+b)}$, which implies that $X \sim$ Binomial with parameters $N = a+b$ and $p = \dfrac{\theta}{\gamma N + \theta}$ and moreover that $S(n \mid n) = \dfrac{a^{(n)}}{(a+b)^{(n)}}$, This is Theorem* 1.2(*a*).

Corollary 2.3. *Set* $c = 1$ *in* (2.3). *Then* $A(\theta) = d\left\{1-\frac{\theta}{\gamma(a+b)}\right\}^{-(a+b)}$ *which implies that* $X \sim$ *Negative Binomial with parameters* $N = a+b$ *and* $p = 1-\frac{\theta}{\gamma N} < 1$ *for* $\gamma > \frac{\theta}{N}$ *and moreover that* $S(n\,|\,n) = \frac{a_{(n)}}{(a+b)_{(n)}}$. *This is Theorem* 1.2(*b*)

Remark 2.1. *It is worth noting that the above characterizations in Corollaries* 2.2 *and* 2.3 *are still valid for* $c < 0$ *and* $c > 0$ *respectively.*

The following generalization of Theorem 2.1 can be proved along the same lines and its proof is omitted.

Theorem 2.2. *Under the conditions of Theorem* 2.1,

$$E(Y^{(i)}) = E(Y^{(i)}\,|\,X = Y+k),\ i = 1, 2 \qquad \dots \quad (2.4)$$

iff $X \sim PED(a+b, c, .)$ *and* $S(n\,|\,n+k) = \binom{n+k}{n}\frac{a_{(n,c)}}{(a+b)_{(n,c)}}S(0\,|\,k)$, *where* k *is a fixed non-negative integer.*

The corresponding corollaries for Theorem 2.2 provide generalizations of Corollaries 2.1—2.3.

Corollary 2.4. (Srivastava and Singh (1975)). *For* $c = 0$, $X \sim$ *Poisson and* $S(n\,|\,n+k) = \binom{n+k}{n}\left(\frac{a}{a+b}\right)^n S(0\,|\,k)$.

Corollary 2.5. *For* $c = -1$, $X \sim$ *Binomial and* $S(n\,|\,n+k) = \binom{n+k}{n}\frac{a_{(n)}}{(a+b)^{(n)}}S(0\,|\,k)$.

Corollary 2.6. *For* $c = 1$, $X \sim$ *Negative Binomial and* $S(n\,|\,n+k) = \binom{n+k}{n}\frac{a_{(n)}}{(a+b)_{(n)}}S(0\,|\,k)$.

In what follows, we give our main extensions of Shanbhag and Clark (1972), and Patil and Ratnaparkhi (1977) results.

Theorem 2.3. *Let* $X \sim PSD(\theta)$ *and the survival distribution* $\{S(r\,|\,n)\}$ *be such that for some fixed* k, *its* k^{th}, $(k+1)^{th}$ *and* $(k+2)^{th}$ *factorial moments are given respectively by* $n^{(k)}\,\alpha^k$, $n^{(k+1)}\,\alpha^{k+1}$ *and* $n^{(k+2)}\,\alpha^{k+2}$ *for some fixed* $\alpha \in (0, 1)$ *that is independent of* θ; *assume that* $S(n\,|\,n)$ *is independent of* θ. If the factorial *moments* $E(Y^{(k)})$ *and* $E(Y^{(k+1)})$ *corresponding to* Y *are non-zero and* $0 < P(X = Y) < 1$, *then*

$$E(Y^{(r)}) = E(Y^{(r)}\,|\,X = Y),\ r = k, k+1, k+2 \qquad \dots \quad (2.5)$$

iff $X \sim$ *Poisson and* $S(n\,|\,n) = \alpha^n$.

