Supplement to: "Testing equality of autocovariance operators for functional time series"

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This supplement contains the proofs of Lemma 5.1 and Lemma 5.2.

1 Proof of Lemma 5.1

We proceed in two steps. First, we proof that, as $n \to \infty$,

$$\left\| \sum_{s=-b+1}^{b-1} g_b(s)\tilde{\Gamma}_s - \sum_{t=-\infty}^{\infty} \mathbb{E}[Z_0 \otimes Z_t] \right\|_{HS} = o_p(1), \tag{1.1}$$

where $\tilde{\Gamma}_s = n^{-1}\sum_{t=1}^{n-s} Z_t \otimes Z_{t+s}$ for $0 \le s \le b-1$ and $\tilde{\Gamma}_s = n^{-1}\sum_{t=1}^{n+s} Z_{t-s} \otimes Z_t$ for $-b+1 \le s < 0$. Then, we prove that, as $n \to \infty$,

$$\left\| \sum_{s=-b+1}^{b-1} g_b(s)\left(\tilde{\Gamma}_s - \hat{\Gamma}_s\right) \right\|_{HS} = o_p(1). \tag{1.2}$$

Consider (1.1). Since $\|n^{-1}\sum_{t=1}^{n} Z_t \otimes Z_t - \mathbb{E}[Z_0 \otimes Z_0]\|_{HS} = o_p(1)$ as $n \to \infty$, it suffices to show that, as $n \to \infty$,

$$\left\| \sum_{s=1}^{b-1} g_b(s)\tilde{\Gamma}_s - \sum_{t\ge 1} \mathbb{E}[Z_0 \otimes Z_t] \right\|_{HS} = o_p(1). \tag{1.3}$$

Let $c_\infty^+ = \sum_{t\ge1}\mathbb{E}[Z_0 \otimes Z_t]$, $c_m^+ = \sum_{t=1}^{m}\mathbb{E}[Z_{0,m} \otimes Z_{t,m}]$ and $\tilde{\Gamma}_s^{(m)} = n^{-1}\sum_{t=1}^{n-s} Z_{t,m} \otimes Z_{t+s,m}$. Then,

$$\left\| \sum_{s=1}^{b-1} g_b(s)\tilde{\Gamma}_s - c_\infty^+ \right\|_{HS} \le \|c_m^+ - c_\infty^+\|_{HS} + \left\| \sum_{s=1}^{b-1} g_b(s)\tilde{\Gamma}_s^{(m)} - c_m^+ \right\|_{HS}$$

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$$+\left\|\sum_{s=1}^{b-1} g_b(s)\tilde{\Gamma}_s - \sum_{s=1}^{b-1} g_b(s)\tilde{\Gamma}_s^{(m)}\right\|_{HS}. \tag{1.4}$$

Assertion (1.3) is proved by showing that there exists $m_0 \in \mathbb{N}$ such that all three terms on the right hand side of (1.4) can be made arbitrarily small, in probability, as $n \to \infty$ for $m = m_0$.

For the first term of the right hand side of the above inequality, we use the bound

$$\left\|\sum_{t=1}^{m} \mathbb{E}\left[Z_{0,m} \otimes Z_{t,m} - Z_0 \otimes Z_t\right]\right\|_{HS} + \left\|\sum_{t=m+1}^{\infty} \mathbb{E}\left[Z_0 \otimes Z_t\right]\right\|_{HS} \tag{1.5}$$

and the decomposition

$$Z_{0,m} \otimes Z_{t,m} - Z_0 \otimes Z_t = (Z_{0,m} - Z_0) \otimes Z_{t,m} + Z_0 \otimes (Z_{t,m} - Z_t).$$

By Cauchy-Schwarz's inequality, we get, for the first term of (1.5), that

$$\begin{aligned}\left\|\sum_{t=1}^{m} \mathbb{E}\left[(Z_{0,m} - Z_0) \otimes Z_{t,m}\right]\right\|_{HS} &+ \left\|\sum_{t=1}^{m} \mathbb{E}\left[Z_0 \otimes (Z_{t,m} - Z_t)\right]\right\|_{HS} \\ &\leq 2\left(\mathbb{E}\|Z_0\|_{HS}^2\right)^{1/2} \sum_{t=1}^{m} \left(\mathbb{E}\|Z_{0,m} - Z_0\|_{HS}^2\right)^{1/2} \\ &= 2\left(\mathbb{E}\|Z_0\|_{HS}^2\right)^{1/2} m \left(\mathbb{E}\|Z_{0,m} - Z_0\|_{HS}^2\right)^{1/2}.\end{aligned}$$

Therefore, by Assumption 1 of the main paper, we get that, for every $\epsilon_1 > 0$, there exists $m_1 \in \mathbb{N}$ such that the last quantity above is less than ϵ_1 for every $m \geq m_1$. Consider the second term of the right hand side of (1.5). Since Z_0 and $Z_{t,t}$ are independent for $t \geq m+1$ and $\mathbb{E}[Z_0] = 0$, we get, using Cauchy-Schwarz's inequality,

$$\left\|\sum_{t=m+1}^{\infty} \mathbb{E}\left[Z_0 \otimes Z_t\right]\right\|_{HS} \leq \left(\mathbb{E}\|Z_0\|_{HS}^2\right)^{1/2} \sum_{t=m+1}^{\infty} \left(\mathbb{E}\|Z_0 - Z_{0,t}\|_{HS}^2\right)^{1/2}.$$

Using (8) of the main paper, it follows that, for every $\epsilon_2 > 0$, there exists $m_2 \in \mathbb{N}$ such that the above quantity is less than ϵ_2 for every $m \geq m_2$.

