SUPPLEMENT TO THE ARTICLE

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Minimax Signal Detection in Ill-Posed Inverse Problems¶

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6. Illustrative examples. Here, we briefly mention some illustrative examples arising in various scientific fields that lead to the GSM (1.2) (or the equivalent model (1.3)). The various models presented below are scattered throughout the literature, see, e.g., [1], [2], [3], [5], [6] and [12].

- *differentiation* ($f \in L^2([0,1])$, periodic on $[0,1]$, $\{\varphi_k\}_{k\in\mathbb{N}}$ being the complex trigonometric system on $[0,1]$). The goal is to detect or estimate the m-th derivative $f(t) = g^{(m)}(t)$ (for some $m \in \mathbb{N}$), based on the observation of a trajectory $\{Y_\varepsilon = Y_\varepsilon(t)\}$, $t \in [0,1]$, obeying the Gaussian white noise model (1.1) with $D = [0,1]$, $\mathcal{H} = \{f : f \in L^2([0,1]), \int_0^1 f(t)dt = 0\}$, $\mathcal{H} \subset L^2([0,1])$ and $Af(t) = Ag^{(m)}(t) = g(t)$. This problem corresponds to a mildly ill-posed inverse problem since $b_k \to 0$ (or, equivalently, $\sigma_k \to \infty$) polynomially (with $\beta = m$) fast as $k \to \infty$.
- *the Dirichlet problem of the Laplacian on the unit circle* ($f \in L^2([0,2\pi])$, periodic on $[0,2\pi]$, $\{\varphi_k\}_{k\in\mathbb{N}}$ being the trigonometric system on $[0,2\pi]$). The goal is to detect or estimate the boundary condition f based on the observation of a trajectory $\{Y_\varepsilon = Y_\varepsilon(\varphi)\}$, $\varphi \in [0,2\pi]$, obeying the Gaussian white noise model (1.1) with φ in place of t, $D = [0,2\pi]$, $\mathcal{H} = L^2([0,2\pi])$ and $Af(t) = u(r_0,\varphi)$, where $u(r,\varphi)$, $r \in [0,1]$, $\varphi \in [0,2\pi]$, is the solution of the Dirichlet problem of the Laplacian on the unit circle in polar coordinates with boundary condition $u(1,\varphi) = f(\varphi)$. This problem corresponds to a severely ill-posed inverse problem since $b_k \to 0$ (or, equivalently, $\sigma_k \to \infty$) exponentially fast as $k \to \infty$.

¶Some of the numbering that appears in this supplement corresponds to numbering used in the article, Sections 1-5. Also, the references cited herein refer to the corresponding references cited in the article. The extra references used, that are not cited in the article, are given at the end of this supplement.

- *the heat conductivity equation* ($f \in L^2([0,1])$, periodic on $[0,1]$, $\{\varphi_k\}_{k\in\mathbb{N}}$ being the complex trigonometric system on $[0,1]$). The goal is to detect or estimate the initial condition f based on the observation of a trajectory $\{Y_\varepsilon = Y_\varepsilon(x)\}$, $x \in [0,1]$, obeying the Gaussian white noise model (1.1) with x in place of t, $D=[0,1]$, $\mathcal{H} = L^2([0,1])$ and $Af(t) = u(T,x)$, where $u(t,x)$, $t>0$, $x\in[0,1]$, is the solution of the heat conductivity equation with periodic boundary conditions and initial condition $u(0,x) = f(x)$. This problem corresponds to an extremely ill-posed inverse problem since $b_k \to 0$ (or, equivalently, $\sigma_k \to \infty$) power-exponentially fast (with $\gamma = 2$) as $k\to\infty$.
- *deconvolution* ($f \in L^2([0,1])$, periodic on $[0,1]$, $\{\varphi_k\}_{k\in\mathbb{N}}$ being the complex trigonometric system on $[0,1]$). The goal is to detect or estimate the response function f based on the observation of a trajectory $\{Y_\varepsilon = Y_\varepsilon(t)\}$, $t\in[0,1]$, obeying the Gaussian white noise model (1.1) with $D=[0,1]$, $\mathcal{H} = L^2([0,1])$ and $Af(t) = (g\star f)(t) = \int_0^1 f(u)g(t-u)du$, i.e., A is the convolution operator on $L^2([0,1])$, where the unknown kernel (or blurring function) $g\in L^2([0,1])$ is also periodic on $[0,1]$. This problem corresponds to a mildly, severely or extremely ill-posed inverse problem, depending on the decay of $b_k = |\nu_k|$ to zero as $k\to\infty$, where ν_k, $k\in\mathbb{N}$, are the Fourier coefficients of g.

REMARK 6.1. We mention that the GSM (1.2) can also arise in computerized tomography, see, e.g., [1], [2]. However, the methods used in this Supplement to study asymptotics in the considered ill-posed inverse problems cannot be directly applied to deal with minimax hypothesis testing in this particular problem (for the explanation, see Remark 4.2 in [9]). Arguments and techniques used to tackle this problem and to derive analogous results with the ones obtained in this article and the Supplement, were specifically developed in [9], for $q=2$.

7. Mildly ill-posed inverse problems with the class of analytic functions. (*A family of asymptotically minimax adaptive consistent tests of simple structure.*) A family of asymptotically minimax consistent tests of simple structure, that is also adaptive, in the sense that it does not depend on the unknown parameters α, β and $q\in(0,2]$, is constructed as follows.

Let a compact set $\Sigma = \{(\alpha,\beta)\} \subset \mathbb{R}^2_+$ be given. Denote by $\Theta_{\varepsilon,\alpha,\beta}(r)$ the set under the alternative given by (2.4) with $r = r_\varepsilon(\alpha,\beta)$ and $q = 2$. Let $u_{\varepsilon,\alpha,\beta}(r)$ be the value of the extreme problem (3.1) for the set $\Theta_\varepsilon = \Theta_{\varepsilon,\alpha,\beta}(r)$. Observe that, for $a_k = \exp(\alpha k)$ and $\sigma_k = k^\beta$, $k \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$, and for ε small enough, in view of (4.7),

$$c\log(\varepsilon^{-1}) \le m_\varepsilon(\alpha,\beta) \le C\log(\varepsilon^{-1}),$$

as

$$\sup_{(\alpha,\beta)\in\Sigma} |\log(u_{\varepsilon,\alpha,\beta}(r_\varepsilon(\alpha,\beta)))| = o(\log(\varepsilon^{-1})),$$

where the constant c and C satisfy

$$0 < \max_{(\alpha,\beta)\in\Sigma} \alpha^{-1} < C, \quad 0 < c < \min_{(\alpha,\beta)\in\Sigma} \alpha^{-1}.$$

Set

$$u_\varepsilon(\Sigma) = \inf_{(\alpha,\beta)\in\Sigma} u_{\varepsilon,\alpha,\beta}(r_\varepsilon(\alpha,\beta)). \tag{7.1}$$

THEOREM 7.1. *Let $a_k = \exp(\alpha k)$ and $\sigma_k = k^\beta$, $k \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$. Consider the GSM (2.2) and the hypothesis testing problem (2.3) where $\Theta_{\varepsilon,\alpha,\beta}(r)$ denotes the set under the alternative given by (2.4) with $r = r_\varepsilon(\alpha,\beta)$ and $q = 2$, and let $r_\varepsilon(\alpha,\beta)$ be taken in such way that $u_\varepsilon(\Sigma)$ in (7.1) satisfies $u_\varepsilon(\Sigma) \to \infty$. Then, the family of tests $\psi_{\varepsilon,H} = 1\!\!1_{\{t_{\varepsilon,\tilde m} > H\}}$ with $t_{\varepsilon,m}$ given by (5.1) and $\tilde m = C\log(\varepsilon^{-1}) + O(1) \in \mathbb{N}$, $\tilde m \ge C\log(\varepsilon^{-1})$, is adaptive and asymptotically minimax consistent, i.e., $\alpha_\varepsilon(\psi_{\varepsilon,H}) \to 0$ as $H \to \infty$ and one can take $H = H_\varepsilon \to \infty$ such that $\beta_\varepsilon(\psi_{\varepsilon,H}, \Theta_{\varepsilon,\alpha,\beta}(r_\varepsilon(\alpha,\beta))) \to 0$, uniformly over $(\alpha,\beta) \in \Sigma$. In view of the embedding (4.8), these hold true uniformly in $q \in (0,2]$.*

The proof is given in Section 11.8.

REMARK 7.1. As stated in Theorem 7.1, the family of tests $\psi_{\varepsilon,H=1\!\!1_{\{t_{\varepsilon,\tilde m} > H\}}}$ with $t_{\varepsilon,m}$ given by (5.1) and $\tilde m = C\log(\varepsilon^{-1}) + O(1) \in \mathbb{N}$ is adaptive with respect to the unknown parameters α, β and $q \in (0,2]$. Furthermore, there is no price to pay for this adaptation, meaning that both non-adaptive and adaptive procedures for the considered ill-posed problem share, asymptotically, the same separation rates. However, this is usually an exception to the rule and in most cases there is a price to pay for the adaptation which is usually appear in the form of an extra log-log factor in the separation rates. The study of adaptivity in the remaining considered ill-posed problems and the construction of appropriate rate optimal families of tests is the theme of the article, Section 5.