Proof. The "if" part is trivial. For the "only if" part, equation (2.5) is equivalent to

$$\alpha^r \sum_n \frac{n^{(r)} a_n \theta^{n-r}}{A(\theta)} = \frac{\sum_n n^{(r)} a_n S(n\,|\,n)\theta^{n-r}}{\sum_n a_n S(n\,|\,n)\theta^n}, \; r = k, k+1, k+2. \qquad \ldots \quad (2.6)$$

Define $A_r(\theta) = \sum_n n^{(r)} a_n \theta^{n-r}$ and $A_r^*(\theta) = \sum_n n^{(r)} a_n S(n\,|\,n)\,\theta^{n-r}$ and note that

$$A_r'(\theta) = A_{r+1}(\theta) \text{ and } A_r^{*\prime}(\theta) = A_{r+1}^*(\theta), \qquad \ldots \quad (2.7)$$

for all values of $r \geqslant 0$ and in particular for $r = k, k+1$. Algebraic manipulation of (2.6) gives

$$\alpha \frac{A_{r+1}(\theta)}{A_r(\theta)} = \frac{A_{r+1}^*(\theta)}{A_r^*(\theta)}, \; r = k, k+1. \qquad \ldots \quad (2.8)$$

(2.8) now yields

$$\alpha \frac{d}{d\theta} \log A_r(\theta) = \frac{d}{d\theta} \log A_r^*(\theta), \quad r = k, k+1,$$

or what is the same

$$A_r^*(\theta) = c_r\,(A_r(\theta))^\alpha, \; r = k, k+1, \qquad \ldots \quad (2.9)$$

where c_r are positive constants. Since $\alpha \in (0, 1)$ and (2.8) implies in view of (2.9)

$$\alpha \frac{A_{k+1}(\theta)}{A_k(\theta)} = \frac{c_{k+1}}{c_k} \left(\frac{A_{k+1}(\theta)}{A_k(\theta)}\right)^\alpha,$$

it follows that $\dfrac{A_{k+1}(\theta)}{A_k(\theta)}$ is a positive constant (say c_1) or equivalently (using 2.7) that $A_k(\theta) = e^{c_0 + c_1\theta}$ with $c_1 > 0$. We have then in view of (2.9) for $r = k$ and (2.7) for all values of r

$$A_k^*(\theta) = c_k\, e^{c_0\alpha + c_1\alpha\theta},$$

$$A(\theta) \equiv A_0(\theta) = c_1^{-k} A_k(\theta) + p_k(\theta),$$

and

$$A_k(\theta) \equiv \sum_n a_n S(n\,|\,n)\theta^n = (c_1\,\alpha)^{-k} A_k^*(\theta) + p_k^*(\theta),$$

for appropriate polynomials $p_k(\theta)$ and $p_k^*(\theta)$ of θ of degree at most $k-1$. The reciprocal of (2.6) for $r = k$ then implies

$$(c_1\alpha)^{-k} + \alpha^{-k}\, p_k(\theta)\, e^{-c_0 - c_1\theta} = (c_1\alpha)^{-k} + c_k^{-1}\, p_k^*(\theta)\, e^{-c_0\alpha - c_1\alpha\theta},$$

which in turn yields

$$c_k^{-1}\, p_k^*(\theta)\, e^{c_0(1-\alpha)+c_1(1-\alpha)\theta} = \alpha^{-k}\, p_k(\theta). \qquad \ldots \quad (2.10)$$

This relation is valid for all parameter values of θ in $X \sim PSD(\theta)$. If this is true for some non-degenerate θ interval, then (2.10) is valid for all θ. Remembering $\alpha \epsilon (0, 1)$ and $c_1 > 0$ and letting $\theta \to \infty$, we see that there is a contradiction unless $p_k(\theta) \equiv p_k^*(\theta) \equiv 0$. Consequently, the expressions for $A(\theta)$ and $A_0^*(\theta)$ imply that $X \sim$ Poisson and $S(n\,|\,n) = \alpha^n$. This concludes the proof of the theorem.

Corollary 2.7. *$k = 0$ leads to Theorem* 1.1.

