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Some properties of extreme stable laws and related infinitely divisible random variables

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A B S T R A C T

In the course of studying the moment sequence $\{n^n : n = 0, 1, \ldots\}$, Eaton et al. [1971. On extreme stable laws and some applications. J. Appl. Probab. 8, 794–801] have shown that this sequence, which is, indeed, the moment sequence of a log-extreme stable law with characteristic exponent $\gamma = 1$, corresponds to a scale mixture of exponential distributions and hence to a distribution with decreasing failure rate. Following essentially the approach of Shanbhag et al. [1977. Some further results in infinite divisibility. Math. Proc. Cambridge Philos. Soc. 82, 289–295] we show that, under certain conditions, log-extreme stable laws with characteristic exponent $\gamma \in [1, 2)$ are scale mixtures of exponential distributions and hence are infinitely divisible and have decreasing failure rates. In addition, we study the moment problem associated with the log-extreme stable laws with characteristic exponent $\gamma \in (0, 2]$ and throw further light on the existing literature on the subject. As a by-product, we show that generalized Poisson and generalized negative binomial distributions are mixed Poisson distributions. Finally, we address some relevant questions on structural aspects of infinitely divisible distributions, and make new observations, including in particular that certain results appearing in Steutel and van Harn [2004. Infinite Divisibility of Probability Distributions on the Real Line. Marcel Dekker, New York] have links with the Wiener–Hopf factorization met in the theory of random walk.

1. Introduction

A stable law with characteristic exponent $\gamma \in (0, 2]$, on the real line $\mathbb{R} = (-\infty, \infty)$, is normal (or Gaussian) if $\gamma = 2$, and has the characteristic function (ch.f.), ϕ, with the following spectral representation if $\gamma \in (0, 2)$:

$$\phi(t) = \exp\left\{ iat + c_1 \int_0^\infty \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{1}{x^{\gamma+1}}\,dx \right.$$
$$\left. + c_2 \int_{-\infty}^0 \left(e^{itx} - 1 - \frac{itx}{1+x^2} \right) \frac{1}{|x|^{\gamma+1}}\,dx \right\}, \quad t \in \mathbb{R}, \tag{1.1}$$

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where $a \in \mathbb{R}$, $c_1, c_2 \in \mathbb{R}_+ = [0, \infty)$ with $c_1 + c_2 > 0$ (see, e.g., Loève, 1963, p. 330 or Breiman, 1968, p. 200). Clearly, this is infinitely divisible and has, in the case of $\gamma \in (0,2)$, Lévy (spectral) measure, μ_γ, satisfying on $\mathbb{R}\backslash\{0\}$

$$\mu_\gamma(\mathrm{d}x) = \frac{c_x}{|x|^{\gamma+1}}\,\mathrm{d}x, \tag{1.2}$$

where $c_x = c_1$ if $x \in (0,\infty)$ and $c_x = c_2$ if $x \in (-\infty, 0)$, with c_1 and c_2 as in (1.1).

It may be recalled here that a ch.f. ϕ^* is infinitely divisible, i.e., corresponds to an infinitely divisible probability law, if and only if it is of the form

$$\phi^*(t) = \exp\left\{\mathrm{i}at - \frac{ct^2}{2} + \int_{\mathbb{R}}\left(\mathrm{e}^{\mathrm{i}tx} - 1 - \frac{\mathrm{i}tx}{1+x^2}\right)\mu(\mathrm{d}x)\right\}, \quad t \in \mathbb{R}, \tag{1.3}$$

with $a \in \mathbb{R}$, $c \in \mathbb{R}_+$ and μ as a measure on (the Borel σ-field of) $\mathbb{R}$ such that

$$\mu(\{0\}) = 0 \quad \text{and} \quad \int_{\mathbb{R}}(\min\{1, x^2\})\mu(\mathrm{d}x) < \infty;$$

if (1.3) holds, then (a, c, μ) is determined uniquely by ϕ^* and the formula is referred to as the Lévy (or the Lévy–Khintchine) spectral representation of ϕ^* with the measure μ as the corresponding Lévy measure.

Stable distributions arise out of the central limit theorem as a subproblem that involves domains of attraction or limiting distributions relative to affine transforms of partial sums of independent and identically distributed random variables; for an account of standard properties of stable distributions see, e.g., Zolotarev (1986). Stable distributions have several important applications, including those in finance, insurance, physics, and economics; see, e.g., Cont et al. (1997), Furrer (1998) and Fabozzi and Focardi (2003).

Suppose, following Eaton et al. (1971), we understand by an extreme stable law, a stable law that is either normal or that is nonnormal having ch.f. ϕ of the form (1.1) with Lévy measure μ_γ specified by (1.2), concentrated either on $(-\infty, 0)$ or on $(0, \infty)$. We shall also refer to a random variable X_γ distributed according to an extreme stable law with characteristic exponent $\gamma \in (0, 2]$ and the corresponding ch.f. as an extreme stable random variable and an extreme stable ch.f., both with characteristic exponent γ, respectively. The random variable e^{X_γ}, or the corresponding distribution or the ch.f., will be referred to as log-extreme stable with characteristic exponent γ.

Among various findings that we have in the present article, there are some showing that certain log-extreme stable random variables have distributions that are scale mixtures of exponential distributions. Mixtures of distributions play a dominant role in building probability models quite frequently in biological and physical sciences; see, e.g., Prakasa Rao (1982) and Titterington et al. (1985). There are numerous papers devoted to studies of mixtures of probability distributions. Some of these concern preservation of structural properties, such as infinite divisibility, self-decomposability, stability, unimodality, log-concavity or log-convexity, under mixing. Steutel (1967) and Steutel and van Harn (2004) have reviewed and unified many of the important contributions in this respect; in particular, Steutel and van Harn (2004) comprises two results that are crucial to the present study, viz., Theorem VI.3.13 asserting that ZY is infinitely divisible if Z and Y are independent random variables with additionally that Y is exponential, and Theorem VI.7.10, essentially, a discrete version of the former.

At the heart of our approach in this article lies a technique developed by Shanbhag et al. (1977) that deals with infinite divisibility aspects of scale mixtures of some well-known distributions, which we use to obtain certain decomposability properties of extreme stable random variables and hence factorization properties of log-extreme stable random variables with characteristic exponent $\gamma \in [1, 2)$. (For information, we may note here that if X_γ is an extreme stable random variable with Lévy measure concentrated on $(-\infty, 0)$, then $-X_\gamma$ is an extreme stable random variable with Lévy measure concentrated on $(0, \infty)$.) These imply in particular that, under certain conditions, log-extreme stable random variables with characteristic exponent $\gamma \in [1, 2)$ are scale mixtures of exponential distributions and, for any $p \in (0, \infty)$, log-extreme stable random variables with characteristic exponent $\gamma \in (1, 2)$ are scale mixtures relative to the distribution of the p-th power of a gamma random variable.