For the second term of the bound in (1.4), note that, for every $m \geq 1$, we have that for any fixed s, as $n \to \infty$,

$$\left\|\tilde{\Gamma}_s^{(m)} - \mathbb{E}[Z_{0,m} \otimes Z_{s,m}]\right\|_{HS} = o_p(1).$$

Hence, the aforementioned term of interest is $o_p(1)$, if we show that, as $n \to \infty$,

$$\left\|\sum_{s=m+1}^{b-1} g_b(s)\tilde{\Gamma}_s^{(m)}\right\|_{HS} = o_p(1). \tag{1.6}$$

By the definition of $\tilde{\Gamma}_s^{(m)}$, we have that

$$\mathbb{E}\left\|\sum_{s=m+1}^{b-1} g_b(s)\tilde{\Gamma}_s^{(m)}\right\|_{HS}^2 = \mathbb{E}\left\langle \sum_{s_1=m+1}^{b-1} g_b(s_1)\tilde{\Gamma}_{s_1}^{(m)}, \sum_{s_2=m+1}^{b-1} g_b(s_2)\tilde{\Gamma}_{s_2}^{(m)} \right\rangle_{HS}$$

$$= \frac{1}{n^2} \sum_{s_1=m+1}^{b-1} \sum_{s_2=m+1}^{b-1} \sum_{t_1=1}^{n-s_1} \sum_{t_2=1}^{n-s_2} g_b(s_1) g_b(s_2) \mathbb{E}\langle Z_{t_1,m} \otimes Z_{t_1+s_1,m}, Z_{t_2,m} \otimes Z_{t_2+s_2,m} \rangle_{HS}.$$

Since the sequence $\{Z_{t,m},\ t \in \mathbb{Z}\}$ is m-dependent, $Z_{t,m}$ and $Z_{t+s,m}$ are independent for $s \geq m+1$ and, therefore, $\mathbb{E}[Z_{t,m} \otimes Z_{t+s,m}] = 0$ for $s \geq m+1$. Hence, the number of terms $\mathbb{E}\langle Z_{t_1,m} \otimes Z_{t_1+s_1,m}, Z_{t_2,m} \otimes Z_{t_2+s_2,m} \rangle_{HS}$ in the last equation above which do not vanish is of order $O(nb)$ and, consequently, as $n \to \infty$,

$$\mathbb{E} \left\| \sum_{s=m+1}^{b-1} g_b(s) \tilde{\Gamma}_s^{(m)} \right\|_{HS}^2 = O\left(\frac{b}{n}\right) = o(1), \tag{1.7}$$

from which (1.6) follows by Markov's inequality.

For the third term in (1.4) we show that, for $m = m_0$ and for any $\delta > 0$,

$$\limsup_{n \to \infty} P\left(\left\| \sum_{s=1}^{b-1} g_b(s) \left(\tilde{\Gamma}_s - \tilde{\Gamma}_s^{(m)} \right) \right\|_{HS} > \delta \right) = 0. \tag{1.8}$$

Using Markov's inequality, expression (1.8) follows if we show that, for $m = m_0$, as $n \to \infty$,

$$\mathbb{E} \left\| \sum_{s=1}^{b-1} g_b(s) (\tilde{\Gamma}_s - \tilde{\Gamma}_s^{(m)}) \right\|_{HS} = o(1). \tag{1.9}$$

Now, by the definitions of $\tilde{\Gamma}_h$ and $\tilde{\Gamma}_s^{(m)}$, we have

$$\begin{aligned} \mathbb{E} \left\| \sum_{s=1}^{b-1} g_b(s) \left(\tilde{\Gamma}_s - \tilde{\Gamma}_s^{(m)} \right) \right\|_{HS} &\leq \mathbb{E} \left\| \frac{1}{n} \sum_{s=1}^{m} g_b(s) \sum_{t=1}^{n-s} (Z_t \otimes Z_{t+s} - Z_{t,m} \otimes Z_{t+s,m}) \right\|_{HS} \\ &+ \mathbb{E} \left\| \frac{1}{n} \sum_{s=m+1}^{b-1} g_b(s) \sum_{t=1}^{n-s} (Z_t \otimes Z_{t+s} - Z_{t,m} \otimes Z_{t+s,m}) \right\|_{HS}. \end{aligned} \tag{1.10}$$