Theorem 2.4. *Let $X \sim PSD(\theta)$ and the survival distribution $\{S(r\,|\,n)\}$ be such that for some fixed k, its k^{th}, $(k+1)^{th}$ and $(k+2)^{th}$ factorial moments are given respectively by $n^{(k)}\frac{m_{(k)}}{N_{(k)}}$, $n^{(k+1)}\frac{m_{(k+1)}}{N_{(k+1)}}$ and $n^{(k+2)}\frac{m_{(k+2)}}{N_{(k+2)}}$, where $0 < m + max\ \{1,k\} \leqslant N$; assume that $S(n\,|\,n)$ is independent of θ. If the factorial moments $E(Y^{(k)})$ and $E(Y^{(k+1)})$ corresponding to Y are non-zero and $0 < P(X = Y) < 1$, then* (2.5) *is valid iff $X \sim$ Negative Binomial $(N, .)$ and $S(n\,|\,n) = \frac{m_{(n)}}{N_{(n)}}$.*

Proof. The "if" part is trivial. For the "only if" part, equation (2.5) is equivalent to

$$\frac{m_{(r)}}{N_{(r)}}\,\frac{\sum\limits_n n^{(r)}\, a_n \theta^{n-r}}{A(\theta)} = \frac{\sum\limits_n n^{(r)}\, a_n S(n\,|\,n)\, \theta^{n-r}}{\sum\limits_n a_n S(n\,|\,n)\theta^n}\,. \qquad \ldots \quad (2.11)$$

By defining $A_r(\theta)$ and $A_r^*(\theta)$ as in Theorem 2.2 and following the same steps we get

$$A_r^*(\theta) = c_r(A_r(\theta))^{\frac{m+r}{N+r}}\,,\ r = k,\, k+1, \qquad \ldots \quad (2.12)$$

where c_r are positive constants. Moreover, $A_k(\theta) = \delta\left(1 - \frac{\theta}{c(N+k)}\right)^{-(N+k)}$, where c is an arbitrary constant, with $A_k(\theta) = \delta > 0$ and because $\frac{dA_k(\theta)}{d\theta} > 0$, $c > 0$.

Now, since $A_k(\theta) = \frac{dA_{k-1}(\theta)}{d\theta}$ we get

$$A_0(\theta) \equiv A(\theta) = \beta_k\left(1 - \frac{\theta}{c(N+k)}\right)^{-N} + p_k(\theta), \qquad \ldots \quad (2.13)$$

where $\beta_k = \delta c^k \dfrac{(N+k)^k}{N_{(k)}} > 0$ and $p_k(\theta)$ is polynomial of θ of degree at most $k-1$. Equation (2.12), on putting $r = k$ implies that

$$A_k^*(\theta) = \gamma_k \left(1- \frac{\theta}{c(N+k)}\right)^{-(m+k)} \qquad \ldots \quad (2.14)$$

where $\gamma_k = c_k\, \delta^{\frac{m+k}{N+k}} > 0$. Using the same technique as in calculating $A(\theta)$ we can obtain

$$A_0^*(\theta) \equiv \sum_n a_n S(n\,|\,n)\theta^n = \epsilon_k \left(1- \frac{\theta}{c(N+k)}\right)^{-m} + p_k^*(\theta), \quad \ldots \quad (2.15)$$

where $\epsilon_k = \dfrac{\gamma_k\, c^k(N+k)^k}{m_{(k)}} > 0$ and $p_k^*(\theta)$ is polynomial of θ of degree at most $k-1$. The reciprocal of (2.11) for $r = k$ implies

$$\frac{m_{(k)}}{N_{(k)}}\, p_k^*(\theta) = c_k\, \delta^{\frac{m-N}{N+k}} \left(1- \frac{\theta}{c(N+k)}\right)^{N-m} p_k(\theta). \qquad \ldots \quad (2.16)$$

By exploiting the same arguments as in Theorem 2.2 in the above expression (2.16), we can conclude that expressions (2.13) and (2.15) imply $X \sim$ Negative Binomial with index N, and $S(n\,|\,n) = \dfrac{m_{(n)}}{N_{(n)}}$. This concludes the proof of the theorem.

Corollary 2.8. *$k = 0$ leads to Theorem* 1.2(*b*).

Similarly, we can obtain an analogous extension of Theorem 1.2(a). In what follows we only state this result.