Our results referred to above extend Theorem 3 of Eaton et al. (1971) which, in turn, follows also from (2) of Theorem 1 of Zolotarev (1964). These tell us, amongst other things that, under certain conditions, log-extreme stable laws are infinitely divisible and have decreasing failure rates. Moreover, these results enable us to provide an elementary approach to deal with the moment problem relative to log-extreme stable laws with characteristic exponent $\gamma \in [1, 2)$, possessing moment sequences; the problem referred to was addressed in the prior literature, at least partially, involving a deeper result, viz., Theorem 1 of Zolotarev (1964), see, e.g., Rao and Shanbhag (1994, p. 8) and Pakes (2001, Theorem 10).

Among several other revelations that we have made in this article, we have, appealing in particular to Theorem 2 of Eaton et al. (1971) or otherwise, that generalized Poisson and generalized negative binomial distributions are mixed Poisson distributions, and a proof of the assertion that Theorems VI.3.13 and VI.7.10 of Steutel and van Harn (2004) (and hence implicitly of Theorem 2.3.1 in Steutel, 1970) have links with the Wiener–Hopf factorization met in the theory of random walk.

2. Auxiliary lemmas

The following lemmas are useful in the present study.

Lemma 1. *If α is a positive real number, then*

$$\frac{\Gamma(\alpha+\theta)}{\Gamma(\alpha)} = \exp\left\{c_\alpha\theta + \int_{-\infty}^{0}(e^{\theta x}-1-\theta x)\frac{e^{\alpha x}}{|x|(1-e^{x})}\,dx\right\}$$
$$= \exp\left\{c'_\alpha\theta + \int_{-\infty}^{0}\left(e^{\theta x}-1-\frac{\theta x}{1+x^2}\right)\frac{e^{\alpha x}}{|x|(1-e^{x})}\,dx\right\}, \quad \mathrm{Re}(\theta) > -\alpha,$$

where

$$c_\alpha = \int_0^\infty \{e^{-x}(1-e^{-x}) - xe^{-\alpha x}\}\{x(1-e^{-x})\}^{-1}\,dx$$

and

$$c'_\alpha = \int_0^\infty \{e^{-x}(1-e^{-x}) - xe^{-\alpha x}(1+x^2)^{-1}\}\{x(1-e^{-x})\}^{-1}\,dx.$$

Proof. As observed in Shanbhag et al. (1977), the result follows easily from the expression for the gamma function given, as that due to Malmstén, in Example 1 in Whittaker and Watson (1962, p. 249); the expression referred to has also appeared as Entry 8.341.3 in Gradshteyn and Ryzhik (1965). □

Lemma 2. *For each $\gamma \in [1,2]$ and $x \in (0,\infty)$,*

$$e^x - 1 > x^\gamma.$$

Proof. We have, for each $\gamma \in [1,2]$ and $x \in (0,\infty)$,

$$\begin{aligned} e^x - 1 &> x + \frac{x^2}{2} + \frac{x^3}{8} \\ &> \max\left\{x, x^2 + \frac{x}{2}\left(\frac{x}{2}-1\right)^2\right\} \\ &\geqslant \max\{x, x^2\} \\ &\geqslant x^\gamma. \end{aligned}$$

Hence, we have the lemma. □

Lemma 3. *For each $\kappa \in (0,\infty)$ and $\beta \in (0,\infty)$,*

$$\max\{\exp\{-(\beta/e)x\kappa^{-1/\beta}\}x^\beta : x \in (0,\infty)\} = \kappa.$$

Proof. Clearly, for each $\lambda \in (0,\infty)$ and $\beta \in (0,\infty)$, the function $x^\beta \exp\{-\lambda x\}$, $x \in (0,\infty)$, tends to 0 as x tends to 0 or to ∞. Also, its derivative equals 0 if and only if $x = \beta/\lambda$. Consequently, we have that the maximum value of the function equals $(\beta/(e\lambda))^\beta$; this value can indeed be seen to be equal to κ in the case when $\lambda = (\beta/e)\kappa^{-1/\beta}$. Hence, we have the lemma. □

Remark 1. Using essentially a slightly modified version of the proof of Lemma 2, it can further be seen that

$$\sup\{\gamma : e^x - 1 > x^\gamma \text{ for all } x \in (0,\infty)\} \in (2,3).$$

(Hint: $e^2 - 1 < 2^3$ and $(e^x-1)/x^2 > 1 + (x/24) > x^{\gamma-2}$ for all $x \in (0,\infty)$ and $\gamma \in (2, 49/24]$.) However, this additional information is not needed in this paper; we do not even require here the result relative to the case $\gamma = 2$ in Lemma 2.

Lemma 4. *Let F be a distribution function (d.f.) on $\mathbb{R}$ such that it has the following representation:*

$$F(x) = \alpha F_1(x) + (1-\alpha)\int_{(0,\infty)} F_2(x/y)G(dy), \quad x \in \mathbb{R}, \tag{2.1}$$

where $\alpha \in [0,1)$, and F_1, F_2 and G are d.f.'s on $\mathbb{R}$ (with obviously G concentrated on $(0,\infty)$). Then F has all moments (i.e., as defined and finite) if and only if one of the following holds:

(i) $\alpha = 0$, *and F_2 and G have all moments;*
(ii) $\alpha \neq 0$, *and F_1, F_2 and G have all moments.*

Also, if (i) *or* (ii) *holds, then F has a moment sequence that is Hamburger-indeterminate* (*relative to the d.f.'s on* $\mathbb{R}$) *or Stieltjes-indeterminate* (*relative to the d.f.'s on* $\mathbb{R}$ *with these concentrated on* $\mathbb{R}_+$) *according to the moment sequence of* F_2 *is Hamburger-indeterminate or, in conjunction with that* $\mathrm{supp}[F] \subset \mathbb{R}_+$, *is Stieltjes-indeterminate.* (*If a moment sequence is Stieltjes-indeterminate, then it is also Hamburger-indeterminate.*)

Proof. The first assertion of the lemma follows easily via a standard argument based on even order moments since, in obvious notation, (2.1) implies that

$$m_{2n} = \alpha m_{2n}^{(1)} + (1-\alpha) m_{2n}^{(2)} m'_{2n}, \quad n = 0, 1, \ldots . \tag{2.2}$$

To prove the second assertion of the lemma, note that if (i) or (ii) holds, then F has moments m_n for all n, satisfying, in obvious notation, (2.2) with "$2n$" replaced by "n". Refer to the modified version of (2.2) as (2.2)′. Then, in view of (2.2)′, (2.1) implies easily that the assertion holds. Thus, we have the lemma. □