Using Cauchy-Schwarz's inequality and the decomposition

$$Z_t \otimes Z_{t+s} - Z_{t,m} \otimes Z_{t+s,m} = (Z_t - Z_{t,m}) \otimes Z_{t+s} + Z_{t,m} \otimes (Z_{t+s} - Z_{t+s,m}),$$

we get, for the first term of the right hand side of (1.10), the bound

$$\begin{aligned} &\frac{1}{n} \sum_{s=1}^{m} \sum_{t=1}^{n-s} (\mathbb{E}\|(Z_t - Z_{t,m}) \otimes Z_{t+s}\|_{HS} + \mathbb{E}\|Z_{t,m} \otimes (Z_{t+s} - Z_{t+s,m})\|_{HS}) \\ &\quad \leq \frac{1}{n} \sum_{s=1}^{m} \sum_{t=1}^{n-s} (\mathbb{E}\|Z_t - Z_{t,m}\|_{HS}^2 \mathbb{E}\|Z_{t+s}\|_{HS}^2)^{1/2} + (\mathbb{E}\|Z_{t+s} - Z_{t+s,m}\|_{HS}^2 \mathbb{E}\|Z_{t,m}\|_{HS}^2)^{1/2} \\ &\quad \leq m[(\mathbb{E}\|Z_0 - Z_{0,m}\|_{HS}^2 \mathbb{E}\|Z_0\|_{HS}^2)^{1/2} + (\mathbb{E}\|Z_0 - Z_{0,m}\|_{HS}^2 \mathbb{E}\|Z_{0,m}\|_{HS}^2)^{1/2}]. \end{aligned}$$

By Assumption 1 of the main paper, it follows that, for every $\epsilon_3 > 0$, there exists $m_3 \in \mathbb{Z}$ such that, for every $m \geq m_3$, this quantity is less than ϵ_3. For the second term on the right hand side of (1.10), we use the bound

$$\mathbb{E} \left\| \frac{1}{n} \sum_{s=m+1}^{b-1} g_b(s) \sum_{t=1}^{n-s} Z_t \otimes Z_{t+s} \right\|_{HS} + \mathbb{E} \left\| \frac{1}{n} \sum_{s=m+1}^{b-1} g_b(s) \sum_{t=1}^{n-s} Z_{t,m} \otimes Z_{t+s,m} \right\|_{HS}. \tag{1.11}$$

Expression (1.7) implies that the second summand of (1.11) is $o(1)$. For the first term of (1.11), we use the decomposition

$$Z_t \otimes Z_{t+s} = Z_t \otimes Z_{t+s,s} + Z_t \otimes (Z_{t+s} - Z_{t+s,s}),$$

and get the bound

$$\mathbb{E}\left\|\frac{1}{n}\sum_{s=m+1}^{b-1} g_b(s)\sum_{t=1}^{n-s} Z_t \otimes Z_{t+s,s}\right\|_{HS} + \mathbb{E}\left\|\frac{1}{n}\sum_{s=m+1}^{b-1} g_b(s)\sum_{t=1}^{n-s} Z_t \otimes (Z_{t+s} - Z_{t+s,s})\right\|_{HS}. \tag{1.12}$$

For the last term of expression (1.12), we have the bound

$$\frac{1}{n}\sum_{s=m+1}^{b-1}\sum_{t=1}^{n-s}\mathbb{E}\,\|Z_t \otimes (Z_{t+s} - Z_{t+s,s})\|_{HS} \leq \left(\mathbb{E}\|Z_0\|_{HS}^2\right)^{1/2}\sum_{s=m+1}^{b-1}\left(\mathbb{E}\|Z_0 - Z_{0,s}\|_{HS}^2\right)^{1/2}.$$

Therefore, since $\{Z_t,\, t \in \mathbb{Z}\}$ is L^2-m-approximable, with $Z_{0,m}$ be the m-dependent approximation of Z_0, it follows that for every $\epsilon_4 > 0$, there exists $m_4 \in \mathbb{N}$ such that, for every $m \geq m_4$, this term is less than ϵ_4. Consider next the first term of (1.12). We have

$$\begin{aligned}
\mathbb{E}\left\|\frac{1}{n}\sum_{s=m+1}^{b-1} g_b(s)\sum_{t=1}^{n-s} Z_t \otimes Z_{t+s,s}\right\|_{HS} &\leq \sum_{s=m+1}^{b-1}\mathbb{E}\left\|\frac{1}{n}\sum_{t=1}^{n-s} Z_t \otimes Z_{t+s,s}\right\|_{HS} \\
&\leq \sum_{s=m+1}^{b-1}\left(\mathbb{E}\left\|\frac{1}{n}\sum_{t=1}^{n-s} Z_t \otimes Z_{t+s,s}\right\|_{HS}^2\right)^{1/2}.
\end{aligned} \tag{1.13}$$

Since Z_0 and $Z_{s,s}$ are independent, $\|Z_0 \otimes Z_t\|_{HS} = \|Z_0\|_{HS}\|Z_t\|_{HS}$ and $\mathbb{E}\langle Z_0 \otimes Z_{s,s}, Z_t \otimes Z_{t+s,s}\rangle_{HS} = \mathbb{E}\langle Z_0, Z_t\rangle_{HS}\langle Z_{s,s}, Z_{t+s,s}\rangle_{HS} = 0$ for $|t| > s$. Using Cauchy-Schwarz's inequality, we get