8. Mildly ill-posed inverse problems with the Sobolev class of functions. (*A family of asymptotically minimax consistent tests of simple structure.*) If $u_\varepsilon \to \infty$, then one can construct a family of asymptotically minimax consistent tests of simpler structure than (3.3). Indeed, observe that, by (11.10), one has

$$r_\varepsilon a_{\tilde m} \sim c_1^\alpha,\ c_1 = c_1(\alpha,\beta) > 1, \quad u_\varepsilon \asymp \frac{r_\varepsilon^2}{\varepsilon^2\sqrt{\tilde m}\sigma_{\tilde m}^2}, \quad \tilde m = [m] \in \mathbb{N} \tag{8.1}$$

where $[m]$ is the integral part of m. Hence, for an integer-valued family $\tilde m = \tilde m_\varepsilon \to \infty$, one has

$$a_{\tilde m+1}\, r_\varepsilon \geq B + o(1), \quad B > 1, \quad u_\varepsilon \asymp \frac{r_\varepsilon^2}{\varepsilon^2\sqrt{\tilde m}\sigma_{\tilde m}^2}. \tag{8.2}$$

For each $m \in \mathbb{N}$, consider the following families of test statistics and tests

$$t_{\varepsilon,m} = \frac{1}{\sqrt{2m}}\sum_{k=1}^m ((y_k/\varepsilon)^2 - 1), \quad \psi_{\varepsilon,H} = 1\!\!\mathrm{I}_{\{t_{\varepsilon,m} > H\}}. \tag{8.3}$$

THEOREM 8.1. *Consider the GSM (2.2) and the hypothesis testing problem (2.3)-(2.4) with $q = 2$. Let $a_k = k^\alpha$ and $\sigma_k = k^\beta$, $k \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$. Let the value u_ε of the extreme problem (3.1) be determined by (11.10), and assume that $u_\varepsilon \to \infty$. Then, the family of tests (8.3) with $m = \tilde m$ satisfying (8.2) and $H = H_\varepsilon \to \infty$, is asymptotically minimax consistent, i.e., $\alpha_\varepsilon(\psi_{\varepsilon,H_\varepsilon}) \to 0$ and there exists $c > 0$ such that $\beta_\varepsilon(\psi_{\varepsilon,H_\varepsilon}, \Theta_\varepsilon) \to 0$ as $H_\varepsilon < (c + o(1))u_\varepsilon$.*

The proof is given in Section 11.4.

9. Extremely ill-posed inverse problems with the class of generalized analytic functions. (*A family of asymptotically minimax consistent tests of simple structure.*) The family of tests given by (3.3) are determined by the sequence $\{w_k\}_{k\in\mathbb{N}}$ given by (3.2) and are rather complicate. Furthermore, as revealed in Remark 4.13, the condition $w_0 = o(1)$ does not hold under assumption (4.10) and, hence, these families of tests are rate optimal only. We describe below another rate optimal family of tests that is of simpler structure.

This procedure is determined by a family $m = m(r_\varepsilon)$ such that $r_\varepsilon \in \Delta_m^* = [1/a_m, 1/a_{m-1}]$, $m \in \mathbb{N}$, $m \geq 2$. Take $\alpha \in (0,1)$ small enough and consider the collection $T_{m,k},\ 1 \leq k \leq m$, such that

(9.1)
$$T_{m,m} = T_{m,m-1} = \Phi^{-1}((1-\alpha/6)), \quad T_{m,k} = \Phi^{-1}((1-c\alpha/(m-k-1)^2)),$$

where c is taking in such way that $\sum_{k=1}^{m-2} k^{-2} = 1/(6c)$. (Note that this yields $\sum_{k=1}^{m} \Phi(-T_{m,k}) = \alpha/2$.)

Consider now the following families of events and tests

$$\mathcal{Y}_{\varepsilon,\alpha} = \{y : |y_k| > \varepsilon\, T_{m,k},\ k = 1,2,\ldots,m\}, \quad \psi_{\varepsilon,\alpha} = \mathbb{1}_{\mathcal{Y}_{\varepsilon,\alpha}}. \tag{9.2}$$

THEOREM 9.1. *Consider the GSM (2.2) and the hypothesis testing problem (2.3)-(2.4) with $q = 2$. Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{\sigma_k\}_{k\in\mathbb{N}}$ be increasing sequences satisfying (4.10). Then, the family of tests given by (9.2) with the collection $T_{m,k}$, $1 \leq k \leq m$ described by (9.1) is asymptotically minimax consistent, i.e., $\alpha_\varepsilon(\psi_{\varepsilon,\alpha}) \leq \alpha$ and one can take $u_\varepsilon^{lin} = u_\varepsilon^{lin}(r_\varepsilon) \to \infty$ such that $\beta_\varepsilon(\psi_{\varepsilon,\alpha}, \Theta(r_\varepsilon)) \to 0$.*

The proof is given in Section 11.10.

REMARK 9.1. It is evident that the statement $\alpha_\varepsilon(\psi_{\varepsilon,\alpha}) \leq \alpha$ in Theorem 9.1 holds uniformly for each ε small enough such that $r_\varepsilon a_2 \leq 1$. From the proof of Theorem 9.1, it is also evident that this statement does not depend on the assumption (4.10).

10. Possible extensions. One can consider a wider class of sets under the alternative, than the one given by (2.4), i.e., the set under the alternative be determined by the following conditions

$$\Theta_\varepsilon = \Theta_{pq}(r_\varepsilon) \overset{\Delta}{=} \{\eta \in l^2 \;:\; \sum_{k\in\mathbb{N}} |a_k\sigma_k\eta_k|^q \le 1,\; \sum_{k\in\mathbb{N}} |\sigma_k\eta_k|^p \ge r_\varepsilon^p\} \tag{10.1}$$

(in other words, we separate the alternative hypothesis from the null not in the l^2-norm but in the l^p-norm), where $0 < p, q < \infty$, with standard modifications for the case $p = \infty$ and/or $q = \infty$. In what follows, we discuss appropriate methods for the study of these problems and mention some theoretical results.

Observe that for $p = q = 2$, the proof of Theorem 4.1 (see Section 11.1) is based on convexity arguments after the transform $u_k = |\eta_k|^2$, $k \in \mathbb{N}$. A similar property holds true for the cases $p \in (0, 2],\ q \ge p$, after the transform $u_k = |\eta_k|^p,\ v,\ k \in \mathbb{N}$. In particular, if there exists an extreme sequence in the extreme problem (3.1), then this determines sharp or rate asymptotics, and centered and the normalized (under H_0) weighed χ^2 test statistic (3.3) determines sharp- or rate-minimax tests. However, the extreme problem is more complicated for $p \ne 2, q \ne p$. For the polynomial case $a_k = k^\alpha,\ \sigma_k = k^\beta$, $k \in \mathbb{N}$, $\alpha, \beta > 0$, and the exponential case $a_k = e^{\alpha k},\ \sigma_k = e^{\beta k}$, $k \in \mathbb{N}$, $\alpha, \beta > 0$, similar and related problems were studied in [11], Section 4.3.

However, if either $p > 2$ or $p > q$ (for instance, if $p = 2 > q$, as in Section 4.7), then the extreme problem (3.1) is not reduced to a convex problem. Below, we shortly describe the key ideas in order to get a convex problem (see [11], Chapters 5 and 6, for more details).

Setting $v_k = \eta_k/\varepsilon$, $k \in \mathbb{N}$, $V_\varepsilon = \{\eta/\varepsilon,\ \eta \in \Theta_\varepsilon\}$, $x = y/\varepsilon$, let us try to find the asymptotically least favorable priors of product type $\pi^\varepsilon(dv) = \prod_{k\in\mathbb{N}} \pi_{\varepsilon,k}(dv_k)$ that correspond to a sequence of priors $\bar\pi_\varepsilon = \{\pi_{\varepsilon,k}\}$ on $\mathbb{R}$. Here, we consider a sequence $\bar\pi$ as elements of the linear space of singed measures with Hilbertian structure generated by the positive semi-definite bilinear form

$$(\bar\pi, \bar r) = \sum_{k\in\mathbb{N}} (\pi_k, r_k); \quad (\pi, r) = \int_{\mathbb{R}}\int_{\mathbb{R}} (e^{ts} - 1)\pi(dt)r(ds),$$

which is generated by the likelihood ratio. Namely, let $\pi = \pi(dt), r = r(ds)$ be one-dimensional measures and let $P_\pi,\ P_r$ be the mixtures of one-dimensional Gaussian measures $P_t = N(t,1)$, $t \in \mathbb{R}$, i.e., $P_\pi(dx) = \int P_t(dx)\pi(dt)$ (and similarly for P_r). One can easily see that

$$(\pi, r) = E_0\left(\Big(\frac{dP_\pi}{dP_0} - 1\Big)\Big(\frac{dP_r}{dP_0} - 1\Big)\right).$$

We factorize this space by the equivalence $a\delta_0 \sim 0$, $a \in \mathbb{R}$. The bilinear form is positive-defined on the factor-space. Let Π be the set of sequences $\bar\pi$ that consists of probability measures. One can identify this set with the set of corresponding cosets in the factor space, see [11], Section 3.3.3, for more details.

