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Moment properties of multivariate infinitely divisible laws and criteria for multivariate self-decomposability

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ABSTRACT

Ramachandran (1969) [9, Theorem 8] has shown that for any univariate infinitely divisible distribution and any positive real number α, an absolute moment of order α relative to the distribution exists (as a finite number) if and only if this is so for a certain truncated version of the corresponding Lévy measure. A generalized version of this result in the case of multivariate infinitely divisible distributions, involving the concept of g-moments, was given by Sato (1999) [6, Theorem 25.3]. We extend Ramachandran's theorem to the multivariate case, keeping in mind the immediate requirements under appropriate assumptions of cumulant studies of the distributions referred to; the format of Sato's theorem just referred to obviously varies from ours and seems to have a different agenda. Also, appealing to a further criterion based on the Lévy measure, we identify in a certain class of multivariate infinitely divisible distributions the distributions that are self-decomposable; this throws new light on structural aspects of certain multivariate distributions such as the multivariate generalized hyperbolic distributions studied by Barndorff-Nielsen (1977) [12] and others. Various points relevant to the study are also addressed through specific examples.

1. Introduction

Infinite divisibility and their specialized versions, namely, self-decomposability and stability, have generated considerable interest among specialists in probability and statistics. There is huge literature devoted to studies of these topics. Books such as Loève [1], Linnik [2], Feller [3] and Lukacs [4] have been instrumental in providing the audience with the basic material on these. More recent monographs such as Bondesson [5], Sato [6] and Steutel and van Harn [7] have unified and studied further contributions to the expanding literature in this connection.

In view of Kendall and Stuart [8, chapter 3], in which the relations between moments and cumulants are addressed in detail, it follows, under appropriate conditions, that the cumulants corresponding to infinitely divisible distributions that exist have some appealing features and have links with certain moments of Lévy and Kolmogorov measures relative to these distributions. Ramachandran [9, Theorem 8] has shown, in the univariate case, that for an infinitely divisible distribution the existence of the absolute moment of order $\alpha \in (0, \infty)$ is equivalent to the existence of its analogue for a certain truncated version of the corresponding Lévy measure. (By a truncated version of a measure ν on $\mathbb{R}^p$ ($p \geq 1$) we mean the restriction of ν to some proper subset of $\mathbb{R}^p$.) This result plays a crucial role in studies related to cumulants of infinitely divisible distributions, see, e.g., [10] and [7, Chapter IV, Section 7]. Sato [6, Theorem 25.3] has given a multivariate extension of Theorem 8 of [9], involving the so-called g-moments. However, it appears that Theorem 25.3 of [6] is not tailored to meet the immediate needs for cumulant studies.

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In a recent expository article, Gupta et al. [11] have unified the literature on infinitely divisible distributions with special reference to moments and cumulants. In the process of doing this, they have made several illuminating observations on the behavior of cumulants of univariate and multivariate infinitely divisible distributions, and have presented some new results in the area. Gupta et al. [11] also poses an open problem on a multivariate extension of Theorem 8 of [9]. One of the main tasks of the present article is to deal with this problem; the problem that we have referred to here is of particular interest, especially if one is concerned with aspects of cumulants of multivariate infinitely divisible distributions. Interestingly, as a by-product of our solution to the problem, it follows that for any infinitely divisible distribution on $\mathbb{R}^p$ ($p \geq 1$), under a mild assumption, (in standard notation) the cumulant $k_{r_1,\ldots,r_p}$ exists if the moment $\mu_{r_1,\ldots,r_p}$ exists (as a real number); in the univariate case, obviously, this result holds without requiring the distribution to be infinitely divisible.

In a somewhat different direction, there are questions relative to structural aspects of the multivariate hyperbolic distributions of Barndorff-Nielsen [12] and their extensions with densities given by Eq. (7.3) in the cited reference; each of the distributions referred to here is indeed (in the notation of Barndorff-Nielsen [12]) a mixture of $N_n(\mu + u\beta\Delta, u\Delta)$ with respect to u, where u follows a certain generalized inverse Gaussian distribution and μ, β and Δ are fixed with Δ nonsingular. (The extended versions have been termed the generalized hyperbolic distributions, especially in the univariate case by, e.g., Halgreen [13, p. 14] and Jørgensen [14, p. 37]). As claimed by Shanbhag and Sreehari [15, p. 24], there exist members in the class of multivariate generalized hyperbolic distributions that are not self-decomposable. Specific examples illustrating that this is so can be found in, e.g., Pestana [16, p. 54] and Rao and Shanbhag [17, Example 3.3, Remark 3.4]; Rao and Shanbhag [17] consists of further information on the problem telling us, amongst other things, that there exist members also in the smaller class of multivariate hyperbolic distributions that are not self-decomposable. In this article, we attempt a comprehensive solution to a characterization problem that is linked with the question on the structural aspects of multivariate hyperbolic and multivariate generalized hyperbolic distributions being addressed.

The paper is organized as follows. In Section 2, a generalization of Theorem 8 of [9] to the case of multivariate distributions is provided in conjunction with several relevant observations and pertinent examples. In Section 3, a characterization theorem, based on the property of self-decomposability, is established for a certain class of mixtures of multivariate distributions, and its implications are emphasized. As mentioned before, the results of Section 2 are of importance in cumulant studies and the results of Section 3 throw further light on the structural aspects of multivariate generalized hyperbolic distributions and some related distributions.

2. Criteria based on Lévy measure for the existence of moments for multivariate infinitely divisible distributions

From [18], or any other appropriate source such as Feller [3, XVII.11] or Sato [6, Theorem 8.1], it follows that ϕ is the characteristic function (ch.f.) of an infinitely divisible (i.d.) distribution on $\mathbb{R}^p$ ($p \geq 1$) if and only if it is of the form

$$\phi(\boldsymbol{t}) = \exp\left\{\mathrm{i}\langle \boldsymbol{a}, \boldsymbol{t}\rangle - \frac{1}{2}Q(\boldsymbol{t}) + \int_{\mathbb{R}^p}\left(\mathrm{e}^{\mathrm{i}\langle \boldsymbol{t},\boldsymbol{x}\rangle} - 1 - \frac{\mathrm{i}\langle \boldsymbol{t},\boldsymbol{x}\rangle}{1+\|\boldsymbol{x}\|^2}\right)\mathrm{d}\nu(\boldsymbol{x})\right\}, \quad \boldsymbol{t} \in \mathbb{R}^p, \tag{2.1}$$

where $\boldsymbol{a}$ is a real vector, Q is a nonnegative definite quadratic form, and ν is a measure (referred to as Lévy measure) on the Borel σ-field of $\mathbb{R}^p$ such that $\nu(\{\boldsymbol{0}\}) = 0$ and

$$\int_{\mathbb{R}^p} \frac{\|\boldsymbol{x}\|^2}{1+\|\boldsymbol{x}\|^2}\,\mathrm{d}\nu(\boldsymbol{x}) < \infty. \tag{2.2}$$

It is easily seen that (2.2) is equivalent to the condition that

$$\int_{\mathbb{R}^p} (\min\{\|\boldsymbol{x}\|, \tau\})^2\ \mathrm{d}\nu(\boldsymbol{x}) < \infty \quad \text{for any fixed } \tau \in (0,\infty). \tag{2.3}$$

(Here, $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ denote respectively the usual inner product and usual norm on $\mathbb{R}^p$.)

As observed by Gupta et al. [11], using essentially the approach of Loève [1, Complement 9, p. 332], with $\tau \in (0,\infty)$, we can rewrite (2.1) as

$$\begin{aligned}\phi(\boldsymbol{t}) &= \exp\left\{\mathrm{i}\langle \boldsymbol{b}, \boldsymbol{t}\rangle - \frac{1}{2}Q(\boldsymbol{t}) + \int_{\mathbb{R}^p}\left(\mathrm{e}^{\mathrm{i}\langle \boldsymbol{t},\boldsymbol{x}\rangle} - 1 - \mathrm{i}\langle \boldsymbol{t},\boldsymbol{x}\rangle\right)\mathrm{d}\nu_2(\boldsymbol{x})\right\}\\ &\quad\times \exp\left\{-\nu_1(\mathbb{R}^p) + \int_{\mathbb{R}^p}\mathrm{e}^{\mathrm{i}\langle \boldsymbol{t},\boldsymbol{x}\rangle}\mathrm{d}\nu_1(\boldsymbol{x})\right\}, \quad \boldsymbol{t} \in \mathbb{R}^p,\end{aligned} \tag{2.4}$$

where $\nu_1(\cdot) = \nu(\{\boldsymbol{x} : \|\boldsymbol{x}\| \geq \tau\} \cap \cdot)$, $\nu_2 = \nu - \nu_1$, and (in obvious notation)

$$b_r = a_r + \int_{\mathbb{R}^p}\left(\frac{x_r\|\boldsymbol{x}\|^2}{1+\|\boldsymbol{x}\|^2}\right)\mathrm{d}\nu_2(\boldsymbol{x}) - \int_{\mathbb{R}^p}\left(\frac{x_r}{1+\|\boldsymbol{x}\|^2}\right)\mathrm{d}\nu_1(\boldsymbol{x}), \quad r = 1, 2, \ldots, p.$$

(Note that (2.2), or the equivalent condition (2.3), trivially implies that b_r's are well defined as well as that ν is a σ-finite measure and ν_1 is a finite measure.)