$$\begin{aligned}
&\mathbb{E}\left\|\frac{1}{n}\sum_{t=1}^{n-s} Z_t \otimes Z_{t+s,s}\right\|_{HS}^2 \leq \frac{n-s}{n^2}\sum_{|t|<n-s}\mathbb{E}(\langle Z_0 \otimes Z_{s,s}, Z_t \otimes Z_{t+s,s}\rangle_{HS}) \\
&\leq \frac{1}{n}\sum_{t=-s}^{s}|\mathbb{E}\langle Z_0 \otimes Z_{s,s}, Z_t \otimes Z_{t+s,s}\rangle_{HS}| \leq \frac{1}{n}\sum_{t=-s}^{s}\mathbb{E}\|Z_0 \otimes Z_{s,s}\|_{HS}\|Z_t \otimes Z_{t+s,s}\|_{HS} \\
&\leq \frac{1}{n}\sum_{t=-s}^{s}\mathbb{E}\|Z_0 \otimes Z_{s,s}\|_{HS}^2 \frac{1}{n}\sum_{t=-s}^{s}\left(\mathbb{E}\|Z_0\|_{HS}^2\right)^2 \\
&\leq \frac{1}{n}\sum_{t=-s}^{s}\left(\mathbb{E}\|X_0 \otimes X_0\|_{HS}^2\right)^2 \leq \frac{1}{n}\sum_{t=-s}^{s}\left(\mathbb{E}\|X_0\|_{HS}^4\right)^2.
\end{aligned}$$

Therefore, by (1.13), the first term of (1.12) is $O_P(b^{3/2}/n^{1/2})$. The proof is then concluded by choosing $m_0 = \max\{m_1, m_2, m_3, m_4\}$.

Consider (1.2). First note that using Theorem 3 of Kokoszka and Reimherr (2013), we get, as $n \to \infty$,

$$\left\|\frac{1}{n}\sum_{t=1}^{n}[Z_t \otimes Z_t - \hat{Z}_t \otimes \hat{Z}_t]\right\|_{HS} = \|(\hat{\tilde{\mathcal{C}}}_0 - \mathcal{C}_0) \otimes (\hat{\tilde{\mathcal{C}}}_0 - \mathcal{C}_0)\|_{HS} = \frac{1}{n}\left\|\sqrt{n}(\hat{\tilde{\mathcal{C}}}_0 - \mathcal{C}_0)\right\|_{HS}^2 = O_P(1/n).$$

Therefore, it suffices to show that

$$\left\|\sum_{s=1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}[Z_t\otimes Z_{t+s}-\hat{Z}_t\otimes\hat{Z}_{t+s}]\right\|_{HS}=o_p(1).$$

Again, by Theorem 3 of Kokoszka and Reimherr (2013), we get that, as $n\to\infty$,

$$\begin{aligned}
&\left\|\sum_{s=1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}[Z_t\otimes Z_{t+s}-\hat{Z}_t\otimes\hat{Z}_{t+s}]\right\|_{HS}\\
&=\left\|\sum_{s=1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}[(X_t\otimes X_t)\otimes(\hat{\hat{\mathcal{C}}}_0-\mathcal{C}_0)+(\hat{\hat{\mathcal{C}}}_0-\mathcal{C}_0)\otimes(X_{t+s}\otimes X_{t+s})+\mathcal{C}_0\otimes\mathcal{C}_0-\hat{\hat{\mathcal{C}}}_0\otimes\hat{\hat{\mathcal{C}}}_0]\right\|_{HS}\\
&\leq\sum_{s=1}^{b-1}\frac{1}{\sqrt{n}}\|\frac{1}{n}\sum_{t=1}^{n-s}(X_t\otimes X_t)\|_{HS}\|\sqrt{n}(\hat{\hat{\mathcal{C}}}_0-\mathcal{C}_0)\|_{HS}\\
&\qquad+\sum_{s=1}^{b-1}\frac{1}{\sqrt{n}}\|\sqrt{n}(\hat{\hat{\mathcal{C}}}_0-\mathcal{C}_0)\|_{HS}\|\frac{1}{n}\sum_{t=1}^{n-s}(X_{t+s}\otimes X_{t+s})\|\\
&\qquad+\frac{1}{\sqrt{n}}\sum_{s=1}^{b-1}\frac{1}{n}\sum_{t=1}^{n-s}\|\mathcal{C}_0\|_{HS}\|\sqrt{n}(\mathcal{C}_0-\hat{\hat{\mathcal{C}}}_0)|_{HS}\\
&\qquad+\frac{1}{\sqrt{n}}\sum_{s=1}^{b-1}\frac{1}{n}\sum_{t=1}^{n-s}\|\hat{\hat{\mathcal{C}}}_0\|_{HS}\|\sqrt{n}(\mathcal{C}_0-\hat{\hat{\mathcal{C}}}_0)\|_{HS}=O_P(b/\sqrt{n})=o_p(1).
\end{aligned}$$

This completes the proof of the lemma.

2 Proof of Lemma 5.2

Since $\sum_{t=-\infty}^{\infty}\mathbb{E}\iint Z_0(u,v)Z_t(u,v)\mathrm{d}u\mathrm{d}v$ converges and is finite, and since

$$\frac{1}{n}\sum_{t=1}^{n}\iint(Z_t(u,v))^2\mathrm{d}u\mathrm{d}v\xrightarrow{P}\mathbb{E}\iint(Z_0(u,v))^2\mathrm{d}u\mathrm{d}v$$

as $n\to\infty$, it suffices to prove that

$$\sum_{s=1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_t(u,v)Z_{t+s}(u,v)\mathrm{d}u\mathrm{d}v\xrightarrow{P}\sum_{t=1}^{\infty}\mathbb{E}\iint Z_0(u,v)Z_t(u,v)\mathrm{d}u\mathrm{d}v. \tag{2.1}$$