Theorem 2.5. *Let $X \sim PSD(\theta)$ and the survival distribution $\{S(r\,|\,n)\}$ be such that for some fixed k, its k^{th}, $(k+1)^{th}$ and $(k+2)^{th}$ factorial moments are given respectively by $n^{(k)} \dfrac{m^{(k)}}{N^{(k)}}$, $n^{(k+1)} \dfrac{m^{(k+1)}}{N^{(k+1)}}$ and $n^{(k+2)} \dfrac{m^{(k+2)}}{N^{(k+2)}}$, where $0 < m + \max\{1,k\} \leqslant N$; assume that $S(n\,|\,n)$ is independent of θ. If the factorial moments $E(Y^{(k)})$ and $E(Y^{(k+1)})$ corresponding to Y are non-zero and $0 < P(X = Y) < 1$, then* (2.5) *is valid iff $X \sim$ Binomial $(N, .)$ and $S(n\,|\,n) = \dfrac{m^{(n)}}{N^{(n)}}$.*

Corollary 2.9. *$k = 0$ leads to Theorem* 1.2(a).

Remark 2.2. A natural question which arises now is the following. Can we obtain a similar unification of Theorems 2.3, 2.4 and 2.5 on the lines of Theorem 2.1? If we scruitinize closely the arguments to prove these theorems, one can see that the answer to this question is in the affirmative. This gives rise to the following result :

Theorem 2.6. *Let* $X \sim PSD(\theta)$ *and the survival distribution* $\{S(r \mid n)\}$ *be such that for some fixed* k, *its* k^{th}, $(k+1)^{th}$ *and* $(k+2)^{th}$ *factorial moments are equal to the corresponding moments of an* $MPD(n, a, b, c)$, *i.e. given respectively by* $n^{(k)} \frac{a_{(k,c)}}{(a+b)_{(k,c)}}$, $n^{(k+1)} \frac{a_{(k+1,c)}}{(a+b)_{(k+1,c)}}$ *and* $n^{(k+2)} \frac{a_{(k+2,c)}}{(a+b)_{(k+2,c)}}$, *where* $0 < \max \{1,k\} \mid C \mid \leqslant b$; *assume that* $S(n \mid n)$ *is independent of* θ. *If the factorial moments* $E(Y^{(k)})$ *and* $E(Y^{(k+1)})$ *corresponding to* Y *are non-zero and* $0 < P(X = Y) < 1$, *then* (2.5) *is valid iff* $X \sim PED\,(a+b, c, .)$ *and* $S(n \mid n) = \frac{a_{(n,c)}}{(a+b)_{(n,c)}}$.

Corollary 2.10. $k = 0$ *leads to Theorem* 2.1.

3. A result in connection with the Srivastava-Singh (1975) conjecture

Srivastava and Singh (1975) conjectured that the Rao-Rubin $RR(k)$ (i.e. $P(Y = r) = P(Y = r \mid X - Y = k)$) condition under a binomial survival distribution implies the original random variable to be Poisson. However, Patil and Taillie (1979), using a counter example, and more recently Alzaid, Rao and Shanbhag (1986) have revealed that the validity of the $RR(k)$ condition for a fixed $k > 0$ is not sufficient to characterize the Poisson distribution, thus disproving the conjecture of Srivastava and Singh (1975). This counter example also disproves Krishnaji's (1974) claim for $RR(1)$. (Incidentally, Shanbhag and Taillie (1979) have extended the result of Patil and Taillie (1979) to Shanbhag's (1977) set-up.) However, if we put an additional restriction that X has an infinite divisible or compound Poisson generating function, then it follows that $RR(1)$ indeed characterizes the Poisson distribution. It is also interesting to note that if $X \sim PSD(\theta)$ with $P(X = 0) > 0$, then an $RR(k)$ condition implies the $RR(0)$ condition (see Theorem 3.1. below). This in turn implies that a version of the Srivastava-Singh conjecture is valid provided the original random variable $X \sim PSD(\theta)$ with $P(X = 0) > 0$. We shall now state our main claim through the following.