We begin now by giving the following theorem that extends Theorem 8 of Ramachandran [9] to the case of i.d. distributions on $\mathbb{R}^p$. The theorem is clearly in a format which makes it easily applicable in cumulant studies, especially when the constants α_r and β_r, $r = 1, 2, \ldots, p$, defined in it are integers; the expressions for cumulants corresponding to an i.d. distribution on $\mathbb{R}^p$ are related, under appropriate assumptions, to those for the moments of measure ν_1 referred to in (2.4), as observed, e.g., in Gupta et al. [11].

Theorem 1. *Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_p)$, $p \geq 1$, be a p-component i.d. random vector with ch.f. ϕ satisfying (2.1) (and hence also (2.4)), and let $\beta_r \in [0, \infty)$, $r = 1, 2, \ldots, p$. Then,*

$$\mathbb{E}\left(\prod_{r=1}^{p} |X_r|^{\alpha_r}\right) < \infty \quad \text{for all } \alpha_r \in [0, \beta_r], \quad r = 1, 2, \ldots, p, \tag{2.5}$$

if and only if

$$\int_{\mathbb{R}^p} \left(\prod_{r=1}^{p} |x_r|^{\alpha_r}\right) \mathrm{d}\nu_1(\boldsymbol{x}) < \infty \quad \text{for all } \alpha_r \in [0, \beta_r], \quad r = 1, 2, \ldots, p, \tag{2.6}$$

where $\boldsymbol{x} = (x_1, x_2, \ldots, x_p)$ and ν_1 is as in (2.4).

Proof. That (2.5) implies (2.6) is an obvious corollary to the first assertion of Theorem 5.1 of [11]. Now, to prove that (2.6) implies (2.5), we may proceed as follows. Since $\boldsymbol{X}$ is an i.d. random vector with ch.f. ϕ, in view of (2.4), we can see that ϕ is of the form

$$\phi(\boldsymbol{t}) = \phi_2(\boldsymbol{t}) \exp\{-\lambda + \lambda\, \phi_1(\boldsymbol{t})\}, \quad \boldsymbol{t} \in \mathbb{R}^p, \tag{2.7}$$

where $\lambda = \nu_1(\mathbb{R}^p)$,

$$\phi_1(\boldsymbol{t}) = \begin{cases} \lambda^{-1} \int_{\mathbb{R}^p} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{t}, \boldsymbol{x}\rangle} \mathrm{d}\nu_1(\boldsymbol{x}), \quad \boldsymbol{t} \in \mathbb{R}^p, & \text{if } \lambda > 0, \\ 1, \quad \boldsymbol{t} \in \mathbb{R}^p, & \text{if } \lambda = 0 \end{cases}$$

and

$$\phi_2(\mathbf{t}) = \exp\left\{\mathrm{i}\langle \boldsymbol{b}, \boldsymbol{t}\rangle - \frac{1}{2} Q(\boldsymbol{t}) + \int_{\mathbb{R}^p} \left(\mathrm{e}^{\mathrm{i}\langle \boldsymbol{t}, \boldsymbol{x}\rangle} - 1 - \mathrm{i}\langle \boldsymbol{t}, \boldsymbol{x}\rangle\right) \mathrm{d}\nu_2(\boldsymbol{x})\right\}, \quad \boldsymbol{t} \in \mathbb{R}^p.$$

Essentially, adopting the approach of Feller [3, p. 534] or others such as Sato [6, Lemmas 25.6, 25.7] or Steutel and van Harn [7, Chapter IV, Section 7] given in the univariate case, it can be easily seen that the distribution corresponding to the ch.f. ϕ_2 has a (full) moment sequence and, hence, satisfies the analogue of (2.5). Since this disposes of the case of $\lambda = 0$ trivially, assume then that $\lambda > 0$, where λ is as in (2.7). From (2.7), we see, in this case, that

$$\phi(\boldsymbol{t}) = \phi_2(\boldsymbol{t}) \sum_{j=0}^{\infty} \mathrm{e}^{-\lambda} \frac{(\lambda \phi_1(\boldsymbol{t}))^j}{j!}, \quad \boldsymbol{t} \in \mathbb{R}^p, \tag{2.8}$$

which, in turn, implies (on noting, in particular, that ϕ is a mixture of ch.f.'s $\phi_2(\boldsymbol{t})(\phi_1(\boldsymbol{t}))^j$, $\boldsymbol{t} \in \mathbb{R}^p$, for $j \in \{0, 1, \ldots\}$, where the mixing distribution is Poisson with mean λ) in view of Fubini's theorem that, for each $\alpha_r \in [0, \beta_r]$, $r = 1, 2, \ldots, p$,

$$\mathbb{E}\left(\prod_{r=1}^{p} |X_r|^{\alpha_r}\right) = \sum_{j=0}^{\infty} \mathrm{e}^{-\lambda} \frac{\lambda^j}{j!} \mathbb{E}\left(\prod_{r=1}^{p} \left|X_{2r} + \sum_{k=1}^{j} X_{1r}^{(k)}\right|^{\alpha_r}\right), \tag{2.9}$$

where $(X_{21}, X_{22}, \ldots, X_{2p})$ and $(X_{11}^{(k)}, X_{12}^{(k)}, \ldots, X_{1p}^{(k)})$, $k = 1, 2, \ldots, j$, are mutually independent random vectors with the first one having ch.f. ϕ_2 and each of the remaining ones having ch.f. ϕ_1. (In (2.9) we adopt the convention that the summations with respect to k equal 0 if $j = 0$.)

Assume now that (2.6) holds. In view of (2.9) and (2.6), it easily follows that, for each $\alpha_r \in [0, \beta_r]$, $r = 1, 2, \ldots, p$,

$$\begin{aligned} \mathbb{E}\left(\prod_{r=1}^{p} |X_r|^{\alpha_r}\right) &\leq \sum_{j=0}^{\infty} \mathrm{e}^{-\lambda} \frac{\lambda^j}{j!} (j+1)^{\left(\sum_{r=1}^{p} \alpha_r\right)} \mathbb{E}\left\{\prod_{r=1}^{p} \sum_{k=0}^{j} \left|X_{1r}^{(k)}\right|^{\alpha_r}\right\} \\ &\leq \sum_{j=0}^{\infty} \mathrm{e}^{-\lambda} \frac{\lambda^j}{j!} (j+1)^{\left(\sum_{r=1}^{p} \alpha_r\right)} \mathbb{E}\left\{\prod_{k=0}^{j} \prod_{r=1}^{p} \left(1 + \left|X_{1r}^{(k)}\right|^{\alpha_r}\right)\right\} \\ &\leq c \sum_{j=0}^{\infty} \mathrm{e}^{-\lambda} \frac{(c\lambda)^j}{j!} (j+1)^{\left(\sum_{r=1}^{p} \alpha_r\right)} < \infty, \end{aligned} \tag{2.10}$$

where

$$c = \max\left\{\mathbb{E}\left(\prod_{r=1}^{p}\left(1+\left|X_{1r}^{(1)}\right|^{\alpha_r}\right)\right), \qquad \mathbb{E}\left(\prod_{r=1}^{p}\left(1+|X_{2r}|^{\alpha_r}\right)\right)\right\} < \infty,$$

and, for notational convenience, we denote X_{2r} by $X_{1r}^{(0)}$ for each r. (We appeal here to, amongst other things, the relevant independence of the vectors concerned and the elementary inequality, for $\alpha \in [0,\infty)$, $y_k \in \mathbb{R}$, $k = 0, 1, \ldots, j$, that $|\sum_{k=0}^{j} y_k|^\alpha \le (j+1)^\alpha \max_{0\le k\le j} |y_k|^\alpha \le (j+1)^\alpha \sum_{k=0}^{j} |y_k|^\alpha$, and also view the last summation with respect to j in (2.10) as the expected value of a function of a Poisson random variable.) Obviously, (2.10) yields the validity of (2.5). This shows that (2.6) implies (2.5). Hence, Theorem 1 follows. □

The following corollary of Theorem 1 is essentially a version of Theorem 8 of [9]; the cited author considers ν_1 with "$|x| > 1$" in place of "$|x| \ge \tau$".