For the product prior $\pi(dv) = \prod_{k\in\mathbb{N}} \pi_k(dv_k)$, $\bar\pi \in \Pi$, we have the following inequality for the variation distance

$$\begin{aligned}
\Big(E_0\Big|\frac{dP_\pi}{dP_0} - 1\Big|\Big)^2 &\le E_0\Big(\frac{dP_\pi}{dP_0} - 1\Big)^2 = E_0\Big(\frac{dP_\pi}{dP_0}\Big)^2 - 1\\
&= \prod_{k\in\mathbb{N}}\left(E_0\Big(\frac{dP_{\pi_k}}{dP_0} - 1\Big)^2 + 1\right) - 1\\
&\le \exp\left(\sum_{k\in\mathbb{N}}\Big(\frac{dP_{\pi_k}}{dP_0} - 1\Big)^2\right) - 1 = \exp(\|\bar\pi\|^2) - 1.
\end{aligned}$$

This yields that the Hilbertian norm $\|\bar\pi\|$ determines the non-distinguishability conditions in the Bayesian problem, and if the priors π^ε are asymptotically supported on V_ε (i.e., $\pi^\varepsilon(V_\varepsilon) \to 1$), then the relation $\|\bar\pi_\varepsilon\| \to 0$ yields non-distinguishability in the minimax problem. One can show that, under suitable assumptions on the sequence $\bar\pi_\varepsilon$, the quantity $\|\bar\pi_\varepsilon\|$ determines the parameters of the asymptotic normality in the Bayesian log-likelihood ratio for the priors π^ε, i.e., $\log(dP_{\pi^\varepsilon}/dP_0) \sim \mathcal{N}(-\|\bar\pi_\varepsilon\|^2/2, \|\bar\pi_\varepsilon\|^2)$ under P_0-probability. These yield the following asymptotic lower bounds for the error probabilities in the Bayesian problem (testing P_0 against P_{π^ε})

$$\beta_\varepsilon(\alpha) \ge \Phi(H^{(\alpha)} - \|\bar\pi_\varepsilon\|) + o(1), \quad \gamma_\varepsilon \ge 2\Phi(-\|\bar\pi_\varepsilon\|/2) + o(1) \tag{10.2}$$

(with similar inequalities for the minimax problem if $\pi^\varepsilon(V_\varepsilon) \to 1$). Therefore, in order to get "asymptotically best" lower bounds, we have to minimize $\|\bar\pi_\varepsilon\|$ over the sets of sequences $\bar\pi_\varepsilon$ such that product priors π^ε are asymptotically supported on V_ε.

Introduce now the set

$$\Pi_\varepsilon = \{\bar\pi \in \Pi : \sum_{k\in\mathbb{N}} \sigma_k^p E_{\pi_k}|t|^p \geq (r_\varepsilon/\varepsilon)^p,\ \sum_{k\in\mathbb{N}} a_k^q \sigma_k^q E_{\pi_k}|t|^q \leq \varepsilon^{-q}\},$$

with standard modifications for the case $p=\infty$ and/or $q=\infty$ (observe that Π_ε is a convex set), and consider the following extreme problem, analogous to (3.1),

$$u_\varepsilon = \inf_{\bar\pi\in\Pi_\varepsilon} \|\bar\pi\|. \tag{10.3}$$

Often, one can take product priors $\tilde\pi^\varepsilon$ such that

$$\|\tilde\pi_\varepsilon\| = u_\varepsilon + o(1),\ \tilde\pi^\varepsilon(V_\varepsilon) \to 1.$$

This yields the following lower bounds in the minimax problem

$$\beta_\varepsilon(\Theta_\varepsilon, \alpha) \geq \Phi(H^{(\alpha)} - u_\varepsilon) + o(1), \quad \gamma_\varepsilon(\Theta_\varepsilon) \geq 2\Phi(-u_\varepsilon/2) + o(1). \tag{10.4}$$

In order to obtain the upper bounds, let us consider tests $\psi_{\bar\pi,\alpha}$ based on the statistic

$$l_{\bar\pi}(x) = \|\bar\pi\|^{-1} \sum_{k\in\mathbb{N}} \left(\frac{dP_{\pi_k}}{dP_0}(x_k) - 1 \right).$$

One can check that

$$E_0(l_{\bar\pi}) = 0,\ \mathrm{Var}_0(l_{\bar\pi}) = 1,\ E_v(l_{\bar\pi}) = (\bar\pi, \bar v)/\|\bar\pi\| = (\bar\delta_v, \bar r),$$

where $\bar r = \bar\pi/\|\bar\pi\|$, and $\bar\delta_v = \{\delta_{v_k}\}_{k\in\mathbb{N}} \in \Pi_\varepsilon$. Under the additional relations

$$\mathrm{Var}_v(l_{\bar\pi}) = 1 + o((\bar\delta_v, \bar r)^2), \quad l_{\bar\pi} \to \xi \sim \mathcal{N}(0,1) \text{ under } P_0, \tag{10.5}$$

$$l_{\bar\pi} - (\bar\delta_v, \bar r) \to \xi \sim \mathcal{N}(0,1) \text{ under } P_v,\ v \in V_\varepsilon : (\bar\delta_v, \bar r) = O(1), \tag{10.6}$$

we then get, for the tests $\psi_{\bar\pi,H} = 1\!\!1_{l_{\bar\pi} > H}$,

$$\alpha(\psi_{\bar\pi,H}) = \Phi(-H) + o(1), \quad \beta_\varepsilon(\eta, \psi_{\bar\pi,H}) = \Phi(H - (\bar\delta_v, \bar r)) + o(1),$$

and we have to take $\bar\pi$ in order to maximize $h(\bar\pi) \overset{\Delta}{=} \inf_{v\in V_\varepsilon}(\bar\delta_v, \bar r)$. By convexity and applying formally the minimax theorem, we get

$$\begin{aligned} \sup_{\bar r\in\Pi:\|r\|=1} \inf_{v\in V_\varepsilon} (\bar\delta_v, \bar r) &\geq \sup_{\bar r\in\Pi:\|r\|\leq 1} \inf_{\bar\pi\in\Pi_\varepsilon} (\bar\pi, \bar r) = \inf_{\bar\pi\in\Pi_\varepsilon} \sup_{\bar r\in\Pi:\|r\|\leq 1} (\bar\pi, \bar r) \\ &= \inf_{\bar\pi\in\Pi_\varepsilon} \|\bar\pi\| = u_\varepsilon. \end{aligned}$$

Therefore, taking tests $\psi_{\bar{\pi}_\varepsilon,H}$ which correspond to extreme sequences $\bar{\pi}_\varepsilon$ in (10.3), we have

$$\beta(\Theta_\varepsilon, \psi_{\bar{\pi}_\varepsilon,H}) \le \Phi(H - u_\varepsilon) + o(1). \tag{10.7}$$

Comparing (10.2) and (10.7), we get the sharp Gaussian asymptotics of the form (2.8).

Therefore, it suffices to verify only the relations (10.5) and (10.6) for the extreme sequence in the extreme problem (10.3). One can show that this sequence consists of symmetric three-point measures

$$\pi_{h_k,z_k} = (1-h_k)\delta_0 + \frac{h_k}{2}(\delta_{-z_k} + \delta_{z_k}),\ h_k \in (0,1],\ z_k \ge 0, \quad k \in \mathbb{N}$$

(two-point measures for $h_k = 1$, $k \in \mathbb{N}$), see [11], Section 5.4. Therefore, the extreme problem (10.3) is reduced to the extreme problem (4.15) with constraints similar to (4.16), and the statistics $l_{\bar{\pi}_\varepsilon}$ are of the form (4.21) (if $h_k = 1$, $k \in \mathbb{N}$, then the extreme problem (10.3) is reduced to an extreme problem similar to (3.1), and the statistics $l_{\bar{\pi}_\varepsilon}$ can be replaced by (3.3)). The assumptions (10.5)-(10.6) can be verified (for slightly modified test statistics combined with thresholding (4.20)) under some constraints on the extreme sequences $\bar{z}$ and $\bar{h}$ (roughly speaking, it suffices to consider a sequence z_k, $k \in \mathbb{N}$, that is bounded or tends to infinity not too fast).

Thus, one has to study an extreme problem similar to (4.15)-(4.16), and to verify the required properties of the extreme sequences $\bar{z}$ and $\bar{h}$ as well as other required assumptions. These were done in [11] for the polynomial case $a_k = k^\alpha$, $\sigma_k = k^\beta$, $k \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$, with l^q-ellipsoids (bodies), $q \in (0,2]$, for Sobolev classes of functions. Gaussian asymptotics were obtained (namely, "dense" and "sparse" type of asymptotics).

The dense type of Gaussian asymptotics corresponds to the case when the "main mass" of the extreme sequence $\bar{\pi}_\varepsilon$ corresponds to $h_k = 1$, $k \in \mathbb{N}$. In this case, we have the rate relations $u_\varepsilon^2 \asymp m z_0^4$, where the "efficient value" $z_0 = z_{0,\varepsilon}$ and the "efficient dimension" $m = m_\varepsilon$ satisfy

$$m^{\beta+1/p} z_0 \asymp r_\varepsilon/\varepsilon, \quad m^{\alpha+\beta+1/q} z_0 \asymp 1/\varepsilon$$

(these holds for instance for $p = q = 2$). For the sparse type, where the "main mass" corresponds to $h_k \in (0,1)$, $k \in \mathbb{N}$, we have the relation $u_\varepsilon^2 \asymp n h_0^2$, where the "efficient sparsity" $h_0 = h_{0,\varepsilon}$ and the "efficient dimension" $n = n_\varepsilon$ satisfy

$$n^{\beta+1/p} h_0^{1/p} \asymp r_\varepsilon/\varepsilon, \quad n^{\alpha+\beta+1/q} h_0^{1/q} \asymp 1/\varepsilon$$

(these holds for instance for $p = 2 > q > 0$). The separation rates $r_\varepsilon^* = r_\varepsilon^*(\alpha, \beta, p, q)$ are determined by the relation $u_\varepsilon \asymp 1$. Observe that $z_0 \asymp m^{-1/4}$ for the dense type and $h_0 \asymp n^{-1/2}$ for the sparse type, when $u_\varepsilon \asymp 1$.