Remark 2. Suppose that v is a finite measure on (the Borel σ-field of) $\mathbb{R}$ with moment sequence $m_n, n=0, 1, \ldots$, and with absolutely continuous component having Radon–Nikodym derivative g, and $\varphi : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a function such that $\int_{[1,\infty)} \varphi(x) x^{-3/2}\, \mathrm{d}x < \infty$. Then, essentially by Corollary 1 in Slud (1993) (or, in the special case of $\varphi(x) = \lambda x^{\alpha}$, $x \in \mathbb{R}_+$, with $\lambda \in (0,\infty)$ and $\alpha \in (0, \frac{1}{2})$, by Remark 1.1.10(iii) in Rao and Shanbhag, 1994), it follows that in the class of all measures on $\mathbb{R}$ with moment sequences, the measure v is not identified by its moment sequence, even with its support specified, if for some $c \in (0,\infty)$,

$$g(x) \exp\{\varphi(|x|)\} \geqslant 1 \quad \text{for a.a. } x \in (-\infty, -c] \text{ or a.a. } x \in [c, \infty) \tag{2.3}$$

or

$$g(x) \exp\{\varphi(x^2)\} \geqslant 1 \quad \text{for a.a. } x \in (-\infty, -c] \cup [c, \infty), \tag{2.4}$$

where a.a. refers to "almost all". In view of Lemma 4, this can be verified noting that $v = v_1 + v_2$ for some measures v_1 and v_2 such that $\mathrm{supp}[v] = \mathrm{supp}[v_1]$ and v_2 is proportional to the probability measure determined by the distribution function of a random variable X, where in the case of (2.3), $X - c$ or $-X - c$, and, in the case of (2.4), $X^2 - c^2$ have distributions meeting the requirements of the criterion of Slud (1993, Corollary 1), with, also, in the latter case X as symmetric (about 0). The result of Slud (1993) referred to here is in fact a variation of the result of Krein (1945), while the specialized results implied in Remark 1.1.10(iii) of Rao and Shanbhag (1994) are based on elementary results on integrals appearing in Shohat and Tamarkin (1963).

3. Properties of extreme stable laws and related results

This section is devoted to studying properties of extreme stable laws or the corresponding random variables, using essentially the approach of Shanbhag et al. (1977).

For each $\gamma \in [1,2)$, let X_γ be an extreme stable random variable with characteristic exponent γ, such that the corresponding Lévy measure, μ_γ, is concentrated on $(-\infty, 0)$ and satisfies

$$\mu_\gamma(\mathrm{d}x) = \frac{1}{|x|^{\gamma+1}} \mathrm{d}x,$$

which is a specialized version of (1.2) with $c_1 = 0$ and $c_2 = 1$. (It easily follows then that, for each $c \in (0,\infty)$, cX_γ is an extreme stable random variable with characteristic exponent γ and Lévy measure $c^\gamma \mu_\gamma$.)

In the subsequent development, "$\stackrel{\mathrm{d}}{=}$" denotes equality in distribution, and, also, we refer to each gamma random variable Y with density $(\lambda^a/\Gamma(\alpha)) y^{\alpha-1} \mathrm{e}^{-\lambda y}$, $y \in (0,\infty)$, in obvious notation, as a gamma random variable with parameter vector (λ, α).

In Sections 3.1 and 3.2, we now present results on log-extreme stable laws and on the generalized Poisson and the generalized negative binomial distributions, respectively.

3.1. Factorizations of log-extreme random variables and a related moment problem

Theorem 1. *For each* $\gamma \in [1,2)$ *and* $c \in [1,\infty)$,

$$\exp\{cX_\gamma\} \stackrel{\mathrm{d}}{=} Y \exp\{V_\gamma^{(c)}\},$$

where $V_\gamma^{(c)}$ *is a certain infinitely divisible random variable, Y is an exponential random variable* (*with unit scale parameter*), *and* $V_\gamma^{(c)}$ *and Y are independent.*

Proof. In view of Lemmas 1 and 2, it follows that for each $\gamma \in [1,2)$ and $c \in [1,\infty)$,

$$\mathbb{E}(\exp\{itcX_\gamma\}) = \Gamma(1+it)\phi_\gamma^{(c)}(t), \quad t \in \mathbb{R},$$

where $\phi_\gamma^{(c)}$ is a certain infinitely divisible ch.f. with Lévy measure, $v_\gamma^{(c)}$, concentrated on $(-\infty,0)$ such that

$$v_\gamma^{(c)}(dx) = \frac{c^\gamma}{|x|^{\gamma+1}}\left\{1 - \frac{|x|^\gamma}{c^\gamma(e^{|x|}-1)}\right\} dx.$$

Since $\Gamma(1+it)$, $t \in \mathbb{R}$, is the ch.f. of the logarithm of an exponential random variable (with unit scale parameter), we can hence see that there exist random variables $V_\gamma^{(c)}$ and Y that are independent with $\phi_\gamma^{(c)}$ as the ch.f. of $V_\gamma^{(c)}$ and Y as an exponential random variable (with unit scale parameter), for which

$$cX_\gamma \stackrel{d}{=} V_\gamma^{(c)} + \log Y.$$

This completes the proof of Theorem 1. □

Corollary 1. *Suppose $\gamma \in [1,2)$, $c \in [1,\infty)$ and W is a random variable that is independent of X_γ. Then, the random variable We^{cX_γ} is infinitely divisible, and, if $W>0$ almost surely, the distribution of We^{cX_γ} has a decreasing failure rate.*

Proof. The first result follows as an obvious corollary to Theorem VI.3.13 of Steutel and van Harn (2004); it can also be viewed as a corollary to Theorem 5 in Shanbhag and Sreehari (1977). The second result follows from Barlow et al. (1963), in which it is proved that scale mixtures of exponential distributions have decreasing failure rates. Thus, we have Corollary 1. □

Theorem 2. *For each $\gamma \in (1,2)$ and $c \in (0,\infty)$, there exist independent random variables $V_\gamma^{(c)}$ and $Y_\gamma^{(c)}$ such that $V_\gamma^{(c)}$ is infinitely divisible, $Y_\gamma^{(c)}$ is gamma with parameter vector $(1,((\gamma-1)/e)c^{-\gamma/(\gamma-1)}+1)$, and*

$$\exp\{cX_\gamma\} \stackrel{d}{=} Y_\gamma^{(c)} \exp\{V_\gamma^{(c)}\}.$$

Note that Theorem 2 implies that, given any $p \in (0,\infty)$, since $e^{cX_\gamma} = (e^{cX_\gamma/p})^p$, the random variable e^{cX_γ} follows a distribution that is a scale mixture relative to the distribution of the p-th power of a gamma random variable. We may now give, for Theorem 2, a proof that follows.