Corollary 1. *Let X be an i.d. random variable with ch.f. ϕ, and let $\beta \in [0,\infty)$. Then,*

$$\mathbb{E}\left(|X|^\beta\right) < \infty$$

if and only if

$$\int_{\mathbb{R}} |x|^\beta \, \mathrm{d}\nu_1(x) < \infty,$$

where ϕ and ν_1 are as in (2.4) corresponding to $p = 1$.

Proof. The corollary follows readily from Theorem 1 by taking $p = 1$ because if μ is a finite measure on the Borel σ-field of $\mathbb{R}$ then $\int_{\mathbb{R}} |x|^\beta \, \mathrm{d}\mu(x) < \infty$ if and only if $\int_{\mathbb{R}} |x|^\alpha \, \mathrm{d}\mu(x) < \infty$ for all $\alpha \in [0,\beta]$. □

Remark 1. We point out that Theorem 1 is not covered by Theorem 25.3 of Sato [6] on g-moments (relative to the case of $\tau = 1$); that this statement is true is obvious from the example (i.e., Example 4) given in Remark 5.5 of [11]. However, to illustrate that neither the function $\prod_{r=1}^{p} |x_r|^{\alpha_r}$, met in the statement of Theorem 1, nor its modified version given by the function $\prod_{r=1}^{p} |x_r|^{\alpha_r}\mathbb{I}(A)$, $\boldsymbol{x} \in \mathbb{R}^p$, $\tau > 0$, with $A = \{\boldsymbol{x} : \|\boldsymbol{x}\| \ge \tau\}$ and $\mathbb{I}(A)$ as its indicator function, is assured to be submultiplicative (where the terminology refers to that of Definition 25.2 of [6, p. 159]), we may consider the following example:

Example 1. Let $p = 2$, $\boldsymbol{x} = (x_1, x_2) \in \mathbb{R}^2$ and $\tau > 0$, and consider

$$g_1(\boldsymbol{x}) = |x_1x_2|\mathbb{I}(A) \quad \text{and} \quad g_2(\boldsymbol{x}) = |x_1x_2|, \quad \boldsymbol{x} \in \mathbb{R}^2.$$

For $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$ such that $x_1 \ge \tau$, $y_2 \ge \tau$, $x_2 = x_1^{-1}$ and $y_1 = y_2^{-1}$, we have

$$g_r(\boldsymbol{x}+\boldsymbol{y}) = |(x_1+y_1)(x_2+y_2)| > x_1y_2, \quad r = 1, 2,$$

and

$$g_r(\boldsymbol{x}) = g_r(\boldsymbol{y}) = 1, \quad r = 1, 2.$$

Consequently, it is impossible to have here for each $r \in \{1, 2\}$ a constant $a_r > 0$ such that

$$g_r(\boldsymbol{x}+\boldsymbol{y}) \le a_r\, g_r(\boldsymbol{x}) g_r(\boldsymbol{y}), \quad r = 1, 2,$$

for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^2$. This supports the claim that we have made above. (It is also now clear that the functions $\max\{|x_1x_2|, c\}$ with $c > 0$ are not submultiplicative.)

Remark 2. We may modify the example given in Remark 5.5 of [11] to shed further light on aspects of Theorem 1. In particular, we can show, with appropriate modifications to the example referred to, that if one or more of certain β_r's are positive then (2.6) with "$[0, \beta_r]$ (in respective places)" replaced by "$(0, \beta_r]$" does not imply (2.5) with the same change, and demonstrate some curious phenomena of (X_1, X_2) in this connection. That this is so, is evident from the information supplied by the following two examples:

Example 2. Let (X_1, X_2) be an i.d. random vector with ch.f. ϕ such that

$$\phi(\boldsymbol{t}) = \exp\{-\lambda + \lambda\psi(\boldsymbol{t})\}, \quad \boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2,$$

where $\lambda \in (0,\infty)$ and ψ is the ch.f. of a random vector (Y_1, Y_2) satisfying

$$(Y_1, Y_2) \stackrel{d}{=} (V, V^{-1}),$$

where V is the modulus of a standard Cauchy random variable (and "$\overset{d}{=}$" denotes the equality in distribution). Clearly, we have then for $\alpha_1, \alpha_2 \in [0, 1]$ such that $0 \le |\alpha_1 - \alpha_2| < 1$, and hence, whenever α_1, α_2 both lie in $(0, 1]$ (or both lie in $[0, 1)$),

$$\int_{\mathbb{R}^2} |x_1|^{\alpha_1} |x_2|^{\alpha_2} \, \mathrm{d}\nu_1(\boldsymbol{x}) < \int_{(0,\infty)^2} x_1^{\alpha_1} x_2^{\alpha_2} \, \mathrm{d}\nu(\boldsymbol{x}) = \lambda \, \mathbb{E}\left(V^{|\alpha_1 - \alpha_2|}\right) < \infty.$$

However, in this case, we have

$$\mathbb{E}(|X_1 X_2|) = \mathbb{E}(X_1 X_2) = \infty \quad \text{and} \quad \mathbb{E}(|X_r|) = \mathbb{E}(X_r) = \infty, \quad r = 1, 2,$$

violating the condition that $\mathbb{E}(|X_1|^{\alpha_1} |X_2|^{\alpha_2}) < \infty$ for all $\alpha_1, \alpha_2 \in (0, 1]$. (Appealing to Theorem 1, we can also see in this case that $\mathbb{E}(|X_1|^{\alpha_1} |X_2|^{\alpha_2}) < \infty$ for each $\alpha_1, \alpha_2 \in [0, 1)$.)

Example 3. Let (X_1, X_2) be as in Example 2 with the exception that ψ in this case refers to the ch.f. of a random vector (Y_1, Y_2) satisfying

$$(Y_1, Y_2) \overset{d}{=} (V_\gamma^{\gamma}, V_\gamma^{-\delta}),$$

with $\gamma \in (0, 1)$, $\delta \in (0, \infty)$ and V_γ as a positive stable random variable with left extremity (i.e., $\inf\{x : F_{V_\gamma}(x) > 0\}$, where F_{V_γ} denotes the distribution function (d.f.) of V_γ) zero and characteristic exponent γ. It now follows that, for $\alpha_1, \alpha_2 \in [0, 1]$ with $\alpha_2 \neq 0$,

$$\int_{\mathbb{R}^2} |x_1|^{\alpha_1} |x_2|^{\alpha_2} \, \mathrm{d}\nu_1(\boldsymbol{x}) < \int_{(0,\infty)^2} x_1^{\alpha_1} x_2^{\alpha_2} \, \mathrm{d}\nu(\boldsymbol{x}) = \lambda \, \mathbb{E}\left(V_\gamma^{(\alpha_1 \gamma - \alpha_2 \delta)}\right) < \infty.$$

(That the expectation appearing above is finite is seen, e.g., from Bondesson [5, p. 85] or from Steutel and van Harn [7, p. 246].) Also, it is clear that, in this case, X_1 and X_2 are nonnegative random variables such that $\mathbb{E}(X_1) = \mathbb{E}(X_1 X_2) = \infty$, with X_2 possessing a (full) moment sequence.

The following theorem enables us to understand the mechanism of Theorem 1 better; the theorem addresses, amongst other things, the problem posed in Remark 5.6 of [11], and its proof that we have produced here is adapted partially from the proof of Theorem 5.1 of [11].