There are regions of parameters (α, β, p, q) where the Gaussian asymptotics do not hold (the main reasons are that either $u_\varepsilon = 0$ or the extreme sequence $\bar{\pi}_\varepsilon$ do not exist). We have different types of asymptotics in these regions. The most interesting seems to be the "degenerate" type: the Bayesian likelihood is not asymptotically Gaussian but asymptotically constant (these holds for instance for the case $p = \infty$). The asymptotically minimax tests $\psi_{\varepsilon,\alpha}^D$ are of the form (4.22) and do not depend on (α, β, p, q) from the region of degenerate asymptotics. The division of the set of (α, β, p, q)-values (in terms of the parameters $r = \beta$, $s = \alpha + \beta$) to the regions of various types of asymptotics is given in [11], Section 6.4.

Finally, let us compare the separation rates $r_\varepsilon^*(\alpha, \beta, p, q)$ with the rates $R_\varepsilon = R_\varepsilon(\alpha, \beta, p, q)$ of accuracy in the estimation problem for the loss function determined by a similar norm of l^p-type, i.e.,

$$R_\varepsilon = \inf_{\hat{\eta}} \sup_{\eta \in \Theta_{q,\alpha+\beta}} \left(E_{\varepsilon,\eta} \sum_{k \in \mathbb{N}} |k^\beta (\hat{\eta}_k - \eta_k)|^p \right)^{1/p}, \quad 0 < p < \infty,$$

where the infimum is taken over all possible estimators $\hat{\eta}$ of η, the later belonging to the following class of sequences

$$\Theta_{q,\alpha+\beta} = \{\eta \in l^2 : \sum_{k \in \mathbb{N}} |k^{(\alpha+\beta)} \eta_k|^q \le 1\}, \quad 0 < q < \infty$$

(with standard modification for the case $p = \infty$ and/or $q = \infty$). Using known results on the asymptotics of $R_\varepsilon(\alpha, \beta, p, q)$ (see, for instance, Delyon & Juditsky (1996), Donoho *et al.* (1995), Lepski *et al.* (1997)), one can see that

$$r_\varepsilon^*(\alpha, \beta, p, q) \ll R_\varepsilon(\alpha, \beta, p, q) \quad \text{for the regions of Gaussian asymptotics}$$

and

$$r_\varepsilon^*(\alpha,\beta,p,q) \asymp R_\varepsilon(\alpha,\beta,p,q) \quad \text{for the region of degenerate asymptotics.}$$

We believe that a similar study is possible for a wider class of ill-posed inverse problems (for instance, the exponential case $a_k = e^{\alpha k}$ and/or $\sigma_k = e^{\beta k}$, $k \in \mathbb{N}$, $\alpha, \beta > 0$, for various l_p and l_q-norms, $0 < p, q \leq \infty$, considered in the set under the alternative). This is a possible topic of future research that we hope to address it elsewhere.

11. Proofs.

We present below detailed proofs of Theorems 4.1-4.9, Theorems 5.1-5.4, Theorems 7.1, 8.1 and 9.1, along with detailed proofs of the auxiliary statements used in their proofs.

11.1. *Proof of Theorem 4.1.* To prove the theorem, we utilize techniques and results presented in Chapters 3 and 4 of [11] for minimax hypothesis testing in infinite dimensional settings. In particular, in order to get the lower bounds, we replace the minimax problem by a Bayesian one. Let $\pi = \pi_\varepsilon$ be a prior (probability measure) on the sequence space such that $\pi(\Theta(r_\varepsilon)) = 1$. Let $P_{\varepsilon,\pi} = E_\pi(P_{\varepsilon,\eta})$ be the mixture over π and $L_{\varepsilon,\pi} = dP_{\varepsilon,\pi}/dP_{\varepsilon,0} = E_\pi dP_{\varepsilon,\eta}/dP_{\varepsilon,0}$ be the likelihood ratio. Denote by $\beta(P_0,P_1,\alpha), \gamma(P_0,P_1)$ the minimal type I error probability for a given level α and the minimal total error probability, respectively, for testing the simple null hypothesis $H_0 : P = P_0$ against the simple alternative hypothesis $H_1 : P = P_1$, for the measure P of the observations. It is well known (see, e.g., [11], Section 2.4.2) that, for any $\alpha \in [0,1]$,

$$\beta_\varepsilon(\Theta(r_\varepsilon),\alpha) \geq \beta(P_{\varepsilon,0},P_{\varepsilon,\pi},\alpha),\ \gamma_\varepsilon(\Theta(r_\varepsilon)) \geq \gamma(P_{\varepsilon,0},P_{\varepsilon,\pi}) = 1-\frac{1}{2}\|P_{\varepsilon,0}-P_{\varepsilon,\pi}\|_1, \tag{11.1}$$

where $\|P_{\varepsilon,0}-P_{\varepsilon,\pi}\|_1 = E_{\varepsilon,0}|L_{\varepsilon,\pi}-1|$ is the variation distance. Therefore, as $\varepsilon \to 0$, in order for $\gamma_\varepsilon(\Theta(r_\varepsilon)) \to 1$, it suffices $\|P_{\varepsilon,0}-P_{\varepsilon,\pi}\|_1 \to 0$. Since

$$\|P_{\varepsilon,0}-P_{\varepsilon,\pi}\|_1^2 \leq \|P_{\varepsilon,0}-P_{\varepsilon,\pi}\|_2^2 = E_{\varepsilon,0}(L_{\varepsilon,\pi})^2 - 1,$$

it suffices $E_{\varepsilon,0}(L_{\varepsilon,\pi})^2 \to 1$. By (2.5), this leads to $\beta_\varepsilon(\Theta(r_\varepsilon)),\alpha) \to 1-\alpha$. Also the relation $E_{\varepsilon,0}(L_{\varepsilon,\pi})^2 = O(1)$ implies $\liminf \gamma_\varepsilon(r_\varepsilon) > 0$ and $\beta_\varepsilon(r_\varepsilon,\alpha) > 1-\alpha$ for any $\alpha \in (0,1)$, see [11], Proposition 2.12.

To proceed, we need a definition. A set V is called *sign-symmetric* (or *orthosymmetric*) if $v = \{v_i\}_{i\in\mathbb{N}} \in V$, then $\tilde{v} = \{\pm v_i\}_{i\in\mathbb{N}} \in V$ for all changes of signs of the coordinates. Observe now that $\Theta(r_\varepsilon)$ is a sign-symmetric set. For $\eta \in \Theta(r_\varepsilon)$, let us consider the product prior $\pi_\varepsilon = \prod_{k\in\mathbb{N}} \pi_{\varepsilon,k}$, where $\pi_{\varepsilon,k} = \frac{1}{2}(\delta_{\eta_k} + \delta_{-\eta_k})$, $k \in \mathbb{N}$, are symmetric two-points priors. Note that, $\pi_\varepsilon(\Theta(r_\varepsilon)) = 1$ by the sign-symmetric condition. Let $P^{(k)}_{\varepsilon,\eta_k}$ be the measure for $y_k = \eta_k + \varepsilon\xi_k,\ \xi_k \sim \mathcal{N}(0,1)$, $k \in \mathbb{N}$, and

$$\begin{aligned} L^{(k)}_{\varepsilon,\eta_k} &= dP^{(k)}_{\varepsilon,\eta_k}/dP^{(k)}_{\varepsilon,0} = \exp((-\eta_k^2/2\varepsilon^2 + \eta_k y_k/\varepsilon^2)), \\ L^{(k)}_{\varepsilon,\pi_k} &= E_{\pi_k}(L^{(k)}_{\varepsilon,\eta_k}) = e^{-\eta_k^2/2\varepsilon^2}\cosh(y_k\eta_k/\varepsilon^2). \end{aligned}$$

Simple calculations give $E_{\varepsilon,0}(E_{\pi_k}(L^{(k)}_{\varepsilon,\eta_k}))^2 = \cosh(\eta_k^2/\varepsilon^2)$, $k \in \mathbb{N}$. Therefore, using the inequality $\cosh(t) \le \exp(t^2/2)$ (which follows from Taylor's expansions), we have

$$\begin{aligned} E_{\varepsilon,0}(L^2_{\varepsilon,\pi}) &= E_{\varepsilon,0}\prod_{k\in\mathbb{N}} E_{\pi_k}(L^{(k)}_{\varepsilon,\eta_k})^2 = \prod_{k\in\mathbb{N}} E_{\varepsilon,0}E_{\pi_k}(L^{(k)}_{\varepsilon,\eta_k})^2 \\ &= \prod_{k\in\mathbb{N}}\cosh(\eta_k^2/\varepsilon^2) \le \exp(\varepsilon^{-4}\sum_{k\in\mathbb{N}}\eta_k^4/2) \to 1, \end{aligned}$$

if $u^2_\varepsilon(\eta) = \frac{1}{2}\varepsilon^{-4}\sum_{k\in\mathbb{N}}\eta_k^4 \to 0$, and in order to get the best $\eta \in \Theta(r_\varepsilon)$ we use the extreme problem (3.1). Also if $u^2_\varepsilon(\eta) = O(1)$, then we get $E_{\varepsilon,0}(L^2_{\varepsilon,\pi}) = O(1)$. This completes part (1)(a) of the theorem.