Since

$$\begin{aligned}
&\left|\sum_{s=1}^{b-1} g_b(s)\frac{1}{n}\sum_{i=1}^{n-s}\iint Z_t(u,v)Z_{t+s}(u,v)\mathrm{d}u\mathrm{d}v-\sum_{t=1}^{\infty}\mathbb{E}\iint Z_0(u,v)Z_t(u,v)\mathrm{d}u\mathrm{d}v\right|\\
&\leq\left|\sum_{t=1}^{m}\mathbb{E}\iint Z_{0,m}(u,v)Z_{t,m}(u,v)\mathrm{d}u\mathrm{d}v-\sum_{t=1}^{\infty}\mathbb{E}\iint Z_0(u,v)Z_t(u,v)\mathrm{d}u\mathrm{d}v\right|
\end{aligned}$$

$$+\left|\sum_{s=1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_{t,m}(u,v)Z_{t+s,m}(u,v)\mathrm{d}u\mathrm{d} - \sum_{t=1}^{m}\mathbb{E}\iint Z_{0,m}(u,v)Z_{t,m}(u,v)\mathrm{d}u\mathrm{d}v\right|$$
$$+\left|\sum_{s=1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_{t}(u,v)Z_{t+s}(u,v)\mathrm{d}u\mathrm{d}v - \sum_{s=1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_{t,m}(u,v)Z_{t+s,m}(u,v)\mathrm{d}u\mathrm{d}v\right|, \quad (2.2)$$

assertion (2.1) is proved by showing that there exists $m_0 \in \mathbb{N}$ such that all three terms on the right hand side of (2.2) can be made arbitrarily small in probability as $n \to \infty$ for $m = m_0$.

For the first term, we use the bound

$$\left|\sum_{t=1}^{m}\left(\mathbb{E}\iint Z_{0,m}(u,v)Z_{t,m}(u,v)\mathrm{d}u\mathrm{d}v - \mathbb{E}\iint Z_0(u,v)Z_t(u,v)\mathrm{d}u\mathrm{d}v\right)\right|$$
$$+\left|\sum_{t=m+1}^{\infty}\mathbb{E}\iint Z_0(u,v)Z_t(u,v)\mathrm{d}u\mathrm{d}v\right|. \quad (2.3)$$

By Cauchy-Schwarz's inequality and the decomposition

$$Z_{0,m}(u,v)Z_{t,m}(u,v)-Z_0(u,v)Z_t(u,v) = [Z_{0,m}(u,v)-Z_0(u,v)]Z_{t,m}(u,v)+Z_0(u,v)[Z_{t,m}(u,v)-Z_t(u,v)],$$

we get that the first term of (2.3) is bounded by

$$\left|\sum_{t=1}^{m}\mathbb{E}\iint [Z_{0,m}(u,v)-Z_0(u,v)]Z_{t,m}(u,v)\mathrm{d}u\mathrm{d}v\right| + \left|\sum_{t=1}^{m}\mathbb{E}\iint Z_0(u,v)[Z_{t,m}(u,v)-Z_t(u,v)]\mathrm{d}u\mathrm{d}v\right|$$
$$\leq 2\sum_{t=1}^{m}\mathbb{E}\left\{\left[\iint [Z_{0,m}(u,v)-Z_0(u,v)]^2\mathrm{d}u\mathrm{d}v\right]^{1/2}\left[\iint [Z_{t,m}(u,v)]^2\mathrm{d}u\mathrm{d}v\right]^{1/2}\right\}$$
$$\leq 2\sum_{t=1}^{m}\left[\mathbb{E}\iint [Z_{0,m}(u,v)-Z_0(u,v)]^2\mathrm{d}u\mathrm{d}v\right]^{1/2}\left[\mathbb{E}\iint [Z_{t,m}(u,v)]^2\mathrm{d}u\mathrm{d}v\right]^{1/2}$$
$$\leq 2\sum_{t=1}^{m}\left[\mathbb{E}\iint [X_{0,m}(u)X_{0,m}(v)-X_0(u)X_0(v)]^2\mathrm{d}u\mathrm{d}v\right]^{1/2}\left[\mathbb{E}\iint [X_{0,m}(u)X_{0,m}(v)-c(u,v)]^2\mathrm{d}u\mathrm{d}v\right]^{1/2}$$
$$= 2\sum_{t=1}^{m}\left[\mathbb{E}\|X_{0,m}\otimes X_{0,m}-X_0\otimes X_0\|_{HS}^2\right]^{1/2}\left[\mathbb{E}\|X_{0,m}\otimes X_{0,m}-C\|_{HS}^2\right]^{1/2}$$
$$= \left[\mathbb{E}\|X_0\otimes X_0 - C\|_{HS}^2\right]^{1/2}\left(m\left[\mathbb{E}\|X_{0,m}\otimes X_{0,m}-X_0\otimes X_0\|_{HS}^2\right]^{1/2}\right).$$