Theorem 2. *Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_p)$ and β_r, $r = 1, 2, \ldots, p$, be as in Theorem 1, but for that there is an additional restriction now that $\mathbb{P}(X_r = 0) < 1$, $r = 1, 2, \ldots, p$. Then, (2.5) is equivalent to the condition that*

$$\mathbb{E}\left(\prod_{r=1}^{p} |X_r|^{\beta_r}\right) < \infty. \tag{2.11}$$

Moreover, we now have

$$\mathbb{P}\left\{\prod_{r=1}^{p} |X_r| > 0\right\} > 0. \tag{2.12}$$

Proof. Trivially, the theorem is true for $p = 1$. We shall follow the method of induction with respect to p to prove it, noting that each subvector of $\boldsymbol{X}$ is i.d. Assume then that $p > 1$ and the theorem holds in the case when p', with $p' \in \{1, 2, \ldots, p-1\}$, appears in place of p in (2.11) and (2.12). The theorem is clearly valid if $\boldsymbol{X}$ is p-variate normal and hence it is sufficient if we show that the theorem holds when ν is non-null. Consequently, we can assume, without loss of generality, that $\nu_1(\mathbb{R}_+^p) > 0$, where $\mathbb{R}_+ = [0, \infty)$, and observe that the theorem follows if, under the assumption, it is proved just that (2.12) is valid and, in view of Theorem 1 and the prevailing symmetry, that (2.11) implies (2.6) with $\mathbb{R}^p$ replaced by $\mathbb{R}_+^p$.

Let $\lambda = \nu_1(\mathbb{R}_+^p) > 0$; then (2.4) implies that, for $n = 1, 2, \ldots$, we have p-variate ch.f.'s $\phi_2^{(n)}$ and $\phi_3^{(n)}$ such that

$$\phi(\boldsymbol{t}) = \gamma_n \, \phi_1(\boldsymbol{t}) \phi_2^{(n)}(\boldsymbol{t}) + (1 - \gamma_n) \, \phi_3^{(n)}(\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{R}^p, \tag{2.13}$$

where

$$\phi_2^{(n)}(\boldsymbol{t}) = \phi_2^{(1)}(\boldsymbol{t}) \, (\phi_1(\boldsymbol{t}))^{n-1}, \quad \boldsymbol{t} \in \mathbb{R}^p,$$

$$\phi_2^{(1)}(\boldsymbol{t}) = \frac{\phi(\boldsymbol{t})}{\exp\{-\lambda + \lambda \, \phi_1(\boldsymbol{t})\}}, \quad \boldsymbol{t} \in \mathbb{R}^p,$$

$$\phi_1(\boldsymbol{t}) = \lambda^{-1} \int_{\mathbb{R}_+^p} \mathrm{e}^{\mathrm{i}\langle \boldsymbol{t}, \boldsymbol{x} \rangle} \, \mathrm{d}\nu_1(\boldsymbol{x}), \quad \boldsymbol{t} \in \mathbb{R}^p,$$

with $\gamma_n = \mathrm{e}^{-\lambda} \lambda^n / n!$. (Note that λ and ϕ_1 considered here are not implied to be the same as those met in the proof of Theorem 1.)

Letting $(X_{11}, X_{12}, \ldots, X_{1p})$ and $(X_{21}^{(n)}, X_{22}^{(n)}, \ldots, X_{2p}^{(n)})$, $n = 1, 2, \ldots$, denote independent random vectors with ch.f.'s ϕ_1 and $\phi_2^{(n)}$, $n = 1, 2, \ldots$, respectively, we get from (2.13) that

$$\mathbb{E}\left(\prod_{r=1}^{p} |X_r|^{\beta_r}\right) \geq \gamma_n\, \mathbb{E}\left(\prod_{r=1}^{p} \left|X_{1r} + X_{2r}^{(n)}\right|^{\beta_r}\right), \quad n = 1, 2, \ldots, \tag{2.14}$$

and (noting especially that, for each n, $\phi_2^{(n+1)}(\boldsymbol{t}) = \phi_1(\boldsymbol{t})\phi_2^{(n)}(\boldsymbol{t})$, $t \in \mathbb{R}^p$)

$$\mathbb{P}\left\{\prod_{r=1}^{p} |X_r| > 0\right\} \geq \gamma_n\, \mathbb{P}\left\{\prod_{r=1}^{p} \left|X_{2r}^{(n+1)}\right| > 0\right\}, \quad n = 1, 2, \ldots. \tag{2.15}$$

If $k < p$ of the X_{1r}, $r = 1, 2, \ldots, p$, are equal to zero almost surely and the remainder satisfy the condition that $\mathbb{P}\{X_{1r} > 0\} > 0$, then we can take without loss of generality that $\mathbb{P}\{X_{11} = X_{12} = \cdots = X_{1k} = 0\} = 1$ and $\mathbb{P}\{X_{1r} > 0\} > 0$, $r = k+1, k+2, \ldots, p$; we take here $k = 0$ to mean that $\mathbb{P}\{X_{1r} > 0\} > 0$ for all $r \in \{1, 2, \ldots, p\}$. For a sufficiently large integer n_0, the distribution corresponding to $\phi_2^{(n_0)}$ has at least one support point, say $(c_1, c_2, \ldots, c_p)$ with $c_r \neq 0$ for each $r \in \{1, 2, \ldots, p\}$ and $c_r > 0$ for each $r \in \{k+1, k+2, \ldots, p\}$; to see this note that, by assumption,

$$\mathbb{P}\left\{\prod_{r=1}^{k} |X_{2r}^{(1)}| > 0\right\} > 0 \quad \text{if } k \geq 1, \tag{2.16}$$

and $\mathbb{P}\{X_{11} = X_{12} = \cdots = X_{1k} = 0\} = 1$ together with $\mathbb{P}\{X_{1r} > 0\} > 0$, $r = k+1, k+2, \ldots, p$. (Obviously, (2.16) is a consequence of the assumption in the inductive argument, especially because we have now that, for each n, $(X_{21}^{(n)}, X_{22}^{(n)}, \ldots, X_{2k}^{(n)}) \stackrel{d}{=} (X_1, X_2, \ldots, X_k)$.)

In view of the aforementioned observation on the existence of the support point $(c_1, c_2, \ldots, c_p)$ of the distribution relative to $\phi_2^{(n_0)}$ with the stated properties, it follows that

$$\mathbb{P}\{X_{2r}^{(n_0)} \in A_r,\ r = 1, 2, \ldots, p\} > 0, \tag{2.17}$$

where

$$A_r = \begin{cases} \left(\dfrac{3c_r}{2}, \dfrac{c_r}{2}\right) & \text{if } c_r < 0, \\ \left(\dfrac{c_r}{2}, \dfrac{3c_r}{2}\right) & \text{if } c_r > 0. \end{cases} \tag{2.18}$$

Consequently, by (2.15) we get that (2.12) is valid, and, by (2.14) in conjunction with (2.11), that, for some constant $\eta \in (0, \infty)$ and A_r's as in (2.18),

$$\infty > \mathbb{E}\left(\prod_{r=1}^{p} |X_r|^{\beta_r}\right) > \eta\, \mathbb{E}\left(\prod_{r=1}^{p} \left|X_{1r} + X_{2r}^{(n_0)}\right|^{\beta_r} \mid X_{2r}^{(n_0)} \in A_r,\ r = 1, 2, \ldots, p\right). \tag{2.19}$$

In view of the properties of $(X_{11}, X_{12}, \ldots, X_{1p})$ and $(X_{21}^{(n_0)}, X_{22}^{(n_0)}, \ldots, X_{2p}^{(n_0)})$, (2.19) implies then that

$$\mathbb{E}\left(\prod_{r=1}^{p} \left|X_{1r} + c_r^{\star}\right|^{\beta_r}\right) < \infty, \tag{2.20}$$

where $c_r^{\star} = |c_r|/2$, $r = 1, 2, \ldots, p$. Since, for $\alpha_r \in [0, \beta_r]$, $r = 1, 2, \ldots, p$,

$$|X_{1r}|^{\alpha_r} \leq (c_r^{\star})^{\alpha_r - \beta_r} \left|X_{1r} + c_r^{\star}\right|^{\beta_r}, \quad r = 1, 2, \ldots, p,$$

it is hence obvious from (2.20) that

$$\mathbb{E}\left(\prod_{r=1}^{p} |X_{1r}|^{\alpha_r}\right) < \infty \quad \text{for all } \alpha_r \in [0, \beta_r], \quad r = 1, 2, \ldots, p,$$

asserting that, as required for the completion of the proof of the theorem, (2.6) with $\mathbb{R}_{+}^{p}$ in place of $\mathbb{R}^p$, is met. Hence, Theorem 2 follows. □

Remark 3(i). Theorem 2 is not valid if the assumption of i.d. is dropped. This is obvious from the following examples in which we use an indirect approach based on the first assertion of Theorem 2 to ascertain that the $\boldsymbol{X}$ considered are non-i.d.:

Example 4A. A. Let V and $V^\star$ be independent random variables such that $\mathbb{E}(|V|) = \infty$ and $V^\star$ is $\{0, 1\}$-valued Bernoulli. Define $X_1 = VV^\star$ and $X_2 = V(1 - V^\star)$. Clearly, the random vector $\boldsymbol{X} = (X_1, X_2)$ is such that $\mathbb{P}\{X_r = 0\} < 1$, $r = 1, 2$, with $\mathbb{P}\{X_1X_2 = 0\} = 1$ and hence with $\mathbb{E}(|X_1X_2|) = 0 < \infty$. Also, in this case, obviously $\mathbb{E}(|X_1|) = \mathbb{E}(|X_2|) = \infty$. That $\boldsymbol{X}$ is non-i.d. and the claim of Remark 3 is valid follows then trivially from the first assertion of Theorem 2. However, it is interesting to observe that in this example, we have $\mathbb{E}(|X_1|^{\alpha_1}|X_2|^{\alpha_2}) = 0 < \infty$ for all $\alpha_1, \alpha_2 \in (0, 1]$.