In order to get the sharp lower bounds, take π that corresponds to the extreme sequence $\tilde{\eta}_\varepsilon$ of the problem (3.1). In order to get the relation

$$\beta_\varepsilon(P_{\varepsilon,0}, P_{\varepsilon,\pi}, \alpha) \ge \Phi(H^{(\alpha)} - u_\varepsilon) + o(1),\ \ \gamma_\varepsilon(P_{\varepsilon,0}, P_{\varepsilon,\pi}) \ge 2\Phi(-u_\varepsilon/2) + o(1),$$

it suffices to show that

$$\log(L_{\varepsilon,\pi}) = -u^2_\varepsilon/2 + u_\varepsilon\xi_\varepsilon + \delta_\varepsilon, \tag{11.2}$$

where $u_\varepsilon = u_\varepsilon(\tilde{\eta}_\varepsilon) = O(1)$ and $\xi_\varepsilon \to \xi \sim \mathcal{N}(0,1)$, $\delta_\varepsilon \to 0$ in $P_{\varepsilon,0}$-probability (see [11], Section 4.3.1). Setting $x_{\varepsilon,k} = y_k/\varepsilon,\ v_{\varepsilon,k} = \tilde{\eta}_{\varepsilon,k}/\varepsilon$, $k \in \mathbb{N}$, we note that $\sum_{k\in\mathbb{N}} v^4_{\varepsilon,k} = 2u^2_\varepsilon$. We have

$$\log(L_{\varepsilon,\pi}) = \sum_{k\in\mathbb{N}}(-v^2_{\varepsilon,k}/2 + \log(\cosh(x_{\varepsilon,k}v_{\varepsilon,k}))).$$

Using the inequality

$$|\log(\cosh(t)) - t^2/2 + t^4/12| \le Bt^6,\ t \in \mathbb{R},$$

for some $B > 0$, we have (11.2) with $\delta_\varepsilon = \delta_{\varepsilon,1} + \delta_{\varepsilon,2}$, where

$$\xi_\varepsilon = \frac{1}{2u_\varepsilon} \sum_{k\in\mathbb{N}} v_{\varepsilon,k}^2 (x_{\varepsilon,k}^2 - 1), \quad \delta_{\varepsilon,1} = \frac{1}{12} \sum_{k\in\mathbb{N}} v_{\varepsilon,k}^4 (3 - x_{\varepsilon,k}^4),$$

$$|\delta_{\varepsilon,2}| \le \delta_{\varepsilon,3} = B \sum_{k\in\mathbb{N}} v_{\varepsilon,k}^6 x_{\varepsilon,k}^6.$$

Since $x_{\varepsilon,k} \sim \mathcal{N}(0,1),\ k \in \mathbb{N}$, under $P_{\varepsilon,0}$, the relations $\delta_{\varepsilon,1} \to 0,\ \delta_{\varepsilon,2} \to 0$ follow from $E_{\varepsilon,0}\delta_{\varepsilon,1} = 0$ and, for some constants $B_l > 0,\ l = 1,2,3,4$,

$$\mathrm{Var}_{\varepsilon,0}\delta_{\varepsilon,1} = B_1 \sum_{k\in\mathbb{N}} v_{\varepsilon,k}^8 \le B_2 u_\varepsilon^4 w_0^2 = o(1),$$

$$E_{\varepsilon,0}\delta_{\varepsilon,3} = B_3 \sum_{k\in\mathbb{N}} v_{\varepsilon,k}^6 \le B_4 u_\varepsilon^3 w_0 = o(1),$$

since $\delta_{\varepsilon,3} \ge 0$. Also, $E_{\varepsilon,0}\xi_\varepsilon = 0$, $\mathrm{Var}_{\varepsilon,0}\xi_\varepsilon = 1$ and the asymptotic $\mathcal{N}(0,1)$ normality of ξ_ε under $P_{\varepsilon,0}$ follows from Lyapunov condition: if $z_{\varepsilon,k} = v_{\varepsilon,k}^2(x_{\varepsilon,k}^2 - 1)$ then $\sum_{k\in\mathbb{N}} E_{\varepsilon,0} z_{\varepsilon,k}^4 / (\sum_k E_{\varepsilon,0} z_{\varepsilon,k}^2)^2 \le Bw_0^2 \to 0$ for some $B > 0$. This completes the lower bounds of part (1)(b) of the theorem.

In order to obtain the upper bounds let us calculate the expectations and variances of the statistics t_ε of the form (3.3) for a sequence $w_k = \tilde{\eta}_k/u_\varepsilon,\ w_k \ge 0,\ k \in \mathbb{N},\ \sum_{k\in\mathbb{N}} w_k^2 = 1/2;\ w_0 = \sup_{k\in\mathbb{N}} w_k \in (0, 2^{-1/2}]$, where $\tilde{\eta}$ and u_ε^2 are the extreme sequence and the extreme value in (3.1). We have

$$E_{\varepsilon,0}t_\varepsilon = 0, \quad \mathrm{Var}_{\varepsilon,0}t_\varepsilon = 1, \quad E_{\varepsilon,\eta}t_\varepsilon = \varepsilon^{-2} \sum_{k\in\mathbb{N}} w_k \eta_{\varepsilon,k}^2 =: h_\varepsilon(\eta), \tag{11.3}$$

$$\mathrm{Var}_{\varepsilon,\eta}t_\varepsilon = 1 + 4\varepsilon^{-2} \sum_{k\in\mathbb{N}} w_k^2 \eta_k^2; \quad 1 \le \mathrm{Var}_{\varepsilon,\eta}t_\varepsilon \le 1 + 4h_\varepsilon(\eta)w_0. \tag{11.4}$$

The key point is the following lemma.

LEMMA 11.1.

$$\inf_{\eta\in\Theta(r_\varepsilon)} h_\varepsilon(\eta) = u_\varepsilon.$$

Proof. Denote $\tau_k = \eta_k^2,\ \tilde{\tau}_k = \tilde{\eta}_k^2,\ k \in \mathbb{N}$, $w = \{w_k\}_{k\in\mathbb{N}}$ and consider the set $\Upsilon = \{\tau = \{\tau_k\}_{k\in\mathbb{N}} : \eta = \{\eta_k\}_{k\in\mathbb{N}} \in \Theta(r_\varepsilon)\}$. Observe that Υ is a convex set (i.e., $\Theta(r_\varepsilon)$ is a quadratically convex set) in the sequence space l_2 and $h_\varepsilon(\eta) = \varepsilon^{-2}(\tau, w) = \varepsilon^{-2}(\tau,\tilde{\tau})/(\sqrt{2}\|\tilde{\tau}\|)$, where $(\cdot,\cdot)$ and $\|\cdot\|$ stand, respectively, for the scalar product and the norm in l_2. We have to check that $G := \inf_{\tau\in\Upsilon}(\tau,\tilde{\tau}) = \|\tilde{\tau}\|^2$, where

$$\tilde{\tau} \in \Upsilon, \quad \|\tilde{\tau}\| = \inf_{\tau\in\Upsilon}\|\tau\|. \tag{11.5}$$

First observe that $G \le \|\tilde{\tau}\|^2$ since $\tilde{\tau} \in \Upsilon$, and it suffices to check that $G \ge \|\tilde{\tau}\|^2$. Suppose there exists $\tau_0 \in \Upsilon$ such that $(\tau_0,\tilde{\tau}) < \|\tilde{\tau}\|^2$, which is equivalent to $r := ((\tau_0 - \tilde{\tau}),\tilde{\tau}) < 0$. Consider the interval $\tau(t) = \tilde{\tau} + t(\tau_0 - \tilde{\tau}),\ \tau(t) \in \Upsilon$ for all $t \in [0,1]$, by convexity of Υ. We have, for $t \in (0,1)$ small enough,

$$\|\tau(t)\|^2 = \|\tilde{\tau}\|^2 + 2tr + t^2\|\tau_0 - \tilde{\tau}\|^2 < \|\tilde{\tau}\|^2.$$

This contradicts to (11.5). The lemma now follows.

Return to the proof of the upper bounds. Let $u_\varepsilon \to \infty$. Then, applying the Chebyshev inequality and (11.3) we have, for $H \to \infty$,

$$\alpha_\varepsilon(\psi_{\varepsilon,H}) = P_{\varepsilon,0}(t_\varepsilon > H) \le \frac{\mathrm{Var}_{\varepsilon,0} t_\varepsilon}{H^2} \to 0.$$

For the alternative hypothesis, applying the Chebyshev inequality once again, (11.4) and Lemma 11.1, we have, for $H = cu_\varepsilon,\ c \in (0,1)$ and uniformly over $\eta \in \Theta(r_\varepsilon)$,

$$\begin{aligned}\beta_\varepsilon(\eta,\psi_{\varepsilon,H}) &= P_{\varepsilon,\eta}(t_\varepsilon \le H) = P_{\varepsilon,\eta}(h_\varepsilon(\eta) - t_\varepsilon \ge h_\varepsilon(\eta) - H)\\ &\le \frac{\mathrm{Var}_{\varepsilon,\eta} t_\varepsilon}{(h_\varepsilon(\eta) - H)^2} \le \frac{1 + 4w_0 h_\varepsilon(\eta)}{((1-c)h_\varepsilon(\eta))^2} \to 0.\end{aligned}$$

This completes part (2) of the theorem.