Theorem 3.1. *If* $X \sim PSD(\theta)$ *with* $P(X = 0) > 0$ *and the survival distribution is independent of* θ *and* $RR(k)$ *is valid, then* $RR(0)$ *is valid.*

Proof. The $RR(k)$ condition is equivalent to

$$P(Y = y) = P(Y = y \mid \boldsymbol{X} - Y = k), y = 0, 1, \ldots, \qquad \ldots \quad (3.1)$$

with $P(\boldsymbol{X} - Y = k) > 0$. Writing the power series distribution of $\boldsymbol{X}$ as

$$g_x = \frac{a_x \theta^x}{A(\theta)},$$

with $a_0 > 0$, we get from (3.1)

$$\sum_{x=y}^{\infty} \frac{a_x\theta^x}{A(\theta)} S(y\,|\,x) = \frac{a_{y+k}\theta^y S(y\,|\,y+k)}{\sum_{x=0}^{\infty} a_{x+k}\theta^x S(x\,|\,x+k)} \quad y = 0, 1, \ldots, \qquad \ldots \quad (3.2)$$

with $a_k S(0\,|\,k) > 0$ and $a_y S(y\,|\,y) > 0$ for some y (in view of the fact that $a_{x+k}S(x\,|\,x+k) > 0$ for some x and $a_0 > 0$). Substituting for $a_{x+k}S(x\,|\,x+k)$ the expression $a_x S(x\,|\,x) a_k S(0\,|\,k)/a_0$) (and also for $a_{y+k}S(y\,|\,y+k)$, mutatis mutandis) on the right hand side of (3.2), we get

$$P(Y = y) = P(Y = y\,|\,X = Y),\ y = 0,1, \ldots, \qquad \ldots \quad (3.3)$$

with $P(X = Y) > 0$. But (3.3) is the $RR(0)$ condition, and our proof is completed.

Applying this result we can obtain modified versions of Shanbhag (1977) and Alzaid (1986) results. Incidentally, Alzaid (1986) utilized his result to obtain a characterization, through Rao-Rubin condition, of the class of infinitely divisible laws having power series distributions. Our modification of Alzaid's (1986) result gives, among others, Theorem 3.1 of Srivastava and Singh (1975) (which is also an improvement of the result of Srivastava and Srivastava (1970)) as a special case. In the remaining part of this section we give these results.

Theorem 3.2. *If* $X \sim PSD(\theta)$ *with* $P(X = 0) > 0$ *and the survival distri bution of* Y *given* $X = x$ *is given by* $S(y\,|\,x) = \frac{a_y b_{x-y}}{c_x}$, $y = 0, 1, \ldots, x$; $x = 0, 1, \ldots$, *where* $a_x > 0, b_x \geqslant 0$ *for all* $x \geqslant 0$ *with* b_0 *and* $b_1 > 0$, *then an* $RR(k)$ *condition is valid iff* $\{g_x\}$ *can be put in the form*

$$g_x = \frac{c_x\lambda^x}{c(\lambda)},\ x = 0, 1, \ldots,$$

with some parameter λ *and normalizing constant* $c(\lambda)$,

Proof. It follows immediately from our Theorem 3.1 and Theorem 1 of Shanbhag (1977).

Remark 3.1. The above Theorem 3.2 is also a consequence of Remark 11 of Alzaid, Lau, Rao and Shanbhag (1988).

Theorem 3.3. *Let* $\{(X_\lambda, Y_\lambda) : \lambda \in (a, b), 0 \leqslant a < b\}$ *be a family of non-negative, integer-valued random vectors with* $P_\lambda(X_\lambda \geqslant Y_\lambda) = 1$ *such that for all* λ,

$$P_\lambda(X_\lambda = n) = a_n \frac{\lambda^n}{A(\lambda)},\ n = 0, 1, \ldots,$$

where $a_n \geqslant 0$ *and independent of* λ *for every* $n \geqslant 0$, $a_0 > 0$. *Let the survival distribution* $\{S(r \mid n)\}$ *also be independent of* λ. *Then the following are equivalent* :