Example 4B. Let $\gamma \in (0, 1)$ and $\boldsymbol{X} = (X_1, X_2)$ be a random vector such that X_1 is a positive random variable with $\mathbb{E}(X^\gamma) = \infty$ and $X_2 = X_1^{-1}$ almost surely. Then, clearly, we have $\mathbb{E}(|X_1||X_2|) = \mathbb{E}(X_1X_2) = 1 < \infty$. However, in this case, it is not even true that $\mathbb{E}(|X_1|^{\alpha_1}|X_2|^{\alpha_2}) < \infty$ for all $\alpha_1, \alpha_2 \in (0, 1)$, since $\mathbb{E}(|X_1|^{\gamma+\delta}|X_2|^{\delta}) = \mathbb{E}(X_1^\gamma) = \infty$ if $\delta \in (0, 1 - \gamma)$. That $\boldsymbol{X}$ is non-i.d. and the claim of Remark 3 is valid follows again trivially from the first assertion of Theorem 2.

Remark 3(ii). That the $\boldsymbol{X}$ vectors dealt with in Examples 4A and 4B are non-i.d. can also be shown via alternative approaches without involving the findings of Theorem 2; in the remainder of this remark, and in Remark 3(iii), we illustrate as to why this is so. Let (U, W) be a 2-component random vector with nonnegative and nondegenerate components U and W such that $\mathbb{P}\{UW = c\} = 1$ for some (nonnegative) constant c. Since, in this case, as a simple corollary to Theorem 2 of [19], it follows that the distribution of $(U, (UW)^{1/2}, W)$ is indecomposable, it is obvious then that the distribution of (U, W) is indecomposable; to see this, note, in particular, that $(UW)^{1/2}$ is degenerate. Consequently, we have the distribution of $\boldsymbol{X}$ in Example 4A in the case when V is nonnegative and that of $\boldsymbol{X}$ in Example 4B to be indecomposable and hence non-i.d.

Remark 3(iii). Let (U, W) be as in Remark 3(ii), but for a modification that U and W in this case are not necessarily nonnegative and also that c is allowed here to be negative. Applying essentially a simpler version of the argument used in [19] to prove its Theorem 2, one can see that the distribution of (U, W) is decomposable if and only if for some $b_1, b_2 \neq 0$ with $b_1^2 - 4cb_1b_2^{-1} > 0$, and some $\alpha, \beta \in (0, 1)$,

$$P\{(U, W) = \boldsymbol{x}\} = \begin{cases} \alpha\beta & \text{if } \boldsymbol{x} = (a_1, a_2b_1^{-1}b_2), \\ (1-\alpha)\beta & \text{if } \boldsymbol{x} = (a_2, a_1b_1^{-1}b_2), \\ \alpha(1-\beta) & \text{if } \boldsymbol{x} = (-a_2, -a_1b_1^{-1}b_2), \\ (1-\alpha)(1-\beta) & \text{if } \boldsymbol{x} = (-a_1, -a_2b_1^{-1}b_2), \end{cases}$$

where

$$a_1 = 2^{-1}\left(b_1 + \sqrt{b_1^2 - 4cb_1b_2^{-1}}\right) \quad \text{and} \quad a_2 = 2^{-1}\left(b_1 - \sqrt{b_1^2 - 4cb_1b_2^{-1}}\right)$$

(reducing, when $c = 0$, the assertion to that with $a_1 = b_1$ and $a_2 = 0$, respectively). (Note that the "if" part of the assertion follows easily since, under the relevant conditions, there exist independent 2-component random vectors $\boldsymbol{Y}^{(1)}$ and $\boldsymbol{Y}^{(2)}$ so that $\mathbb{P}\{\boldsymbol{Y}^{(1)} = (a_1, a_2b_1^{-1}b_2)\} = \alpha$, $\mathbb{P}\{\boldsymbol{Y}^{(1)} = (a_2, a_1b_1^{-1}b_2)\} = 1 - \alpha$, $\mathbb{P}\{\boldsymbol{Y}^{(2)} = (0, 0)\} = \beta$, $\mathbb{P}\{\boldsymbol{Y}^{(2)} = (-b_1, -b_2)\} = 1 - \beta$ and $\boldsymbol{Y}^{(1)} + \boldsymbol{Y}^{(2)}$ is distributed as (U, W).) Clearly, the characterization met here implies that the distribution of (U, W) is indecomposable if U and W are nonnegative, a result referred to in Remark 3(ii), and also that the distributions of $\boldsymbol{X}$ appearing in both Examples 4A and 4B are indeed indecomposable and hence non-i.d; to see the validity of the claim concerning the distributions of $\boldsymbol{X}$, note, in particular, that each of these examples has $\mathbb{E}(|X_1|) = \infty$ with $\boldsymbol{X}$ meeting the assumptions relative to (U, W).

Remark 4. For any p-component random vector $\boldsymbol{X} = (X_1, X_2, \ldots, X_p)$, $p > 1$, with d.f. F, irrespectively of whether or not it is i.d., (2.12) is met if and only if the support of F includes a point $(c_1, c_2, \ldots, c_p)$ so that $\prod_{r=1}^p |c_r| > 0$. Suppose now that, for some positive integer j,

$$\boldsymbol{X} \stackrel{d}{=} \sum_{n=0}^{j} \boldsymbol{Y}^{(n)},$$

where $\boldsymbol{Y}^{(n)}$, $n = 0, 1, \ldots, j$ are independent p-component random vectors such that, for each n, given any support point $(c_1, c_2, \ldots, c_p)$ of the d.f. of $\boldsymbol{Y}^{(n)}$, there exists a support point $(d_1, d_2, \ldots, d_p)$ of the d.f. of $\sum_{n'(\neq n)=0}^{j} \boldsymbol{Y}^{(n')}$, for which $\prod_{r=1}^p |d_r| \neq 0$ and $c_rd_r > 0$ for each r with $c_r \neq 0$. In this case, $\prod_{r=1}^p |c_r + d_r| > 0$, from which we can see that (2.12) is met, and, following essentially the relevant steps in the proof of Theorem 2, appearing (2.17) onwards, we can further see that (2.11) implies (2.5) with $\boldsymbol{Y}^{(n)}$ in place of $\boldsymbol{X}$ respectively for $n = 0, 1, \ldots, j$, and hence, in view of the inequality referred to under brackets below (2.10), in the proof of Theorem 1, that the first assertion of Theorem 2 holds; this is so irrespectively of whether or not $\boldsymbol{X}$ is i.d. (To see, especially, that if (2.11) holds for this $\boldsymbol{X}$, then, for each n, $\boldsymbol{Y}^{(n)}$ satisfies (2.5), it is sufficient, by symmetry, taking, in obvious notation, without loss of generality that $\mathbb{P}\{\boldsymbol{Y}^{(n)} \in (\{0\})^k \times (0, \infty)^{p-k}\} > 0$, to check that this is so for a vector with distribution the same as the conditional distribution of $\boldsymbol{Y}^{(n)}$ given that $\boldsymbol{Y}^{(n)} \in (\{0\})^k \times (0, \infty)^{p-k}$.) Consequently, we are led now to a new version of Theorem 2.

The following corollary, which is obvious from Theorems 1 and 2, presents an extended version of the first assertion (i.e., the crucial assertion) of Theorem 5.1 of [11].