Let $u_\varepsilon \asymp 1$ and $w_0 = o(1)$. Observe that $t_\varepsilon = \xi_\varepsilon$ where ξ_ε is the statistic from the proof of the lower bounds, and it was shown that t_ε is asymptotically standard Gaussian under $P_{\varepsilon,0}$. This yields $\alpha_\varepsilon(\psi_{\varepsilon,H}) = \Phi(-H) + o(1)$. In order to evaluate type II error probability, let us divide the set $\Theta(r_\varepsilon)$ into two sets

$$\Theta_{\varepsilon,1} = \{\eta \in \Theta(r_\varepsilon) : h_\varepsilon(\eta) < h_\varepsilon\}, \quad \Theta_{\varepsilon,2} = \{\eta \in \Theta(r_\varepsilon) : h_\varepsilon(\eta) > h_\varepsilon\},$$

where $h_\varepsilon \to \infty$, $h_\varepsilon w_0 \to 0$. Similarly to evaluation above, we get $\sup_{\eta\in\Theta_{\varepsilon,2}} \beta_\varepsilon(\psi_{\varepsilon,H},\eta) \to 0$ for any $H = O(1)$. Let $\eta \in \Theta_{\varepsilon,1}$. By (11.4), we have $\mathrm{Var}_{\varepsilon,\eta} t_\varepsilon = 1 + o(1)$. Observe that the statistics $\hat{t}_\varepsilon = (t_\varepsilon - h_\varepsilon(\eta))/\sqrt{\mathrm{Var}_{\varepsilon,\eta} t_\varepsilon}$ are asymptotically standard Gaussian under $P_{\varepsilon,\eta}$. This follows from Lyapunov's condition, since $t_\varepsilon - h_\varepsilon(\eta) = \sum_{k\in\mathbb{N}} \tilde{t}_{\varepsilon,k}$, where $\tilde{t}_{\varepsilon,k}$ are independent and $P_{\varepsilon,\eta}$-distributed as $w_k(\xi_k^2 - 1 + 2v_k\xi_k)$, where $v_k = \eta_k/\varepsilon,\ \xi_k \sim \mathcal{N}(0,1)$, $k \in \mathbb{N}$. Therefore, one has, for some constant $B > 0$, uniformly over $\eta \in \Theta_{\varepsilon,1}$,

$$\begin{aligned}
E_{\varepsilon,\eta}\tilde{t}^4_{\varepsilon,k} &\le B(w_k^4 + w_k^4 v_k^4),\\
\sum_{k\in\mathbb{N}}(w_k^4 + w_k^4 v_k^4) &\le w_0^2\sum_{k\in\mathbb{N}} w_k^2 + w_0^2\left(\sum_{k\in\mathbb{N}} w_k v_k^2\right)^2\\
&\le w_0^2/2 + w_0^2 h_\varepsilon^2 \to 0.
\end{aligned}$$

It also follows from the asymptotic normality of $\hat{t}_\varepsilon$ that, uniformly over $\eta \in \Theta_{\varepsilon,1}$,

$$\begin{aligned}
\beta_\varepsilon(\eta,\psi_{\varepsilon,H}) &= P_{\varepsilon,\eta}(t_\varepsilon \le H) = P_{\varepsilon,\eta}(\hat{t}_\varepsilon \le (H - h_\varepsilon(\eta))/\sqrt{\mathrm{Var}_{\varepsilon,\eta} t_\varepsilon})\\
&= \Phi(H - h_\varepsilon(\eta)) + o(1).
\end{aligned}$$

By Lemma 11.1 and evaluation over $\eta \in \Theta_{\varepsilon,2}$ above, we get

$$\beta(\Theta(r_\varepsilon),\psi_{\varepsilon,H}) = \Phi(H - \inf_{\eta\in\Theta(r_\varepsilon)} h_\varepsilon(\eta)) + o(1) = \Phi(H - u_\varepsilon) + o(1).$$

Taking $H = H^{(\alpha)}$ and $H = u_\varepsilon/2$, it completes the upper bounds for part (1) (b) of the theorem. The theorem now follows.

11.2. *Equations (4.1)-(4.2).* The equations (4.1)-(4.2) are immediately rewritten in the form

$$r_\varepsilon^2 = z_0^2 J_1, \quad 1 = z_0^2 A^{-1} J_2, \tag{11.6}$$

and, hence, the extreme problem (3.1) takes the form

$$u_\varepsilon^2 = \varepsilon^{-4} z_0^4 J_0/2, \tag{11.7}$$

where

$$J_1 = \sum_{k\in\mathbb{N}} \sigma_k^4(1-Aa_k^2)_+,$$
$$J_2 = A\sum_{k\in\mathbb{N}} a_k^2\sigma_k^4(1-Aa_k^2)_+,$$
$$J_0 = J_1 - J_2 = \sum_{k\in\mathbb{N}} \sigma_k^4(1-Aa_k^2)_+^2.$$

It is also convenient to rewrite (11.6) and (11.7) in the form

$$r_\varepsilon^2 = A\,\frac{J_1}{J_2}, \quad u_\varepsilon^2 = \left(\frac{r_\varepsilon}{\varepsilon}\right)^4 \frac{J_0}{2J_1^2}. \tag{11.8}$$

Note that the first equation in (11.8) is used to calculate A.

11.3. *Proof of Theorem 4.2.* Set $A = m^{-2\alpha}$ in (4.1); the quantity $m = m_\varepsilon$ determines the *efficient dimension* in the problem. Then, the extreme sequence (4.1) in the extreme problem (3.1) takes the form

$$\tilde\eta_k^2 = z_0^2 k^{2\beta}(1-(k/m)^{2\alpha})_+, \quad 1\le k\le m, \tag{11.9}$$

while the equations for $z_0 = z_{0,\varepsilon}$, $m = m_\varepsilon$ and u_ε take the form (11.7), (11.8), where

$$J_1 = \frac{1}{m}\sum_{1\le k\le m}\left(\frac{k}{m}\right)^{4\beta}\left(1-\left(\frac{k}{m}\right)^{2\alpha}\right),$$
$$J_2 = \frac{1}{m}\sum_{1\le k\le m}\left(\frac{k}{m}\right)^{2\alpha+4\beta}\left(1-\left(\frac{k}{m}\right)^{2\alpha}\right),$$
$$J_0 = J_1 - J_2 = \frac{1}{m}\sum_{1\le k\le m}\left(\frac{k}{m}\right)^{4\beta}\left(1-\left(\frac{k}{m}\right)^{2\alpha}\right)^2.$$

We consider the situation $m\to\infty$ and $r_\varepsilon\to 0$ as $\varepsilon\to 0$. Let us now find the asymptotics of the sums $J_1,\ J_2$ and J_0 as $m\to\infty$. Replacing the sums J_1, J_2 and J_0 by integrals, after some calculations, we have, as $m\to\infty$,

$$J_1 \sim \frac{2\alpha}{(4\beta+1)(4\beta+2\alpha+1)} := d_1,$$
$$J_2 \sim \frac{2\alpha}{(4\alpha+4\beta+1)(4\beta+2\alpha+1)} := d_2,$$
$$J_0 \sim \frac{8\alpha^2}{(4\beta+1)(4\alpha+4\beta+1)(4\beta+2\alpha+1)} := d_0.$$

These yield

$$r_\varepsilon \sim c_1^\alpha m^{-\alpha}, \quad u_\varepsilon^2 \sim c_2 \varepsilon^{-4} r_\varepsilon^{(4\alpha+4\beta+1)/\alpha}, \tag{11.10}$$

where $c_1 = (d_1/d_2)^{1/(2\alpha)}$, $c_2 = (d_2/d_1)^{(4\beta+1)/(2\alpha)} d_0/(2d_1^2)$. Hence, the value u_ε of the extreme problem (3.1) and the efficient dimensions $m = m_\varepsilon$ satisfy

$$u_\varepsilon \sim c_2^{1/2} \varepsilon^{-2} (c_1/m)^{(4\alpha+4\beta+1)/2}, \quad m \sim c_1 (\varepsilon^4 u_\varepsilon^2/c_2)^{-1/(4\alpha+4\beta+1)}. \tag{11.11}$$

Observe also that, for the extreme sequence determined by (11.9), one has

$$w_0 = \frac{\max_{1\le k\le m} \tilde{\eta}_k^2}{\sqrt{2\sum_{k=1}^m \tilde{\eta}_k^4}} \le \frac{Bz_0^2 m^{2\beta}}{z_0^2 m^{2\beta+1/2}} \asymp m^{-1/2} \to 0, \quad B > 0.$$

The theorem now follows on applying Theorem 4.1.

11.4. *Proof of Theorem 8.1.* To prove the theorem, we need the following proposition. (Note that its validity is true for a wider class of sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{\sigma_k\}_{k\in\mathbb{N}}$, which cover all ill-posed inverse problems of interest.)