Proof. Appealing to Lemmas 1 and 3, we have that for each $\gamma \in (1,2)$ and $c \in (0,\infty)$,

$$\mathbb{E}(\exp\{itcX_\gamma\}) = \frac{\Gamma(((\gamma-1)/e)c^{-\gamma/(\gamma-1)}+1+it)}{\Gamma(((\gamma-1)/e)c^{-\gamma/(\gamma-1)}+1)}\phi_\gamma^{(c)}(t), \quad t \in \mathbb{R},$$

where $\phi_\gamma^{(c)}$ is a certain infinitely divisible ch.f. with Lévy measure, $v_\gamma^{(c)}$, concentrated on $(-\infty,0)$ such that

$$v_\gamma^{(c)}(dx) = \frac{c^\gamma}{|x|^{\gamma+1}}\left\{1 - \frac{|x|^\gamma e^{-\{((\gamma-1)/e)c^{-\gamma/(\gamma-1)}|x|\}}}{c^\gamma(e^{|x|}-1)}\right\} dx.$$

(To see that $v_\gamma^{(c)}$ is well defined as a measure, take, in Lemma 3, $\kappa = c^\gamma$ and $\beta = \gamma-1$, and also note that $|x|/(e^{|x|}-1) < 1$.) Proceeding then along the lines of the proof of Theorem 1, with obvious modifications, we can hence have that Theorem 2 holds. □

Remark 3. Both Theorems 1 and 2 remain valid if "cX_γ" is replaced by "$W+cX_\gamma$" with W as an infinitely divisible random variable independent of X_γ. From Shanbhag and Sreehari (1977), it is also clear that the validity of Theorem 1 follows from the validity of its specialized version with $c=1$ and $\gamma \in [1,2)$ (since $e^{cX_\gamma} = (e^{X_\gamma})^c$).

Remark 4. The second result of Corollary 1 in the case when $c=\gamma=1$ has essentially appeared as Theorem 3 in Eaton et al. (1971). This specialized result was also proved by showing that the distribution in question is a scale mixture of exponential distributions. That the distribution of e^{X_1} is a scale mixture of exponential distributions is implicit in (2) of Theorem 1 in Zolotarev (1964).

Remark 5. Theorem 2.3.1 in Steutel (1967) has implicitly played a crucial role in attaining the two results that we have referred to in the proof of the first part of Corollary 1; the two results in question are clearly extended versions of the so-called Goldie–Steutel result, appearing in both Goldie (1967) and Steutel (1967), which asserts that if X and Y are independent random variables with X as nonnegative and Y as exponential, then XY is infinitely divisible. We shall make some observations of relevance to Theorem 2.3.1 of Steutel (1970) as well as certain other results of Steutel (1970), or Steutel and van Harn (2004), in Section 4 of this article.

Remark 6. While we are on the topic of stable laws, it may be worth pointing out that certain representations in terms of gamma as well as in terms of χ^2 random variables, given in Sections 2 and 3, respectively, of Williams (1977), for the reciprocals of specialized versions of positive stable random variables have appeared explicitly, or follow easily from the information available, in the prior literature; see Brown and Tukey (1946) and, in conjunction especially with the statement and the proof of Theorem 3.4.3 in Zolotarev (1986, pp. 206–207), Karlin (1968, pp. 121–122). Incidentally, the theorem in Zolotarev (1986), just referred to, is due to Williams (1977) (see Zolotarev, 1986, Section 3.6) and, in view of Mitra (1983), Brockwell (1984) (which includes an editorial note from *Mathematical Reviews*) throws, though implicitly, some light on the link between Brown and Tukey (1946) and Williams (1977). It may also be worth pointing out here that a variation of this theorem has appeared as Theorem 1 in Shanbhag et al. (1977) and is of relevance to the assertion of Theorem 5 (see Remarks 14 and 16).

Remark 7. Shanbhag and Gani (1975) gave certain decomposability results relative to infinitely divisible random variables involving, amongst other things, dependent components. It may be noted now that the general result established in Section 4 of Shanbhag and Gani (1975) tells us effectively that a modified version of Theorem 2 of the cited reference with the assumption of finite left extremity for the d.f. of Z appearing in it replaced by that the Lévy measure corresponding to the ch.f. ϕ is concentrated on $(0,\infty)$ (possibly, with the measure as null), holds. This, in turn, implies that given a strictly stable random variable X^*_γ, with characteristic exponent $\gamma \in (0,1)\cup(1,2]$ and Lévy measure concentrated on $(0,\infty)$, and a continuous decreasing function $c:\mathbb{R}\mapsto[0,1]$, there exists a random vector (X',Y'), with $X' \stackrel{\mathrm{d}}{=} X^*_\gamma$ and Y' as positive and independent of X', such that $Y'\exp\{c(\ln(Y'))X'\} \stackrel{\mathrm{d}}{=} \exp\{X'\}$; for simplicity, we have not involved here the details of the given information in the notation for the random vector.

From Theorem 2 in Eaton et al. (1971), or other sources, it is known that for each $\lambda \in (0,\infty)$ and $\gamma \in (0,2]$, the sequence $\{m_n^{(\lambda,\gamma)} : n = 0, 1, \ldots\}$ defined by the following (where we read $0^0 = 1$) is a moment sequence

$$m_n^{(\lambda,\gamma)} = \begin{cases} e^{-\lambda n^\gamma} & \text{if } \gamma \in (0,1), \\ n^{\lambda n} & \text{if } \gamma = 1, \\ e^{\lambda n^\gamma} & \text{if } \gamma \in (1,2]. \end{cases} \tag{3.1}$$

(Incidentally, there are typos in Eaton et al., 1971 that concern the theorem: in (1.6) "$\beta = 1$" should read "$\beta = -1$" and, in line 4 of the second paragraph on page 796, "now" should have appeared "not".)

Indeed, in each of the three cases, the moment sequence defined by (3.1) corresponds to a log-extreme stable law. Note that the moment problem concerning (3.1) was studied, appealing to Theorem 1 of Zolotarev (1964) or otherwise, by Rao and Shanbhag(1994, p. 8) and, more recently, by Bisgaard and Sasvari (2000, pp. 103–106) and Pakes (1997, Lemma 5.2; 2001, Theorem 10); see, also, Heyde (1963) who has dealt with the special case of $\gamma = 2$ (i.e., of the log-normal distribution).

It is easy to verify that for $\gamma \in [1,2)$, the moment sequence appearing in (3.1) is, in fact, that corresponding to the random variable e^{cX_γ}, up to a change of scale, where X_γ is as defined earlier and

$$c = \begin{cases} \lambda & \text{if } \gamma = 1, \\ \left(\dfrac{\gamma(\gamma-1)\lambda}{\Gamma(2-\gamma)}\right)^{1/\gamma} & \text{if } \gamma \in (1,2). \end{cases} \tag{3.2}$$

Using this information (though not explicitly in the case of $\gamma \in (1,2)$), amongst other things, including in particular some standard results on moments and our Theorems 1 and 2, we can now prove the following result without involving directly Theorem 1 of Zolotarev (1964) referred to above.