Corollary 2. *Let* $\boldsymbol{X} = (X_1, X_2, \ldots, X_p)$ *and* β_r, $r = 1, 2, \ldots, p$, *be as in Theorems 1 and 2. Then, (2.11) is equivalent to (2.6).*

Remark 5. In statistical literature, whenever the covariance between two random variables is mentioned, the relevant random variables are often assumed to be square-integrable. The following simple example, which indeed is a slight variation of Example 2 met above, illustrates that the covariance may be well defined even when this standard assumption does not hold and the joint distribution of the random variables is i.d.:

Example 5. Let (X_1, X_2) be an i.d. random vector with ch.f. ϕ such that

$$\phi(\boldsymbol{t}) = \exp\{-\lambda + \lambda\psi(\boldsymbol{t})\}, \quad \boldsymbol{t} = (t_1, t_2) \in \mathbb{R}^2,$$

where $\lambda \in (0, \infty)$ and ψ is the ch.f. of a random vector (V, V^{-1}) with V as a positive random variable such that its square equals the modulus of a standard Cauchy random variable. Then, we have

$$\mathbb{E}\,(|X_1X_2|) = \mathbb{E}\,(X_1X_2) < \infty \quad \text{and} \quad \mathbb{E}\,(|X_r|) = \mathbb{E}\,(X_r) < \infty, \quad r = 1, 2.$$

However, in this case, it is obvious that $\mathbb{E}\left(X_1^2\right) = \mathbb{E}\left(X_2^2\right) = \infty$, hence we have the validity of our claim. (A closer scrutiny of the Example 5 above tells us, actually, in view of Theorems 1 and 2, something more. Indeed, the random variables X_1 and X_2 that we have considered in this example satisfy the condition that $\mathbb{E}\,(|X_1|^{\alpha_1}|X_2|^{\alpha_2}) = \mathbb{E}\left(X_1^{\alpha_1}X_2^{\alpha_2}\right) < \infty$ for each $\alpha_1, \alpha_2 \in (0, 2]$.) Incidentally, it may be worth noting here that, in view of the corollary to Theorem 2 of [19], implied in Remark 3(ii) above, any random vector relative to ch.f. ψ of the present example gives us an example of a random vector for which the conclusion of Remark 5 is valid with "i.d." replaced by "indecomposable".

3. A characteristic property relative to multivariate self-decomposability

Following Urbanik [20, p. 92], let us define a distribution on $\mathbb{R}^p$ ($p \geq 1$) to be self-decomposable (s.d.) if the corresponding ch.f. ϕ satisfies the condition that, for each $c \in (0, 1)$,

$$\phi(\boldsymbol{t}) = \phi(c\boldsymbol{t})\phi_c(\boldsymbol{t}), \quad \boldsymbol{t} \in \mathbb{R}^p, \tag{3.1}$$

where ϕ_c is a ch.f. Essentially, from Sections 8 and 11 of Chapter XVII of [3], it is then clear that if (3.1) is met then both ϕ and ϕ_c are i.d. ch.f.'s; the partial information that ϕ and ϕ_c are nonvanishing also follows, in effect, from the argument given in Lukacs [4, p. 162] to show that this is so in the univariate case. (As a by-product of this, it follows that if ϕ is nonvanishing or, in particular, i.d., then ϕ is a s.d. ch.f. if and only if $\phi(\boldsymbol{t})/\phi(c\boldsymbol{t})$, $t \in \mathbb{R}^p$, is an i.d. ch.f. for each $c \in (0, 1)$.) If a distribution on $\mathbb{R}^p$ is s.d., we refer to the corresponding ch.f., or a random vector with this distribution, also as s.d.

As pointed out in Section 1, the concept of generalized hyperbolic distributions is well documented, especially in the univariate case. Shanbhag and Sreehari [15, p. 24] have claimed that there exist members in the class of multivariate generalized hyperbolic distributions with $\beta \neq 0$ (where the distribution referred to is absolutely continuous with probability density function (p.d.f.) given by (7.3) of [12]) that are not s.d. The original example implied in the cited paper to illustrate this remained unpublished though its version was later revisited in Pestana [16, p. 54]. More recently, Rao and Shanbhag ([17], Example 3.3, Remark 3.4) have produced a simpler example in support of the claim and have made some further relevant comments on the issue. There is also Example 6.8 in [21] that enlightens one with certain related information.

Before discussing the results of the present section, we may give two further pieces of information that are of relevance to these:

(i) Although Feller [3, XII.8] is concerned with properties of s.d. distributions in the univariate case, that the properties of ϕ and ϕ_c referred to above are met in the multivariate case is essentially implied by Feller [3, XII.11]; also, it is of interest to note here that, if ϕ satisfies (3.1), then every linear combination of the components of the corresponding random vector has its ch.f. satisfying the univariate version of (3.1), implying immediately that ϕ and ϕ_c (of the multivariate case) are nonvanishing.

(ii) To be more precise, in the notation of Section 1, we refer in this paper to mixtures of $N_n(\mu + u\beta\Delta, u\Delta)$ as generalized hyperbolic if the mixing distribution relative to u is generalized inverse Gaussian, i.e., if it has p.d.f. of the form

$$f(u \mid \lambda, \xi, \psi) = C(\lambda, \xi, \psi)u^{\lambda-1}\exp\{-(\xi u^{-1} + \psi u)/2\}, \quad u > 0,$$

with $\xi, \psi \geq 0$ for which $\max\{\xi, \psi\} > 0$, C as the normalizing constant depending on the modified Bessel function of the third kind, and λ so that $\int_0^\infty u^{\lambda-1}\exp\{-(\xi u^{-1} + \psi u)/2\}\mathrm{d}u < \infty$. Obviously, the class of hyperbolic distributions is a subclass of the class of these distributions, see, e.g., [12].

The following theorem and its corollary subsume some of the major observations that are made by the examples cited above.

Theorem 3. *Let V be a positive random variable such that*

$$\mathbb{E}\left(e^{sV}\right) = \exp\left\{\int_{(0,\infty)} \left(e^{sv} - 1\right) v^{-(\alpha+1)} g(v) dv\right\}, \quad Re(s) \le 0, \tag{3.2}$$

with $\alpha \in [0, 1)$ and g as a bounded decreasing nonnegative real function on $(0, \infty)$ satisfying

$$\int_{(0,\infty)} \frac{v^{-\alpha} g(v)}{1+v} dv < \infty.$$

Also, let $\boldsymbol{X} = (X_1, X_2, \ldots, X_{p-1})$, $p \ge 2$, be a $(p-1)$-component random vector independent of V, with X_r, $r = 1, 2, \ldots, p-1$, as independent (nondegenerate) symmetric stable random variables with characteristic exponents $\gamma_r \in [1, 2]$, $r = 1, 2, \ldots, p-1$, respectively. Then, the p-component random vector

$$\boldsymbol{Z} = \left(V, V^{\frac{1}{\gamma_1}} X_1, \ldots, V^{\frac{1}{\gamma_{p-1}}} X_{p-1}\right) \tag{3.3}$$

is s.d. if and only if

$$\alpha - p + 1 + \sum_{r=1}^{p-1} \gamma_r^{-1} \ge 0. \tag{3.4}$$

Proof. In view of (3.2), it follows that the ch.f. of $\boldsymbol{Z}$ given by (3.3) is of the form

$$\phi(\boldsymbol{t}) = \exp\left\{\int_{(0,\infty)} \left(e^{it_1 v - v\sum_{r=1}^{p-1} \lambda_r |t_{r+1}|^{\gamma_r}} - 1\right) v^{-(\alpha+1)} g(v) dv\right\}, \quad \boldsymbol{t} = (t_1, t_2, \ldots, t_p) \in \mathbb{R}^p, \tag{3.5}$$

with $\lambda_r > 0$ for all $r \in \{1, 2, \ldots, p-1\}$; this is clear since the ch.f. of each symmetric stable random variable X_r, with characteristic exponent γ_r, is of the form $\phi_r(t) = \exp\{-\lambda_r |t|^{\gamma_r}\}$, $t \in \mathbb{R}$, and we have

$$\phi(\boldsymbol{t}) = \mathbb{E}\left(e^{it_1 V} \prod_{r=1}^{p-1} \mathbb{E}\left[\left(e^{it_{r+1} V^{\frac{1}{\gamma_r}} X_r}\right) \mid V\right]\right), \quad \boldsymbol{t} \in \mathbb{R}^p.$$