1. $P_\lambda(Y_\lambda = r) = P_\lambda(Y_\lambda = r \mid X_\lambda - Y_\lambda = k)$,
   $r = 0, 1, \ldots,$ *for all* n, k *is a fixed non-negative integer* ;
2. $P_\lambda(Y_\lambda = r) = P_\lambda(Y_\lambda = r \mid X_\lambda - Y_\lambda = 0)$, $r = 0, 1, \ldots,$ *for all* λ ;
3. $S(r \mid n)a_n a_0 = a_r S(r \mid r) a_{n-r} S(0 \mid n-r)$, $r = 0, 1, \ldots, n$; $n = 0, 1, \ldots$;
4. *The r.v.'s* Y_λ *and* $X_\lambda - Y_\lambda$ *are independent for all* λ.

Proof. The implication (1) $\Rightarrow$ (2) follows from Theorem 3.3, while implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) have been given by Alzaid (1986). Finally the implication (4) $\Rightarrow$ (1) is obvious.

The following corollary of Theorem 3.3 is Theorem 3.1 of Srivastava and Singh (1975).

Corollary 3.1. *If* X_λ *is Poisson with parameter* λ, *the survival distribution* $\{S(r \mid n)\}$ *is independent of* λ *and such that* $0 < S(n \mid n) < 1$ *for some* n, *then* (1) *holds, iff the survival distribution is binomial* (n, p), *where* $p \epsilon (0, 1)$ *and fixed.*

Proof. It follows immediately from our Theorem 3.3 and Corollary 1 of Alzaid (1986).

Remark 3.2. Multivariate generalizations of Theorems 3.1 and 3.2, and Corollary 3.1 can also be obtained.

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References

ALZAID, A. A. (1986). Some results connected with the Rao-Rubin condition. *Sankhyā*, **48** Series A, pt. 1, 104-106.

ALZAID, A. A., RAO, C.R. and SHANBHAG, D.N. (1986). An application of the Perron-Frobenious theorem to a damage model problems. *Sankhyā* A, **48**, pt. 1, 43-50.

ALZAID, A. A., LAU, K. S., RAO, C. R. and SHANBHAG, D. N. (1988). Solution of Deny convolution equation restricted to a half line via a random walk approach. *J. Multivariate Analysis*, **24**, No. 2, 309-329.

KRISHNAJI, N. (1974). Characterization for some discrete distributions based on damage model. *Sankhyā* A **36**, 204-213.

PATIL, G. P. and RATNAPARKHI, M. V. (1977). Characterizations of certain statistical distribution based on additive damage models involving Rao-Rubin condition and some of its variants. *Sankhyā* B, **39**, pt. 1, 65-75.

PATIL, G. P. and TAILLIE, C. (1979). On a variant of the Rao-Rubin theorem. *Sankhyā* A, **41**, pt. 1 and 2, 129-132.

RAO, C. R. (1963). On discrete distributions arising out of methods of ascertainments. In *Classical and Contagious Discrete Distributions*. (Ed. Patil, G. P.), 320-332, Statistical Publishing Society, Calcutta.

RAO, C. R. and RUBIN, H. (1964). On a characterization of the Poisson distribution. *Sankhyā* A, **25**, 295-298.

SHANBHAG, D. N. (1977). An extension of the Rao-Rubin characterization of the Poisson distribution. *J. Appl. Prob.*, **14**, 640-646.

SHANBHAG, D. N. and TAILLIE, C. (1979). An extension of the Patil-Taillie characterization of the Poisson distribution. (unpublished).

SHANBHAG, D. N. and CLARK, R. M. (1972). Some characterzations for the Poisson distribution starting with a power series distribution. *Proc. Camb. Phil. Soc.*, **71**, 517-522.

SRIVASTAVA, R. C. and SINGH, J. (1975). On some characterizations of the binomial and Poisson distributions based on a damage model. In *Statistical Distributions in Scientific Work*. (Ed. Patil, G. P. etal), Vol. 3, 271-277. D. Reidel Publishing Co., Holland.

SRIVASTAVA, R. C. and SRIVASTAVA, A. B. L. (1970). On a characterization of the Poisson distribution. *J. Appl. Prob.*, **7**, 497-501.

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