Theorem 3. *Let $\{m_n^{(\lambda,\gamma)} : n = 0, 1, \ldots\}$ be as in (3.1) for $\lambda \in (0,\infty)$ and $\gamma \in (0,2]$. Then the following assertions for the moment problem (relative to the moment sequence) hold:*

(i) *If $\gamma \in (0,1)$, then for all $\lambda \in (0,\infty)$, and if $\gamma = 1$, then for all $\lambda \in (0,1]$, the Hamburger moment problem is determined.*
(ii) *If $\gamma = 1$, then for all $\lambda \in (1,2]$, the Stieltjes moment problem is determined.*
(iii) *If $\gamma = 1$, then for all $\lambda \in (2,\infty)$, and if $\gamma \in (1,2]$, then for all $\lambda \in (0,\infty)$, the Stieltjes moment problem is indeterminate.*

Proof. That (i) holds is obvious because of the Riesz criterion appearing in, e.g., Shohat and Tamarkin (1963, p. 20). (Indeed, if $\gamma \in (0,1)$ then, for each $\lambda \in (0,\infty)$, the moment sequence corresponds to a probability distribution with support $[0,1]$.) Also, the version of the Riesz criterion, for the Stieltjes moment problem, implies that (ii) holds. Finally, to prove (iii), restrict first to the case of $\gamma \neq 2$. In view of Theorems 1 and 2, we can then easily see that each of the distributions involved in (iii) is a scale mixture relative to the distribution of a certain power greater than 2 of a gamma random variable. Taking into account the information given in Remark 1.1.10 (iii) in Rao and Shanbhag (1994, pp. 7–8) directly or appealing to Proposition 2 in Slud (1993) in conjunction with Lemma 4, we can hence conclude that the result holds. The remaining case of $\gamma = 2$ has already been dealt with, essentially, by Heyde (1963). Consequently, we have that Theorem 3 holds. □

Remark 8. Several of the existing results in the literature on indeterminacy of moment sequences, such as those of Heyde (1963) and Slud (1993) referred to in the proof of Theorem 3, follow also from the information given in Remark 1.1.10 (iii) of Rao and Shanbhag (1994) that we have come across in the proof of Theorem 3 as well as in Remark 2. More general results involving versions of the Krein criterion, in this connection appear in Slud (1993) (as implied in Remark 2), Simon (1998), Lin (1997), Pakes et al. (2001) and other places. It may be worth pointing out in this place that the proof of Theorem 3 of Lin (1997) has a gap since it assumes implicitly without justifying that if the symmetric distribution G of the proof has a moment sequence, then it is not determined by the moment sequence, in the class of symmetric distributions with moment sequences, if the Krein criterion is met. However, the gap in question can easily be abridged by using some of the information in the proof of Theorem 1 of Lin (1997), noting in particular, that if g in the proof of this latter theorem agrees a.e. with an odd function, then the assertion of the 5th line on page 87 of Lin (1997) is valid with $g(x)$ replaced by $g(x)\sin(tx)$ for any nonnegative t.

3.2. Generalized Poisson and generalized negative binomial as mixed Poisson distributions

We begin this subsection addressing a further problem that is related to the moment sequence (3.1) in the case of $\lambda = \gamma = 1$ or the corresponding Mellin transform. This concerns the so-called generalized Poisson (referred to also in the literature as Lagrangian Poisson) distributions (see, e.g., Consul and Jain, 1973), with support $\{0, 1, \ldots\}$. Their distributions are of the form

$$\mathbb{P}_{\theta,\eta}\{X = x\} = \theta(\theta + \eta x)^{x-1} \mathrm{e}^{-\theta-\eta x}/x!, \quad x = 0, 1, \ldots, \tag{3.3}$$

with $\theta > 0$ and $\eta \in [0, 1]$ (see Remark 9).

The infinite divisibility related questions for (3.3) have been tackled by some authors; see, e.g., Fosam and Shanbhag (1997) and Examples II.11.16 and V.9.13 in Steutel and van Harn (2004). Another question that has been raised on the structure of the family (3.3) enquires as to whether, in the case when $\eta \in [0, 1)$, the distributions obtained from (3.3) could be expressed as mixed Poisson distributions (see Joe, 1997; p. 242). Using an indirect approach, without specifying the mixing distribution, Joe and Zhu (2005) have shown that the answer to the question is in the affirmative.

We shall now see that the information that we have on extreme stable laws enables us to answer the question referred to above, in a slightly modified form (with $\eta \in [0, 1)$ replaced by $\eta \in [0, 1]$), in a more definitive and constructive way; the details in this connection are provided by the following theorem and its proof.

Theorem 4. *The generalized Poisson distributions in* (3.3) *are mixed Poisson distributions.*

Proof. The case of $\eta = 0$ is trivial since, in this case, the distributions given in (3.3) reduce to Poisson with parameter θ (i.e., a mixed Poisson with degenerate mixing distribution).

If $\eta \in (0, 1]$, appealing in particular to Theorem 2 in Eaton et al. (1971), one can see that

$$\begin{aligned} \mathbb{P}_{\theta,\eta}\{X = x\} &= \frac{\theta \mathrm{e}^{-\theta}}{\eta(x!)} (\eta \mathrm{e}^{-\eta})^x \left(\frac{\theta}{\eta} + x\right)^{(\theta/\eta)+x} \left(\frac{\theta}{\eta} + x\right)^{-((\theta/\eta)+1)} \\ &= \theta^{-\theta/\eta} \mathbb{E}\{V^{\theta/\eta}((VW)^x/x!)\}, \quad x = 0, 1, \ldots, \end{aligned} \tag{3.4}$$

with V and W as independent positive random variables such that $\ln V$ is an extreme stable random variable (with moment generating function (m.g.f.) $(\eta \mathrm{e}^{-\eta})^t t^t$, $t \geqslant 0$, where we read $0^0 = 1$) with characteristic exponent $\gamma = 1$ and $-\ln W$ is a gamma random variable with parameter vector $(\theta/\eta, (\theta/\eta) + 1)$. (For simplicity, we do not use here for random variables the notation involving parameters.)