From (3.2), it is obvious that ϕ is i.d. Also, on appealing to Fubini's theorem, it is now clear that the Lévy measure in the present case is concentrated on $(0, \infty) \times \mathbb{R}^{p-1}$ and is absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^p$ with Radon–Nikodym derivative h such that the restriction to $(0, \infty) \times \mathbb{R}^{p-1}$ of h is given by

$$h(\boldsymbol{y}) = y_1^{-(\alpha+1)} g(y_1) \prod_{r=1}^{p-1} \left[f_r\left(y_{r+1}/y_1^{\frac{1}{\gamma_r}}\right) y_1^{-\frac{1}{\gamma_r}}\right], \quad \boldsymbol{y} = (y_1, y_2, \ldots, y_p) \in (0, \infty) \times \mathbb{R}^{p-1}, \tag{3.6}$$

where f_r denotes the p.d.f. of X_r and is implied by the inversion theorem to be bounded (continuous). Recalling then that ϕ is s.d. if and only if, for each $c \in (0, 1)$, $\phi(\boldsymbol{t})/\phi(c\boldsymbol{t})$, $\boldsymbol{t} \in \mathbb{R}^p$, is i.d., we can claim that $\boldsymbol{Z}$ is s.d. if and only if

$$h(\boldsymbol{y}) \ge c^{-p} h(\boldsymbol{y}/c), \quad \boldsymbol{y} \in (0, \infty) \times \mathbb{R}^{p-1}, \tag{3.7}$$

for all $c \in (0, 1)$. Since each symmetric stable distribution is unimodal with vertex 0 (see, e.g., Lemma 5.10.1 of [4]), (3.6) then implies that (3.7) is met for the required c and $\boldsymbol{y}$ if and only if (3.4) is valid; observe in particular that if (3.4) is not valid, then (3.7) is violated for $\boldsymbol{y}$ with $y_{r+1} = o(y_1^{1/\gamma_r})$, $r = 1, 2, \ldots, p-1$, and y_1 sufficiently small. Hence, Theorem 3 follows. □

Remark 6. In view of (3.2) directly, or, trivially, as a corollary to Theorem 3, it follows that the random variable V considered in Theorem 3 is s.d. Positive stable (with left extremity zero), gamma and inverse Gaussian random variables provide us with some specialized versions of V met here; in these cases, we have $g(v) \propto e^{-\lambda v}$ with $\lambda = 0$ for stable and $\lambda > 0$ otherwise, and, also, have the parameter α respectively as positive, equal to 0, and equal to 1/2. Also, in view of the closure property (under weak convergence) of the class of s.d. distributions on $\mathbb{R}^p$, as a corollary to Theorem 3, it now follows that the "if" part of the theorem referred to remains valid without the assumption that g be bounded.

Corollary 3. *Let V and X_r, $r = 1, 2, \ldots, p-1$, be as in Theorem 3, and, additionally, suppose that we have $p \ge 3$ and $\alpha - p + 2 + \sum_{r=1}^{p-2} \gamma_r^{-1} < 0$. Then, there exist real c_r, $r = 1, 2, \ldots, p-1$, such that the $(p-1)$-component random vector $\boldsymbol{W}$ given by*

$$\boldsymbol{W} = \left(c_1 V + V^{\frac{1}{\gamma_1}} X_1, \ldots, c_{p-1} V + V^{\frac{1}{\gamma_{p-1}}} X_{p-1}\right)$$

is not s.d.

Proof. Clearly, it follows via a standard argument that the sequence of the random vectors $\{\boldsymbol{W}_n : n = 1, 2, \ldots\}$, where, for each $n \geq 1$,

$$\boldsymbol{W}_n = \left(\frac{1}{n}V + V^{\frac{1}{\gamma_1}}X_1, \ldots, \frac{1}{n}V + V^{\frac{1}{\gamma_{p-2}}}X_{p-2}, V + \frac{1}{n}V^{\frac{1}{\gamma_{p-1}}}X_{p-1}\right),$$

converges in probability and hence in distribution to the random vector $\boldsymbol{W}^\star$ given by

$$\boldsymbol{W}^\star = \left(V^{\frac{1}{\gamma_1}}X_1, \ldots, V^{\frac{1}{\gamma_{p-2}}}X_{p-2}, V\right).$$

Since, by Theorem 3, we have that $\boldsymbol{W}^\star$ is not s.d., appealing to the closure property (under weak convergence) of the class of s.d. distributions on $\mathbb{R}^{p-1}$ ($p \geq 3$), we can readily conclude that, for some $n \geq 1$, $\boldsymbol{W}_n$ is not s.d. This, in turn, implies that the assertion of the corollary is true; note that if, for some n, W_n is not s.d., then so also is $(\frac{1}{n}V + V^{\frac{1}{\gamma_1}}X_1, \ldots, \frac{1}{n}V + V^{\frac{1}{\gamma_{p-2}}}X_{p-2}, nV + V^{\frac{1}{\gamma_{p-1}}}X_{p-1})$. Hence, Corollary 3 follows. □

Remark 7. Although, the problem of finding non-s.d. multivariate hyperbolic and multivariate generalized hyperbolic distributions of Barndorff-Nielsen [12], touched upon in this article, prompted us to establish Theorem 3, that this latter theorem is of interest in its own right, is obvious. However, it may be worth emphasizing here that Corollary 3, which is a corollary to Theorem 3, identifies certain members of the class of multivariate generalized hyperbolic distributions of [12], that are non-s.d.; this follows on noting especially that the specialized version of the corollary in the case of $\alpha \in \{0, 1/2\}$, $g(v) = \exp\{-\lambda v\}$, $v > 0$, and $\gamma_1 = \gamma_2 = \cdots \gamma_{p-1} = 2$, concerns these distributions. The examples of non-s.d. distributions given by Corollary 3 are obviously in the spirit of those discussed earlier in Pestana [16, p. 54] and Rao and Shanbhag [17, Example 3.3 & Remark 3.4, pp. 2882–2883].

Corollary 4. *Given an integer $p \geq 2$, there exists a p-component random vector of the form of (3.3) that is non-s.d. such that all its lower dimensional subvectors are s.d.*

Proof. Given $p \geq 2$, choose, e.g., $\gamma \in (1, 2]$ and $\alpha \in [0, 1)$ such that $\alpha = (p-2)(1-\gamma^{-1})$ and, hence, satisfying also that $\alpha - (p-1)(1-\gamma^{-1}) < 0$. Then, Theorem 3 implies that in the special case of $\gamma_r = \gamma$, $r = 1, 2, \ldots, p-1$, with α as stated, the random vector $\boldsymbol{Z}$ is not s.d., but, for each $r \in \{2, 3, \ldots, p\}$, the $(p-1)$-component subvector of $\boldsymbol{Z}$ that does not include the rth component of $\boldsymbol{Z}$ is s.d. Also, since the specialized version in this case of $\boldsymbol{Z}$ with its first component deleted satisfies (3.1) trivially for each $c \in (0, 1)$, it is obvious that this is s.d. Hence we have the corollary. □

Corollary 5. *If $\alpha = 0$, the random vector $\boldsymbol{Z}$ is s.d. if and only if $\gamma_1 = \gamma_2 = \cdots = \gamma_{p-1} = 1$, i.e., if and only if $X_1, X_2, \ldots, X_{p-1}$ are Cauchy (up to scale changes) random variables.*

Proof. The result follows trivially from Theorem 3; also, the "if" part of the assertion is immediate on noting that the concerned ch.f. satisfies (3.1) for each $c \in (0, 1)$. □

Remark 8. Clearly, any random vector $\boldsymbol{Z}^\star$ is i.d. if its ch.f. is of the form

$$v_\alpha^\star\left(it_1 - \sum_{r=1}^{p-1}\lambda_r|t_{r+1}|^{\gamma_r}\right), \quad \boldsymbol{t} = (t_1, t_2, \ldots, t_p) \in \mathbb{R}^p,$$

with, $p > 1$, $\alpha \in [0, 1)$, $\lambda_r > 0$ and $\gamma_r \in (0, 2]$ (i.e., p, α, λ_r and γ_r are as in Theorem 3 but for that γ_r is now allowed to lie in $(0, 1)$), and $v_\alpha^\star$ as the function defined by the left hand side of (3.2); note that $\boldsymbol{Z}^\star \stackrel{d}{=} \boldsymbol{Z}$, where $\boldsymbol{Z}$ is as in (3.3) if $\gamma_r \in [1, 2]$, $r = 1, 2, \ldots, p-1$. Moreover, essentially as in Example 6.8 of [21], it is now seen that each (one-dimensional) projection of $\boldsymbol{Z}^\star$ is s.d. if $\gamma_1 = \gamma_2 = \cdots = \gamma_{p-1} = 2$ and, additionally, g in (3.2) is completely monotone, or, equivalently, by Bernstein's theorem, denotes the Laplace transform of a measure on the Borel σ-field of $\mathbb{R}_+$; this follows since in the present case also we have the distribution of V to be a member of the generalized gamma convolution family of Bondesson [5], on observing that if g is a Laplace transform as above, then so also is $v^{-\alpha}g(v)$, $v \in (0, \infty)$. However, by Theorem 3, we have, in this case, obviously $\boldsymbol{Z}^\star$ (and hence $\boldsymbol{Z}$) to be non-s.d., unless $p = 2$ and $\alpha \in [1/2, 1)$.