Using (9) of the main paper, and since $\{X_t\otimes X_t,\, t\in\mathbb{Z}\}$ is L^2-m-approximable, it follows that for every $\epsilon_1 > 0$ there exists $m_1 \in \mathbb{N}$ such that the above term is less than ϵ_1 for every $m \geq m_1$. Consider the second term of (2.3). Since $Z_0(u,v)$ and $Z_{t,t}(u,v)$ are independent for $t \geq m+1$, using Cauchy-Schwarz's inequality, we get

$$\left|\sum_{t=m+1}^{\infty}\mathbb{E}\iint Z_0(u,v)Z_t(u,v)\mathrm{d}u\mathrm{d}v\right| = \left|\sum_{t=m+1}^{\infty}\mathbb{E}\iint Z_0(u,v)[Z_t(u,v)-Z_{t,t}(u,v)]\mathrm{d}u\mathrm{d}v\right|$$
$$\leq \sum_{t=m+1}^{\infty}\left[\mathbb{E}\iint [Z_0(u,v)]^2\mathrm{d}u\mathrm{d}v\right]^{1/2}\left[\mathbb{E}\iint [Z_t(u,v)-Z_{t,t}(u,v)]^2\mathrm{d}u\mathrm{d}v^2\right]^{1/2}$$

$$= \left[\mathbb{E}\iint [X_0(u)X_0(v) - c(u,v)]^2 \mathrm{d}u\mathrm{d}v\right]^{1/2} \sum_{t=m+1}^{\infty} \left[\mathbb{E}\iint [X_t(u)X_t(v) - X_{t,t}(u)X_{t,t}(v)]^2 \mathrm{d}u\mathrm{d}v\right]^{1/2}$$

$$= \left[\mathbb{E}\|X_{0,m}\otimes X_{0,m} - C\|_{HS}^2\right]^{1/2} \sum_{t=m+1}^{\infty} \left[\mathbb{E}\|X_{0,m}\otimes X_{0,m} - X_0\otimes X_0\|_{HS}^2\right]^{1/2}.$$

From (8) of the main paper, it follows that for every $\epsilon_2 > 0$, there exists $m_2 \in \mathbb{N}$ such that the above quantity is less than ϵ_2 for every $m \geq m_2$.

Consider next the second term of the the right-hand side of the inequality (2.2). Note that for every $m \geq 1$, we have that, for any fixed s, as $n \to \infty$,

$$\left|\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_{t,m}(u,v)Z_{t+s,m}(u,v)\mathrm{d}u\mathrm{d} - \mathbb{E}\iint Z_{0,m}(u,v)Z_{s,m}(u,v)\mathrm{d}u\mathrm{d}v\right| = o_p(1).$$

Therefore, the aforementioned term is $o_p(1)$ if we show that

$$\left|\sum_{s=m+1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_{t,m}(u,v)Z_{t+s,m}(u,v)\mathrm{d}u\mathrm{d}\right| = o_p(1). \tag{2.4}$$

For this, notice first that

$$\mathbb{E}\left[\sum_{s=m+1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_{t,m}(u,v)Z_{t+s,m}(u,v)\mathrm{d}u\mathrm{d}\right]^2$$
$$= \frac{1}{n^2}\sum_{s_1=m+1}^{b-1}\sum_{s_2=m+1}^{b-1} g_b(s_1)g_b(s_2)\sum_{t_1=1}^{n-s_1}\sum_{t_2=1}^{n-s_1}\mathbb{E}\Bigg[\iint Z_{t_1,m}(u_1,v_1)Z_{t_1+s_1,m}(u_1,v_1)\mathrm{d}u_1\mathrm{d}v_1\times$$
$$\iint Z_{t_2,m}(u_2,v_2)Z_{t_2+s_2,m}(u_2,v_2)\mathrm{d}u_2\mathrm{d}v_2\Bigg].$$

Since the sequence $\{Z_{t,m}(u,v),\ t\in\mathbb{Z}\}$ is m-dependent, $Z_{t,m}(u,v)$ and $Z_{t+s,m}(u,v)$ are independent for $s \geq m+1$, therefore using $\mathbb{E}(Z_{0,m}(u,v)) = 0$ we get that, $\mathbb{E}\iint Z_{t,m}(u,v)Z_{t+s,m}(u,v)\mathrm{d}u\mathrm{d}v = 0$. Hence, the number of terms

$$\mathbb{E}\left[\iint Z_{t_1,m}(u_1,v_1)Z_{t_1+s_1,m}(u_1,v_1)\mathrm{d}u_1\mathrm{d}v_1 \times \iint Z_{t_2,m}(u_2,v_2)Z_{t_2+s_2,m}(u_2,v_2)\mathrm{d}u_2\mathrm{d}v_2\right]$$

in the last equation above which do not vanish is of order $O(nb)$ and, consequently, as $n \to \infty$,

$$\mathbb{E}\left[\sum_{s=m+1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_{t,m}(u,v)Z_{t+s,m}(u,v)\mathrm{d}u\mathrm{d}\right]^2 = O\left(\frac{b}{n}\right) = o(1), \tag{2.5}$$

from which (2.4) follows by Markov's inequality.