Summing (3.4) over $x \in \{0, 1, \ldots\}$, we get then that

$$\theta^{-\theta/\eta} \mathbb{E}(V^{\theta/\eta} \mathrm{e}^{VW}) = 1. \tag{3.5}$$

In view of (3.5), (3.4) implies that

$$\begin{aligned} \mathbb{P}_{\theta,\eta}\{X = x\} &= \mathbb{E}\left\{\mathrm{e}^{-VW} \frac{(VW)^x}{x!} V^{\theta/\eta} \mathrm{e}^{VW}\right\} \Big/ \mathbb{E}\{V^{\theta/\eta} \mathrm{e}^{VW}\} \\ &= \int_{(0,\infty)} \mathrm{e}^{-\lambda} \frac{\lambda^x}{x!} B_{\theta,\eta}(\mathrm{d}\lambda), \quad x = 0, 1, \ldots, \end{aligned}$$

where $B_{\theta,\eta}$ is the d.f. given for all $\lambda \in (0,\infty)$ by

$$B_{\theta,\eta}(\lambda) = \frac{\mathbb{E}\{V^{\theta/\eta} e^{VW} \mathbb{I}_{\{VW \leqslant \lambda\}}\}}{\mathbb{E}\{V^{\theta/\eta} e^{VW}\}},$$

where $\mathbb{I}_A$ denotes the indicator function of the set A. This completes the proof of Theorem 4. □

The next theorem that we are to present in this section throws further light on the mechanism of Theorem 4 (see Remark 10) and reveals that the generalized negative binomial distributions (see, e.g., Consul and Gupta, 1980), with support $\{0, 1, \ldots\}$, are also mixed Poisson distributions. Their distributions are of the form

$$\mathbb{P}_{\alpha,\beta,\rho}\{X = x\} = \alpha(1-\rho)^{\alpha} \frac{\Gamma(\alpha+\beta x)}{x!\,\Gamma(\alpha+(\beta-1)x+1)} (\rho(1-\rho)^{\beta-1})^{x}, \quad x = 0, 1, \ldots, \tag{3.6}$$

with $\alpha > 0$, $\rho \in (0, 1)$ and $\beta \in [1, \rho^{-1}]$ (see Remark 9).

Theorem 5. *The generalized negative binomial distributions in* (3.6) *are mixed Poisson distributions*.

Proof. The case of $\beta = 1$ is trivial since, in this case, the distributions given in (3.6) reduce to negative binomial with parameter vector (α, ρ) (i.e., a mixed Poisson distribution with gamma mixing distribution).

If $\beta \in (1, \rho^{-1}]$, essentially, by the property that each gamma random variable has its logarithm to be self-decomposable (see, e.g., Shanbhag and Sreehari, 1977) which, in turn, is a corollary to Lemma 1, and that

$$\mathbb{E}(e^{-xV_{\alpha,\beta}}) = \frac{\alpha}{\alpha+(\beta-1)x}, \quad x = 0, 1, \ldots,$$

where $V_{\alpha,\beta}$ as an exponential random variable with mean $(\beta-1)/\alpha$, we get that there exists a positive random variable $Y_{\alpha,\beta,\rho}$ such that

$$\mathbb{E}(Y^{x}_{\alpha,\beta,\rho}) = \frac{\Gamma(\alpha+\beta x)}{\Gamma(\alpha+\beta^{-1}(\beta-1)\beta x)} \frac{\alpha}{(\alpha+(\beta-1)x)} (\rho(1-\rho)^{\beta-1})^{x}, \quad x = 0, 1, \ldots\ . \tag{3.7}$$

In view of (3.7), (3.6) implies that

$$\mathbb{P}_{\alpha,\beta,\rho}\{X = x\} = (1-\rho)^{\alpha}\, \mathbb{E}(Y^{x}_{\alpha,\beta,\rho}/x!), \quad x = 0, 1, \ldots \tag{3.8}$$

and, summing (3.8) over $x \in \{0, 1, \ldots\}$, we get then that

$$(1-\rho)^{\alpha}\, \mathbb{E}(e^{Y_{\alpha,\beta,\rho}}) = 1. \tag{3.9}$$

From (3.8) and (3.9), we can then conclude that

$$\begin{aligned}\mathbb{P}_{\alpha,\beta,\rho}\{X = x\} &= \mathbb{E}\left\{ e^{-Y_{\alpha,\beta,\rho}} \frac{Y^{x}_{\alpha,\beta,\rho}}{x!} e^{Y_{\alpha,\beta,\rho}} \right\} \Big/ \mathbb{E}\{e^{Y_{\alpha,\beta,\rho}}\} \\ &= \int_{(0,\infty)} e^{-\lambda} \frac{\lambda^{x}}{x!} B_{\alpha,\beta,\rho}(d\lambda), \quad x = 0, 1, \ldots,\end{aligned}$$

where $B_{\alpha,\beta,\rho}$ is the d.f. given for all $\lambda \in (0,\infty)$ by

$$B_{\alpha,\beta,\rho}(\lambda) = \frac{\mathbb{E}\{e^{Y_{\alpha,\beta,\rho}} \mathbb{I}_{\{Y_{\alpha,\beta,\rho} \leqslant \lambda\}}\}}{\mathbb{E}\{e^{Y_{\alpha,\beta,\rho}}\}}.$$

Hence, we have that Theorem 5 holds. □

Remark 9. In the statistical literature, the cases $\eta = 1$ in (3.3) and $\beta = \rho^{-1}$ in (3.6) are usually left out since in these cases even though the corresponding distributions are proper, they have infinite means. (A relevant observation in this matter appears in Remark 13.) However, as the presence of these cases do not affect the arguments that we have used to prove Theorems 4 and 5, we have opted in this article for the modified versions of the relevant classes of distributions, incorporating the cases referred to. Incidentally, for $\eta \in (0,\infty)$ and $\beta \in (1,\infty)$, the distributions met here can in fact be viewed as those corresponding to the lengths of wet periods initiated by random contents that are, respectively, Poisson (θ) and negative binomial (α, ρ), in certain infinite-capacity discrete dams with the output parameter equal to 1 and the input distributions to be, respectively, Poisson (η)

and negative binomial $(\beta-1,\rho)$. (If the initial content is equal to 0, we consider the length in that case to be equal to 0; also, note that the dams that we have referred to here are to be assumed essentially as versions of those introduced by Moran, 1959.)

Remark 10. Using an appropriate limiting argument, we can arrive at the part for $\eta \in (0,1]$ (i.e., the crucial part) of Theorem 4 as a corollary to Theorem 5 by substituting θ/α for ρ and $\alpha\eta/\theta$ for β, respectively, in (3.6) and letting $\alpha \to \infty$; that the limiting distribution in this case is such that the corresponding mixing distribution (is absolutely continuous or, in particular) has 0 as its continuity point follows trivially from (3.4).

Remark 11. From an observation in Fosam and Shanbhag (1997, p. 176), it is clear that the distributions defined by (3.3) and (3.6) are Poisson (for (3.3) with $\eta=0$) or compound negative binomial (in other cases), and hence infinitely divisible; further details in this connection, which are also of relevance to the observations in Remark 9, are provided in Remark 12. Consequently, in view of a general result in Shanbhag (1977b), these distributions can be seen to have characterizations under damage model set-up of the type met in Rao and Rubin (1964), provided we take appropriate survival distributions.