Remark 9. The example provided in the proof of Corollary 4 gives us in particular the existence of yet another type of i.d. distribution on $\mathbb{R}^p$ ($p \geq 3$) that is non-s.d., of which each projection is s.d. This, in conjunction with the information provided by Corollary 4 and Remark 8, compares well with certain findings of Lévy [22] and Shanbhag [23]; the results in the cited references show us, amongst other things, that there exist indecomposable distributions on $\mathbb{R}^p$ ($p = 2, 3$) with all marginals (univariate or otherwise) and projections i.d. Further material of relevance to the findings referred to here has appeared or been cited in, e.g., [24–27,19,28–30].

Remark 10. Since the class of distributions on $\mathbb{R}^p$ that are s.d. is closed under convolution, from an observation in Remark 3.4 of Rao and Shanbhag [17, p. 2883], it is clear that any ch.f. of the form

$$v_\alpha^\star\left(\mathrm{i}c_1t_1+\mathrm{i}c_2t_2-\lambda_1t_1^2-\lambda_2t_2^2\right),\quad \boldsymbol{t}=(t_1,t_2)\in\mathbb{R}^2,$$

with $v_\alpha^\star$ as in Remark 8, $(c_1,c_2)\neq(0,0)$ and $\lambda_1,\lambda_2>0$, is s.d., provided that $\alpha\in[1/2,1)$ and g is completely monotone; to see this, use the fact that, in this case, $v^{-(\alpha-1/2)}g(v)$, $v\in(0,\infty)$, is completely monotone. The cited remark of Rao and Shanbhag ([17]) also gives some further relevant information on the subject.

Remark 11. In view of Sato [6, Example 25.10, pp. 162–164], by (3.6), it follows by Fubini's theorem (in the notation of Theorem 3) that if $\alpha-p+1+\sum_{r=1}^{p-1}\gamma_r^{-1}<0$, then, for each $\beta\in(0,p-1-\alpha-\sum_{r=1}^{p-1}\gamma_r^{-1})$ and $\theta\in(\alpha/(\alpha+\beta),1)$, we have a function G on $(0,\infty)$ such that for all $x\in(0,\infty)$,

$$G(x)=\int_{(0,x]}y_1^{\left(\alpha+\beta+\sum\limits_{r=1}^{p-1}\gamma_r^{-1}\right)\theta}\left(\prod_{r=1}^{p-1}|y_{r+1}|^{-\theta}\right)\mathrm{d}\nu(\boldsymbol{y})<\infty,\tag{3.8}$$

where ν is the Lévy measure relative to the distribution of $\boldsymbol{Z}$ (with $\boldsymbol{Z}$ as in (3.3)). For each $c\in(0,1)$, in obvious notation, the analogue $G^{(c)}$ of G with respect to ch.f. $\phi(c\boldsymbol{t})$, $\boldsymbol{t}\in\mathbb{R}^p$, can easily be seen to be given by $c^{(\alpha+\beta-p+1+\sum_{r=1}^{p-1}\gamma_r^{-1})\theta}G(x/c)$, $x\in(0,\infty)$, with G as in (3.8); obviously, since, for each $c\in(0,1)$ and $x>0$, $G^{(c)}(x)>G(x)$, it then follows that, for none of $c\in(0,1)$, we have, in this case, $\phi(\boldsymbol{t})/\phi(c\boldsymbol{t})$, $\boldsymbol{t}\in\mathbb{R}^p$, to be i.d. Consequently, we have now an alternative argument for proving the "only if" part of Theorem 3.

Remark 12. Suppose, we define (in the notation in (3.2))

$$\theta^\star=\sup\left\{\theta>0:\int_{[1,\infty)}v^{\theta-\alpha-1}g(v)\mathrm{d}v<\infty\right\}.$$

Then, by Corollary 1, and the result from Sato [6, Example 25.10, pp. 162–164], we have, in view of Fubini's theorem, that, for each $(\alpha_1,\alpha_2,\ldots,\alpha_p)$, $p\geq 2$, such that $\alpha_r\in(-1,\gamma_{r-1}^\star)$, $r=2,3,\ldots,p$, and $\alpha_1+\sum_{r=2}^{p}\alpha_r\gamma_{r-1}^{-1}\in[0,\theta^\star)$,

$$\mathbb{E}\left(Z_1^{\alpha_1}\prod_{r=2}^{p}|Z_r|^{\alpha_r}\right)<\infty,$$

where $\boldsymbol{Z}=(Z_1,Z_2,\ldots,Z_p)$, $p\geq 2$, as in (3.3) but for a modification that, in this case, γ_r, $r=1,2,\ldots,p-1$, $p\geq 2$, are allowed to be less than 1, and $\gamma_r^\star$, $r=1,2,\ldots,p-1$, are so that $\gamma_r^\star=\gamma_r$ if $\gamma_r<2$ and $\gamma_r^\star=\infty$ if $\gamma_r=2$. In this remark, as in Remark 11, we have come across arguments essentially in the spirit of those met in Section 2 of the paper.

Remark 13. Theorem 3 does not hold if the assumption that $\gamma_r\in[1,2]$, $r=1,2,\ldots,p-1$, is dropped. This is obvious from the following example:

Example 6. Let $\{\phi_n:\ n=1,2,\ldots\}$ be a sequence of i.d. ch.f's on $\mathbb{R}^3$, such that

$$\phi_n(\boldsymbol{t})=v_{\alpha_0}^\star\left(\mathrm{i}t_1-t_2^2-\frac{1}{n}|t_3|^\gamma\right),\quad \boldsymbol{t}=(t_1,t_2,t_3)\in\mathbb{R}^3,$$

with $v_\alpha^\star$ as in Remark 8, $\alpha_0\in[0,1/2)$ and $\gamma\in(0,2/(3-2\alpha_0))$. Clearly, the sequence $\{\phi_n\}$ converges to a ch.f. on $\mathbb{R}^3$, which, by Theorem 3, is not s.d.; note that the limiting random vector (Y_1,Y_2,Y_3) in this case is so that the ch.f. of (Y_1,Y_2) equals $v_{\alpha_0}^\star(\mathrm{i}t_1-t_2^2)$, $(t_1,t_2)\in\mathbb{R}^2$, which is indeed non-s.d. by Theorem 3. Consequently, appealing to the closure property (under weak convergence) of the class of s.d. distributions on $\mathbb{R}^3$, we can claim that there exist, for some large n, ch.f.'s ϕ_n that are not s.d. on $\mathbb{R}^3$ despite the fact that $\alpha_0-2+2^{-1}+\gamma^{-1}>0$.

Remark 14. Applying a result of Zolotarev [31] (which has appeared also as Theorem 5.8.4 of [4]), in conjunction with a standard result stating that each stable distribution is unimodal (an obvious consequence of a result of [32] stating that each s.d. distribution is unimodal), one can easily see, with appropriate scrutiny of Lévy measures, that there exist i.d. ch.f.'s on $\mathbb{R}^2$ of the form

$$\phi(\boldsymbol{t})=v_\alpha^\star\left(\mathrm{i}t_1-\lambda_1|t_2|^\gamma\right),\quad \boldsymbol{t}=(t_1,t_2)\in\mathbb{R}^2,$$

with $v_\alpha^\star$ as in Remark 8, $\lambda_1>0$ and $\gamma\in[1/2,1)$, that are not s.d. for certain α and g (e.g., if $\alpha<1-\gamma$ and g is identically equal to a constant). This sheds further light on the observation of Remark 13.

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