PROPOSITION 11.1. *Let $\{a_k\}_{k\in\mathbb{N}}$ and $\{\sigma_k\}_{k\in\mathbb{N}}$ be positive increasing sequences. Assume that there exists $B > 1$, $a > 0$, $\tilde{m} = \tilde{m}_\varepsilon \in \mathbb{N}$, $\varepsilon_0 > 0$ such that, as $0 < \varepsilon < \varepsilon_0$,*

$$r_\varepsilon a_{\tilde{m}+1} \ge B, \quad u_\varepsilon \le \frac{a r_\varepsilon^2}{\varepsilon^2 \sqrt{\tilde{m}} \sigma_{\tilde{m}}^2}. \tag{11.12}$$

Set

$$h_{\tilde{m}}(\eta) = \frac{1}{\varepsilon^2 \sqrt{2\tilde{m}}} \sum_{k=1}^{\tilde{m}} \eta_k^2, \quad h_{\tilde{m}} = \inf_{\eta\in\Theta_\varepsilon} h_{\tilde{m}}(\eta). \tag{11.13}$$

Then, there exists $b = b(B, a) > 0$ such that $h_{\tilde{m}} \ge b u_\varepsilon$ as $0 < \varepsilon < \varepsilon_0$.

Proof. By definition of Θ_ε, since the sequences $\{a_k\}_{k\in\mathbb{N}}$ and $\{\sigma_k\}_{k\in\mathbb{N}}$ increase as $k\to\infty$, and by (11.12), we have, for $\eta\in\Theta(r_\varepsilon)$,

$$\begin{aligned}\sum_{k=1}^{\tilde m}\eta_k^2 &\geq \frac{1}{\sigma_{\tilde m}^2}\sum_{k=1}^{\tilde m}\sigma_k^2\eta_k^2 \geq \frac{1}{\sigma_{\tilde m}^2}\left(r_\varepsilon^2-\sum_{k=\tilde m+1}^{\infty}\sigma_k^2\eta_k^2\right)\\ &\geq \frac{1}{\sigma_{\tilde m}^2}\left(r_\varepsilon^2-\frac{1}{a_{\tilde m+1}^2}\sum_{k=\tilde m+1}^{\infty}a_k^2\sigma_k^2\eta_k^2\right)\geq \frac{1}{\sigma_{\tilde m}^2}\left(r_\varepsilon^2-\frac{1}{a_{\tilde m+1}^2}\right)\\ &= \frac{r_\varepsilon^2}{\sigma_{\tilde m}^2}\left(1-\frac{1}{r_\varepsilon^2 a_{\tilde m+1}^2}\right)\geq \frac{b_1 r_\varepsilon^2}{\sigma_{\tilde m}^2},\quad b_1=1-B^{-2}>0.\end{aligned}\tag{11.14}$$

Therefore, we have

$$h_{\tilde m}\geq \frac{b_1 r_\varepsilon^2}{\varepsilon^2\sqrt{2\tilde m}\sigma_{\tilde m}^2}\geq b u_\varepsilon,\quad b=b_1/(\sqrt{2}a).\tag{11.15}$$

The proposition now follows.

We are now ready to prove Theorem 8.1. By the asymptotic normality of $t_{\tilde m}$ under $P_{0,\varepsilon}$ as $\tilde m\to\infty$ (see [11], Lemma 3.1), we have $\alpha(\psi_{\varepsilon,H})=\Phi(-H)+o(1)\to 0$ as $H=H_\varepsilon\to\infty$.

In order to evaluate type II error probability for the test $\psi_{\varepsilon,H}$, take $h_{\tilde m}(\eta)$ and $h_{\tilde m}$ as in (11.13). By the asymptotic normality of $t_{\tilde m}-h_{\tilde m}(\eta)$ under $P_{\eta,\varepsilon}$ as $\tilde m\to\infty$ (see [11], Lemma 3.1), we have

$$\beta_\varepsilon(\eta,\psi_{\varepsilon,H})\leq\Phi(H-h_{\tilde m}(\eta))+o(1),\quad \beta_\varepsilon(\Theta_\varepsilon,\psi_{\varepsilon,H})\leq\Phi(H-h_{\tilde m})+o(1).\tag{11.16}$$

Proposition 11.1 implies that $\beta_\varepsilon(\psi_{\varepsilon,H_\varepsilon},\Theta_\varepsilon)\to 0$ as $H_\varepsilon\leq(c+o(1))u_\varepsilon\to\infty$, $c\in(0,b)$. The theorem now follows.

11.5. *Proof of Theorem 4.3.* We first consider the "standard" case $q=2$. Let the *efficient dimension* $m=m_\varepsilon$ be determined by $A=\exp(-2\alpha m)$ in (4.1). Then, the extreme sequence (4.1) in the extreme problem (3.1) takes the form

$$\tilde\eta_k^2=z_0^2\exp(2\beta k)(1-\exp(2\alpha(k-m)))_+,\quad 1\leq k\leq m,\tag{11.17}$$

while the equations for $z_0 = z_{0,\varepsilon}$, $m = m_\varepsilon$ and u_ε take the form (11.7), (11.8) where

$$\begin{aligned} J_1 &= \sum_{1\le k\le m} \exp(4\beta k)(1-\exp(2\alpha(k-m))), \\ J_2 &= \exp(-2\alpha m)\sum_{1\le k\le m} \exp((2\alpha+4\beta)k)(1-\exp(2\alpha(k-m))), \\ J_0 &= J_1 - J_2 = \sum_{1\le k\le m} \exp(4\beta k)(1-\exp(2\alpha(k-m)))^2. \end{aligned}$$

We consider the situation $m \to \infty$ and $r_\varepsilon \to 0$ as $\varepsilon \to 0$. Let us now find the asymptotics of the sums J_1, J_2 and J_0 as $m \to \infty$. After some calculations, we have, as $m \to \infty$,

$$J_1 \asymp J_2 \asymp J_0 \asymp \exp(4\beta m)$$

and, hence, using (11.8), we get the the relations

$$r_\varepsilon \asymp \exp(-\alpha m)), \quad u_\varepsilon^2 \asymp \varepsilon^{-4} r_\varepsilon^{4(\alpha+\beta)/\alpha}. \tag{11.18}$$

Hence, the value u_ε of the extreme problem (3.1) and the efficient dimensions $m = m_\varepsilon$ satisfy

$$u_\varepsilon \asymp \varepsilon^{-2}\exp(-2(\alpha+\beta)m), \quad m = \frac{2\log(\varepsilon^{-1}) - \log(u_\varepsilon)}{2(\alpha+\beta)} + O(1). \tag{11.19}$$

Hence, the "standard" case $q = 2$ for the theorem follows on applying Theorem 4.1.

Consider now the "sparse" case $q \in (0, 2)$. The embedding (4.8) yields $\gamma(\psi, \Theta_q(r_\varepsilon)) \le \gamma(\psi, \Theta_2(r_\varepsilon))$. Therefore, it suffices to establish the lower bounds. Take $m = \max\{k : r_\varepsilon \exp(\alpha k) \le 1\}$, and consider the vector η_m that contains only one non-zero coordinate, the value $z_n = r_\varepsilon \exp(-\beta n)$ at position m. One can easily check that $\eta_m \in \Theta_q(r_\varepsilon)$ for any $q \in (0, 2)$. Therefore, one cannot distinguish between H_0 and H_1 if $z_n = o(\varepsilon)$, which is equivalent to $r_\varepsilon = o(r_\varepsilon^*)$, where r_ε^* is obtained by combining $u_\varepsilon \asymp 1$ and (11.18). In view of the above and the results for the "standard" case $q = 2$, the "sparse" case $q = 2$ for the theorem also follows. Hence, the theorem follows.

11.6. *Proof of Theorem 4.4.* We first consider the "standard" case $q=2$. Let the efficient dimension $m = m_\varepsilon$ be determined by $A = m^{-2\alpha}$ in (4.1). Then, the extreme sequence (4.1) in the extreme problem (3.1) takes the form

$$\tilde{\eta}_k^2 = z_0^2 \exp(2\beta k)\left(1-(k/m)^{2\alpha}\right)_+, \quad 1 \le k \le m, \tag{11.20}$$

while the equations for $z_0 = z_{0,\varepsilon}$, $m = m_\varepsilon$ and u_ε take the form (11.7), (11.8) where

$$\begin{aligned} J_1 &= \sum_{1\le k\le m} \exp(4k\beta)\left(1-\left(\frac{k}{m}\right)^{2\alpha}\right), \\ J_2 &= \sum_{1\le k\le m} \exp(4k\beta)\left(\frac{k}{m}\right)^{2\alpha}\left(1-\left(\frac{k}{m}\right)^{2\alpha}\right), \\ J_0 &= J_1 - J_2 = \sum_{1\le k\le m} \exp(4k\beta)\left(1-\left(\frac{k}{m}\right)^{2\alpha}\right)^2. \end{aligned}$$