For the third term in (2.2), we show that, for $m = m_0$,

$$\limsup_{n\to\infty} P\left(\left|\sum_{s=1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_t(u,v)Z_{t+s}(u,v)\mathrm{d}u\mathrm{d}v\right.\right.$$

$$-\sum_{s=1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_{t,m}(u,v)Z_{t+s,m}(u,v)\mathrm{d}u\mathrm{d}v\Bigg| > \delta\Bigg) = 0, \qquad (2.6)$$

for any $\delta > 0$. By Markov's inequality, expression (2.6) follows if we show that, for $m = m_0$,

$$\mathbb{E}\left|\sum_{s=1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_t(u,v)Z_{t+s}(u,v)\mathrm{d}u\mathrm{d}v - Z_{t,m}(u,v)Z_{t+s,m}(u,v)\mathrm{d}u\mathrm{d}v\right| = o(1). \qquad (2.7)$$

For the above quantity we have the bound

$$\mathbb{E}\left|\sum_{s=1}^{m} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_t(u,v)Z_{t+s}(u,v) - Z_{t,m}(u,v)Z_{t+s,m}(u,v)\mathrm{d}u\mathrm{d}v\right|$$
$$+\,\mathbb{E}\left|\sum_{s=m+1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_t(u,v)Z_{t+s}(u,v) - Z_{t,m}(u,v)Z_{t+s,m}(u,v)\mathrm{d}u\mathrm{d}v\right|. \qquad (2.8)$$

For the first term of the right hand side of the above inequality, using the decomposition

$$\begin{aligned}&Z_t(u,v)Z_{t+s}(u,v) - Z_{t,m}(u,v)Z_{t+s,m}(u,v)\\ &\quad= [Z_t(u,v) - Z_{t,m}(u,v)]Z_{t+s}(u,v) + [Z_{t+s}(u,v) - Z_{t+s,m}(u,v)]Z_{t,m}(u,v)\end{aligned}$$

we get the bound,

$$\sum_{s=1}^{m}\frac{1}{n}\sum_{t=1}^{n-s}\mathbb{E}\iint |[Z_t(u,v) - Z_{t,m}(u,v)]Z_{t+s}(u,v)|\,\mathrm{d}u\mathrm{d}v$$
$$+\,\mathbb{E}\iint |[Z_{t+s}(u,v) - Z_{t+s,m}(u,v)]Z_{t,m}(u,v)|\,\mathrm{d}u\mathrm{d}v. \qquad (2.9)$$

Using Cauchy-Schwarz's inequality, we have

$$\begin{aligned}&\mathbb{E}\iint |[Z_t(u,v) - Z_{t,m}(u,v)]Z_{t+h}(u,v)|\,\mathrm{d}u\mathrm{d}v\\
&\quad\le \mathbb{E}\Big[\iint [Z_t(u,v) - Z_{t,m}(u,v)]^2\mathrm{d}u\mathrm{d}v\Big]^{1/2}\Big[\iint [Z_{t+s}(u,v)]^2\mathrm{d}u\mathrm{d}v\Big]^{1/2}\\
&\quad\le \Big[\mathbb{E}\iint [Z_t(u,v) - Z_{t,m}(u,v)]^2\mathrm{d}u\mathrm{d}v\Big]^{1/2}\Big[\mathbb{E}\iint [Z_{t+s}(u,v)]^2\mathrm{d}u\mathrm{d}v\Big]^{1/2}\\
&\quad= \Big[\mathbb{E}\|X_t\otimes X_t - X_{t,m}\otimes X_{t,m}\|_{HS}^2\Big]^{1/2}\Big[\mathbb{E}\|X_{t+h}\otimes X_{t+s} - \mathcal{C}_0\|_{HS}^2\Big]^{1/2}. \end{aligned}\qquad (2.10)$$

Using the same arguments, we get

$$\begin{aligned}&\mathbb{E}\iint |[Z_{t+s}(u,v) - Z_{t+s,m}(u,v)]Z_{t,m}(u,v)|\,\mathrm{d}u\mathrm{d}v\\
&\quad\le \Big[\mathbb{E}\|X_{t+s}\otimes X_{t+s} - X_{t+s,m}\otimes X_{t+s,m}\|_{HS}^2\Big]^{1/2}\Big[\mathbb{E}\|X_t\otimes X_t - C\|_{HS}^2\Big]^{1/2}.\end{aligned}$$

Therefore, (2.9) is bounded by

$$2(\mathbb{E}\|X_0\otimes X_0 - \mathcal{C}_0\|_{HS}^2)^{1/2}\left[m(\mathbb{E}\|X_0\otimes X_0 - X_{0,m}\otimes X_{0,m}\|_{HS}^2)^{1/2}\right].$$

Hence, by (9) of the main paper, it follows that, for every $\epsilon_3 > 0$, there exists $m_3 \in \mathbb{Z}$ such that, for every $m \geq m_3$, this quantity is bounded by ϵ_3. For the second term on the right hand side of (2.8), we use the bound

$$\mathbb{E}\left|\sum_{s=m+1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_t(u,v)Z_{t+s}(u,v)\mathrm{d}u\mathrm{d}v\right| + \mathbb{E}\left|\sum_{s=m+1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_{t,m}(u,v)Z_{t+s,m}(u,v)\mathrm{d}u\mathrm{d}v\right|. \quad (2.11)$$

Expression (2.5) implies that the second summand of (2.11) is $o(1)$, while for the first term of (2.11) we use the decomposition

$$Z_t(u,v)Z_{t+s}(u,v) = Z_t(u,v)Z_{t+s,s}(u,v) + Z_t(u,v)[Z_{t+s}(u,v) - Z_{t+s,s}(u,v)]$$

to get the bound

$$\mathbb{E}\left|\sum_{s=m+1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_t(u,v)Z_{t+s,s}(u,v)\mathrm{d}u\mathrm{d}v\right| + \mathbb{E}\left|\sum_{s=m+1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_t(u,v)[Z_{t+s}(u,v) - Z_{t+s,s}(u,v)]\mathrm{d}u\mathrm{d}v\right|. \quad (2.12)$$