Remark 12. Given a probability generating function (p.g.f.) f with $f(0)\in(0,1)$ and $f'(1-)\leqslant 1$, the first observation in Remark 9 of Biggins and Shanbhag (1981) implies that the p.g.f. g satisfying $g(s)=f(sg(s))$, $s\in[-1,1]$ (met essentially in storage theory and branching processes) is compound geometric or, equivalently, corresponds to a distribution (on $\{0,1,\ldots\}$) that is proportional to a renewal sequence. Hence, it follows that for every $c\in(0,\infty)$, g^c, with g as stated, is a compound negative binomial p.g.f. and by, e.g., Corollary 4.2 in Letac and Mora (1990), corresponds to the distribution $\{[c/(c+j)]p_j^{(c+j)} : j=0,1,\ldots\}$, where for each $r\in(0,\infty)$ and $j\in\{0,1,\ldots,\}$, $p_j^{(r)}$ denotes the coefficient of s^j in the powerseries expansion of $(f(s))^r$ in an appropriate neighbourhood of the origin (e.g., in $(-s_0,s_0)$ with $s_0=\sup\{s\in(0,1] : (f(s)-f(0))/f(0)<1\}$). The partial result that the g referred to is infinitely divisible, implied here, also follows from Dwass (1968) or Corollary 4.2 in Letac and Mora (1990). Moreover, we can now see that (3.3) with $\eta\neq 0$ refers to the d.f. corresponding to the p.g.f. g^c with $c=\theta/\eta$ and f as the p.g.f. of a Poisson random variable with mean η, and (3.6) with $\beta\neq 1$ refers to the distribution corresponding to the p.g.f. g^c with $c=\alpha/(\beta-1)$ and f as the p.g.f. of a negative binomial random variable with parameter vector $(\beta-1,\rho)$.

Remark 13. To supplement the information on g in Remark 12, we have the following: (i) In view of the functional equation relative to g, one can obtain inductively, under appropriate assumptions, for each $c\in(0,\infty)$, the moments of g^c in terms of those of f; in particular, essentially as seen in the literature, from the equation, we get the mean, m_c, of g^c here as

$$m_c=\begin{cases} cm(1-m)^{-1} & \text{if } m\in(0,1),\\ \infty & \text{if } m=1,\end{cases}$$

where $m=f'(1-)$. (ii) Because of the revelations made in Remark 15 appearing in Section 4.1, if f is Poisson or negative binomial (with obviously mean in $(0,1]$), it follows that g is mixed geometric and hence, with the mixing distribution itself as a scale-mixture of exponential distributions, is mixed Poisson. Consequently, by the Goldie–Steutel theorem mentioned in Remark 5, we get that, in this case, for each $c\in(0,\infty)$ there exists a positive infinitely divisible random variable X_c such that

$$(g(s))^c=\mathbb{E}(\mathrm{e}^{-1(1-s)X_c}),\quad s\in[0,1],$$

implying that the assertions of Theorems 4 and 5, additionally with the respective mixing distributions as infinitely divisible, are valid. (That the assertions referred to above are valid in the cases $\eta=0$ and $\beta=1$, respectively, follows trivially.)

4. Relevant observations on Steutel and van Harn (2004)

The present section is devoted to making some observations on Steutel and van Harn (2004), of relevance to certain results that we have met in Section 3 of the present article, as well as to some results given in Steutel (1970).

4.1. Theorems involving exponential distributions as corollaries to those involving geometric distributions

Theorem VI.3.13 in Steutel and van Harn (2004) asserts that the ch.f., ϕ, of the form given below is infinitely divisible

$$\phi(t)=\alpha+\beta\int_{(0,\infty)}\frac{\lambda}{\lambda+\mathrm{i}t}G_1(\mathrm{d}\lambda)+\gamma\int_{(0,\infty)}\frac{\lambda}{\lambda-\mathrm{i}t}G_2(\mathrm{d}\lambda),\quad t\in\mathbb{R}, \tag{4.1}$$

where $\alpha,\beta,\gamma\in[0,1]$ with $\alpha+\beta+\gamma=1$ and G_1 and G_2 are d.f.'s concentrated on $(0,\infty)$, and also notes that this is equivalent to stating that if Z and Y are independent random variables with Y as exponential, then ZY is infinitely divisible.

To verify that the theorem referred to above holds, it is obviously sufficient if one proves its first assertion; it can now be seen that the latter is a simple corollary to Theorem VI.7.10 of Steutel and van Harn (2004). Define then, for each $n \in \{1, 2, \ldots\}$,

$$\phi_n(t) = \alpha + \beta \int_{(0,\infty)} \frac{\lambda}{\lambda + n(1 - \mathrm{e}^{-\mathrm{i}t/n})} G_1(\mathrm{d}\lambda) + \gamma \int_{(0,\infty)} \frac{\lambda}{\lambda + n(1 - \mathrm{e}^{\mathrm{i}t/n})} G_2(\mathrm{d}\lambda), \quad t \in \mathbb{R}, \tag{4.2}$$

where $\alpha, \beta, \gamma, G_1$ and G_2 are as in (4.1). Theorem VI.7.10 in Steutel and van Harn (2004) implies that each ϕ_n is infinitely divisible. By the Lebesgue dominated convergence theorem, we have that the sequence $\{\phi_n : n = 1, 2, \ldots\}$ converges pointwise to ϕ and, hence, by the closure property of the class of infinitely divisible distributions (under weak convergence), it follows that ϕ is infinitely divisible.

Thus, it is clear that Theorem VI.7.10 in Steutel and van Harn (2004) implies its Theorem VI.3.13. (Note that the same argument as above essentially shows that Theorem VI.7.9 in Steutel and van Harn, 2004 implies its Theorem VI.3.10.)

Remark 14. Goldie (1967) proves implicitly that any nondegenerate mixture of the degenerate distribution at the origin and geometric distributions is proportional (on its support) to a Kaluza sequence, and hence that it is a compound geometric distribution; for the relevant information on Kaluza sequences, see Kingman (1972, Section 1.5) and also Shanbhag (1977a). Further results or observations in this connection appear in Steutel (1970) and Steutel and van Harn (2004, Chapter VI); in particular, Theorem VI.7.9 of Steutel and van Harn (2004) implies that any mixture of the degenerate distribution at the origin and geometric distributions is infinitely divisible. It may be pointed out in this place that the observation that we have made in Remark 12, involving an earlier observation of Biggins and Shanbhag (1981), compares well with the result that we have attributed above to Goldie (1967).