Using same arguments as those applied in (2.10), we get the bound

$$\mathbb{E}\iint |Z_t(u,v)[Z_{t+s}(u,v) - Z_{t+s,s}(u,v)]|\,\mathrm{d}u\mathrm{d}v \leq \left[\mathbb{E}\|X_{t+s}\otimes X_{t+s} - X_{t+s,s}\otimes X_{t+s,s}\|_{HS}^2\right]^{1/2}\left[\mathbb{E}\|X_t\otimes X_t - \mathcal{C}_0\|_{HS}^2\right]^{1/2}.$$

Hence, for the last term of expression (2.12), we have

$$\mathbb{E}\left|\sum_{s=m+1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_t(u,v)[Z_{t+s}(u,v) - Z_{t+s,s}(u,v)]\mathrm{d}u\mathrm{d}v\right| \leq \left[\mathbb{E}\|X_0\otimes X_0 - \mathcal{C}_0\|_{HS}^2\right]^{1/2}\sum_{s=m+1}^{\infty}\left[\mathbb{E}\|X_0\otimes X_0 - X_{0,s}\otimes X_{0,s}\|_{HS}^2\right]^{1/2}.$$

Therefore, using (8) of the main paper, we get that for every $\epsilon_4 > 0$, there exists $m_4 \in \mathbb{N}$ such that, for every $m \geq m_4$, this term is bounded by ϵ_4. Consider next the first term of (2.12). Using the decomposition

$$Z_t(u,v)Z_{t+s,s}(u,v) = [Z_t(u,v) - Z_{t,s}(u,v)]Z_{t+s,s}(u,v) + Z_{t,s}(u,v)Z_{t+s,s}(u,v),$$

we get the bound

$$\mathbb{E}\left|\sum_{s=m+1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint [Z_t(u,v) - Z_{t,s}(u,v)]Z_{t+s,s}(u,v)\mathrm{d}u\mathrm{d}v\right|$$

$$+\,\mathbb{E}\left|\sum_{s=m+1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_{t,s}(u,v)Z_{t+s,s}(u,v)\mathrm{d}u\mathrm{d}v\right|. \tag{2.13}$$

For the first term of this bound, and by Cauchy-Schwarz's inequality, we get the bound

$$\mathbb{E}\left|\sum_{s=m+1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint [Z_t(u,v)-Z_{t,s}(u,v)]Z_{t+s,s}(u,v)\mathrm{d}u\mathrm{d}v\right|$$
$$\leq \left[\mathbb{E}\|X_t\otimes X_t - X_{t,m}\otimes X_{t,m}\|_{HS}^2\right]^{1/2}\left[\mathbb{E}\|X_{t+s}\otimes X_{t+s}-\mathcal{C}_0\|_{HS}^2\right]^{1/2}.$$

Hence, by (8) of the main paper, it follows that, for every $\epsilon_5 > 0$, there exists $m_3 \in \mathbb{Z}$ such that, for every $m \geq m_3$, this quantity is bounded by ϵ_5. Consider the last term of the expression given in (2.13) and note that $\{\iint Z_{t,s}(u,v)Z_{t+s,s}(u,v)\mathrm{d}u\mathrm{d}v,\ t\in\mathbb{Z}\}$ is a $2s$-dependent sequence. Also note that since $Z_{t,s}(u,v)$ and $Z_{t+s}(u,v)$ are independent $\mathbb{E}\iint Z_{t,s}(u,v)Z_{t+s,s}(u,v)\mathrm{d}u\mathrm{d}v = 0$. Therefore, as $n\to\infty$, $n^{-1/2}\sum_{t=1}^{n}\iint Z_{t,s}(u,v)Z_{t+s,s}(u,v)\mathrm{d}u\mathrm{d}v = O_P(1)$. Hence, using Portmanteau's theorem, and since $f(x)=|x|$ is a Lipschitz function, we get that, as $n\to\infty$,

$$\mathbb{E}\left|\frac{1}{\sqrt{n}}\sum_{t=1}^{n}\iint Z_{t,s}(u,v)Z_{t+s}(u,v)\mathrm{d}u\mathrm{d}v\right| = O(1).$$

Therefore, as $n\to\infty$,

$$\mathbb{E}\left|\sum_{s=m+1}^{b-1} g_b(s)\frac{1}{n}\sum_{t=1}^{n-s}\iint Z_{t,s}(u,v)Z_{t+s,s}(u,v)\mathrm{d}u\mathrm{d}v\right|$$
$$\leq \frac{1}{\sqrt{n}}\sum_{s=m+1}^{b-1}\mathbb{E}\left|\frac{1}{\sqrt{n}}\sum_{t=1}^{n-s}\iint Z_{t,s}(u,v)Z_{t+s,s}(u,v)\mathrm{d}u\mathrm{d}v\right| = O(b/\sqrt{n}) = o(1).$$

The proof of the lemma is concluded by choosing $m_0 = \max\{m_1, m_2, m_3, m_4, m_5\}$.

References

[1] Kokoszka, P. and Reimherr, M. (2013). Asymptotic normality of the principal components of functional time series. *Stochastic Processes and their Applications*, Vol. **123**, 1546–1562.