Remark 15. Indeed, it is true that in the cases of $\eta = \theta$ and $\beta - 1 = \alpha$, the distributions given by (3.3) and (3.6), respectively, are mixed geometric. To see this, it is sufficient to verify that the sequences

$$\left\{\frac{(1+x)^{x-1}}{x!} : x = 0, 1, \ldots\right\} \quad \text{and} \quad \left\{(\beta - 1)\frac{\Gamma(\beta(x+1) - 1)}{x!\Gamma((\beta - 1)(x+1) + 1)} : x = 0, 1, \ldots\right\}$$

are moment sequences of distributions that are concentrated on $(0, \infty)$. However, since

$$\frac{(1+x)^{x-1}}{x!} = \frac{(1+x)^{x+1}}{\Gamma(x+2)} \frac{1}{x+1}, \quad x = 0, 1, \ldots$$

and

$$\frac{(\beta - 1)\Gamma(\beta(x+1) - 1)}{x!\Gamma((\beta - 1)(x+1) + 1)} = \frac{(\beta - 1)}{(\beta - 1) + \beta x} \frac{\beta^{-1}\Gamma(\beta(x+1) + 1)}{\Gamma((x+1) + 1)\Gamma((x+1)(\beta - 1) + 1)}, \quad x = 0, 1, \ldots,$$

especially by the specialized version of Theorem 1 for $c = \gamma = 1$ (implied also by the prior literature as observed in Remark 4) and Eq. (3.6) in Shanbhag et al. (1977) (with simple notational alterations), in conjunction with the relevant information on the moment sequence of e^{-Y}, where Y as exponential, it is now obvious that the two sequences referred to are as required. Consequently, we can now claim that the assertion on (3.3) and (3.6) is valid, from which it follows that, in each of the cases on hand, we have the sequence $\{\mathbb{P}\{X = x\} : x = 0, 1, \ldots\}$ to be completely monotone and, hence, proportional to a Kaluza sequence; partial information in this connection appears also in Example V.9.13 of Steutel and van Harn (2004).

4.2. Proofs based on the Wiener–Hopf factorization

We may point out here that Theorems VI.3.13 and VI.7.10 in Steutel and van Harn (2004) have links with the Wiener–Hopf factorization met in the theory of random walk. In view of what we have already revealed in Section 4.1, it is sufficient then if we show that this is so for Theorem VI.7.10 in Steutel and van Harn (2004).

Theorem VI.7.10 in Steutel and van Harn (2004) follows readily, in view especially of the closure property of the class of infinitely divisible distributions, if it is shown that for given $n \in \{1, 2, \ldots\}$, each m.g.f. M of the following form is infinitely divisible:

$$M(t) = \left(\sum_{i=1}^{2}\sum_{j=1}^{n} c_{ij}(1 - p_{ij}\mathrm{e}^{(-1)^i t})^{-1}\right) - \min\{c_1^*, c_2^*\}, \quad t \in \mathscr{D}, \tag{4.3}$$

where $c_{ij} \in (0, \infty)$, $p_{ij} \in (0, 1)$, $c_i^* = \sum_{j=1}^{n} c_{ij}$ and $\mathscr{D}$ is the domain of definition of M. (Note that we have $\mathscr{D} = (\ln p_1^*, -\ln p_2^*)$, where $p_i^* = \max_j\{p_{ij}\}$.)

Taking without loss of generality $c_1^* > c_2^*$, we can now define the m.g.f. M^* with domain of definition $\mathscr{D}$, such that

$$M^*(t) = K\left\{\left(\sum_{i=1}^{2}\sum_{j=1}^{n} c_{ij}^*(1 - p_{ij}e^{(-1)^i t})^{-1}\right) + \alpha e^t\right\}, \quad t \in \mathscr{D}, \tag{4.4}$$

where $c_{ij}^* = ((1-p_{ij})^2 c_{ij})/p_{ij}$, $\alpha = c_1^* - c_2^*$ and K is the normalizing constant (to give $M^*(0)=1$). Clearly, it is now an easy exercise to see that

$$\frac{(1 - M^*(t))}{(1-e^{-t})(1-e^t)} = KM(t), \quad t \in \mathscr{D}\backslash\{0\}. \tag{4.5}$$

In view of equation (XII.3.11) of Feller (1971), denoting the probability measure relative to M^* by $\mathbb{P}$, it is seen that

$$\mathbb{P} = \nu_1 + \nu_2 - \nu_1 * \nu_2, \tag{4.6}$$

where $\nu_r, r=1,2$, are, respectively, the weak descending and the ascending ladder height measures; note that Feller (1971) denotes the measure functions corresponding to $\mathbb{P}$, ν_1 and ν_2 by F, ρ and H, respectively. Essentially, due to the (strong) memoryless property of geometric distributions, equation (XII.3.4) for ρ, appearing in Feller (1971), and its obvious analogue for H, imply easily that ν_r, $r=1,2$, are indeed of the forms of $\mathbb{P}(\{\ldots,-1,0\}\bigcap\cdot)$ and $\mathbb{P}(\{1,2,\ldots\}\bigcap\cdot)$, respectively, but with constants c_{ij}'s and α replaced by appropriate larger real numbers. It is hence obvious that we can define functions $m_r : \mathscr{D}\mapsto(0,\infty)$, $r=1,2$, where

$$m_r(t) = \int_{\mathbb{R}} \exp\{xt\}\nu_r(dx), \quad t \in \mathscr{D}, \quad r = 1,2,$$

and observe that, by (4.6),

$$(1 - M^*(t)) = (1 - m_1(t))(1 - m_2(t)), \quad t \in \mathscr{D}. \tag{4.7}$$

Appealing to (4.5) and (4.7), since $KM(0+) \in (0,\infty)$, it then follows that for some positive constants b_{1j}, $j=1,2,\ldots,n$, and b_{2j}, $j=0,1,\ldots,n$,

$$\begin{aligned} KM(t) &= \left(\frac{1-m_1(t)}{1-e^{-t}}\right)\left(\frac{1-m_2(t)}{1-e^t}\right) \\ &= \left\{\sum_{j=1}^{n} b_{1j}(1-p_{1j}e^{-t})^{-1}\right\}\left\{b_{20} + \sum_{j=1}^{n} b_{2j}(1-p_{2j}e^t)^{-1}\right\}, \quad t \in \mathscr{D}\backslash\{0\}. \end{aligned} \tag{4.8}$$

As observed in Remark 13, essentially Goldie (1967) implies that each mixture of the degenerate distribution at the origin and geometric distributions on $\{0,1,\ldots\}$ is infinitely divisible or, indeed, compound geometric and, consequently, from (4.8), we can conclude that M, or the corresponding distribution, is infinitely divisible. Hence, we have the validity of our claim.

Remark 16. From (4.8), it is implicit that ν_r, $r=1,2$, are both probability measures; on noting that the mean corresponding to M^* is zero, one can also verify directly from (4.7), or from Theorem XII.2.2 of Feller (1971), that the measures referred to are so. (Incidentally, one may view $m_r, r=1,2$, defined above as the restrictions to $\mathscr{D}$ of the m.g.f.'s of the relevant probability measures.) Also, it is worth pointing out in this place that although our argument for proving Theorem VI.7.10 of Steutel and van Harn (2004) relies upon certain nontrivial properties of ladder heights, it still helps one to explain the underlying mechanism of the theorem better.

Remark 17. With obvious modifications to the arguments employed above, one can easily produce a direct proof based on the Wiener–Hopf factorization (without involving Theorem VI.7.10 in Steutel and van Harn, 2004) for Theorem VI.3.13 in Steutel and van Harn (2004), and hence for Theorem 2.3.1 in Steutel (1970).

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