We consider the situation $m\to\infty$ and $r_\varepsilon \to 0$ as $\varepsilon\to 0$. Let us now find the asymptotics of the sums $J_1,\ J_2$ and J_0 as $m\to\infty$. Take $\delta>0,\ \delta\to 0$, such that $m\delta\to\infty,\ m\delta \gg \log(m),\ m\delta^2\to 0$ as $m\to\infty$. Set $k=m-l$. Then,

$$\begin{aligned} J_1 &= \sum_{1\le k\le m,\ l=m-k} e^{4\beta(m-l)}\Big(1-\left(1-\frac{l}{m}\right)^{2\alpha}\Big) \\ &= e^{4m\beta} \sum_{m(1-\delta)\le k\le m,\ l=m-k} e^{-4l\beta}\Big(\frac{2l\alpha}{m} + O(l^2/m^2)\Big) \\ &\quad + e^{4m\beta} \sum_{1\le k< m(1-\delta),\ l=m-k} e^{-4l\beta}\Big(1-\left(1-\frac{l}{m}\right)^{2\alpha}\Big) \\ &:= \exp(4m\beta)(A+B). \end{aligned}$$

For the term A, we have

$$\begin{aligned} A &= \frac{2\alpha}{m}\Big(\sum_{m(1-\delta)\le k\le m,\ l=m-k} le^{-4l\beta} \\ &+ m^{-1}O\Big(\sum_{m(1-\delta)\le k\le m,\ l=m-k} l^2e^{-4l\beta}\Big)\Big) = \frac{2\alpha}{m}\Big(A_1 + O(m^{-1})\Big). \end{aligned}$$

Set $t = \exp(-4\beta)$. Then, the sum A_1 can be rewritten in the form

$$A_1 = A_1(m, \beta) = \exp(-4\beta)\big(\sum_{m(1-\delta)\le k\le m,\ l=m-k} t^l\big)'_t \asymp 1,$$

where $()'_t$ denotes differentiation with respect to t. Therefore,

$$A = \frac{2\alpha A_1(m,\beta)}{m} + O(m^{-2}) \asymp m^{-1}.$$

For the term B, we have

$$\begin{aligned} B &= \sum_{1\le k< m(1-\delta),\ l=m-k} \exp(-4l\beta)\Big(1-\Big(1-\frac{l}{m}\Big)^{2\alpha}\Big) \\ &= O(\exp(-4\beta m\delta)) = o(1/m^k),\ k\in\mathbb{N}. \end{aligned}$$

Hence, combining the two terms, we get the asymptotics

$$J_1 = \frac{2\alpha\exp(4\beta m)A_1(m,\beta)}{m}(1+O(m^{-1})) \asymp \frac{\exp(4\beta m)}{m}.$$

For the asymptotics of J_2, let us first rewrite it in the form

$$J_2 = -J_1 + \sum_{1\le k\le m} \exp(4\beta k)\Big(1-\Big(\frac{k}{m}\Big)^{4\alpha}\Big) = -J_1(\alpha,\beta) + J_1(2\alpha,\beta).$$

The asymptotics of $J_1(2\alpha, \beta)$ are studied similar to ones of $J_1 = J_1(\alpha, \beta)$, and we get

$$J_2 = 2\alpha m^{-1}e^{4\beta m}A_1(m,\beta)(1+O(m^{-1})) \sim J_1.$$

For the asymptotics of J_0, we use the second order Taylor's formula to get

$$\Big(1-\frac{l}{m}\Big)^{2\alpha} = 1 - \frac{2\alpha\, l}{m} + \frac{\alpha(2\alpha-1)\, l^2}{m^2} + O\left(\frac{l^3}{m^3}\right).$$

Repeating the considerations that we used for $J_1,\ l=m-k$ we have for

$$\begin{aligned} J_0 &= \exp(4m\beta)\left(\frac{4\alpha^2}{m^2}\sum_{m(1-\delta)\le k\le m}\exp(-4l\beta)\left(l^2+O\left(\frac{l^3}{m}\right)\right)\right)\\ &+ O(\exp(4m\beta(1-\delta)))\\ &= \exp(4m\beta)\left(\frac{4\alpha^2}{m^2}\sum_{m(1-\delta)\le k\le m}\exp(-4l\beta)\,l^2+O(m^{-3})\right)\\ &+ o(\exp(4m\beta)\,m^{-3})\\ &= \exp(4m\beta)\left(\frac{4\alpha^2}{m^2}A_2+O(m^{-3})\right)+o(\exp(4m\beta)\,m^{-3}). \end{aligned}$$

Taking derivatives as in the calculation of A_1, we get $A_2=A_2(m,\beta)\asymp 1$, which implies

$$J_0\asymp\frac{\exp(4\beta m)}{m^2}.$$

Thus, using (11.8), we obtain the following asymptotics

$$\begin{aligned} J_1 &\sim J_2\asymp\frac{\exp(4\beta m)}{m},\quad J_0\asymp\frac{\exp(4\beta m)}{m^2},\\ r_\varepsilon^2 &= m^{-2\alpha}\left(1+\frac{J_0}{J_1}\right)=m^{-2\alpha}\left(1+\frac{B}{m}\right),\ B\asymp 1, \end{aligned}$$

and, hence, we get the relations

$$r_\varepsilon^{-1/\alpha}=m+O(1),\quad u_\varepsilon^2\asymp\varepsilon^{-4}r_\varepsilon^4\exp(-4\beta r_\varepsilon^{-1/\alpha}). \tag{11.21}$$

Hence, the value u_ε of the extreme problem (3.1) and the efficient dimensions $m=m_\varepsilon$ satisfy

$$u_\varepsilon^2 \sim \varepsilon^{-4}m^{-4\alpha}\exp(-4m\beta)A_2/(2A_1^2), \tag{11.22}$$

$$m \sim \frac{2\log(\varepsilon^{-1})-2\alpha\log\log(\varepsilon^{-1})-(\log(u_\varepsilon)}{2\beta}+D, \tag{11.23}$$

where $D\asymp 1$ hold true uniformly over $(\alpha,\beta)\in\Sigma$ for any compact set $\Sigma\subset(0,\infty)\times(0,\infty)$. Hence, the "standard" case $q=2$ for the theorem follows on applying Theorem 4.1.

Consider now the "sparse" case $q\in(0,2)$. In view of the embedding (4.8), it suffices to establish the lower bounds. Take $m=\max\{k: r_\varepsilon k^\alpha\le 1\}, m=$

$r^{-1/a} + O(1)$, and consider the vector η_m that contains only one non-zero coordinate, the value $z_n = r_\varepsilon \exp(-\beta n)$ at position m. One can easily check that $\eta_m \in \Theta_q(r_\varepsilon)$ for any $q > 0$. Therefore, one cannot distinguish between H_0 and H_1 if $z_n = o(\varepsilon)$, which is equivalent to

$$r_\varepsilon/\varepsilon = o(\exp(-\beta r_\varepsilon^{-1/\alpha} + O(1)).$$

However, this is equivalent to $u_\varepsilon \to 0$, where u_ε is determined by (11.21). In view of the above and the results for the "standard" case $q = 2$, the "sparse" case $q = 2$ for the theorem also follows. Hence, the theorem follows.

11.7. *Proof of Theorem 4.5.* Let the efficient dimension $m = m_\varepsilon$ be determined by $A = \exp(-2m\alpha)$ in (4.1). Then, the extreme sequence (4.1) in the extreme problem (3.1) takes the form

$$\tilde{\eta}_k^2 = z_0^2 k^{2\beta}(1 - \exp(2\alpha(k-m)))_+, \quad 1 \le k \le m, \tag{11.24}$$

while the equations for $z_0 = z_{0,\varepsilon}$, $m = m_\varepsilon$ and u_ε take the form (11.7), (11.8), where

$$\begin{aligned} J_1 &= \sum_{1\le k\le m} k^{4\beta}(1 - \exp(-2\alpha(m-k))), \\ J_2 &= \sum_{1\le k\le m} k^{4\beta}\exp(-2\alpha(m-k))(1 - \exp(-2\alpha(m-k))), \\ J_0 &= J_1 - J_2 = \sum_{1\le k\le m} k^{4\beta}(1 - \exp(-2\alpha(m-k)))^2. \end{aligned}$$

We consider the situation $m \to \infty$ and $r_\varepsilon \to 0$ as $\varepsilon \to 0$. Let us now find the asymptotics of the sums $J_1,\ J_2$ and J_0 as $m \to \infty$. We have

$$J_1 = \sum_{1\le k\le m} k^{4\beta} - \sum_{1\le k\le m} k^{4\beta}\exp(-2\alpha(m-k)) = A - B, \tag{11.25}$$

where

$$A = \sum_{1\le k\le m} k^{4\beta} = m^{4\beta+1}\sum_{1\le k\le m}\left(\frac{k}{m}\right)^{4\beta}\frac{1}{m} \sim \frac{m^{4\beta+1}}{4\beta+1}.$$

Let us now evaluate the term B in the sum (11.25). Let $k = m - l$. Then,

$$\begin{aligned} B &= \sum_{1\le k\le m} k^{4\beta}\exp(-2\alpha(m-k)) \\ &= m^{4\beta}\sum_{1\le k\le m,\ l=m-k}(1 - l/m)^{4\beta}\exp(-2\alpha l) = m^{4\beta}B_1. \end{aligned}$$

Let $\alpha > 0$. Using the Taylor's formula, and since the series $\sum_{l=1}^{\infty} l^k \exp(-2\alpha l)$, $k = 1, 2$, converges, we get

$$B_1 = \sum_{1\le k\le m,\ l=m-k} \left(1 - \frac{4\beta\, l}{m} + O\left(\frac{l^2}{m^2}\right)\right) \exp(-2\alpha l) \asymp 1.$$

Therefore, combining the terms A, B and B_1, we get

$$J_1 \sim \frac{m^{4\beta+1}}{4\beta+1}. \tag{11.26}$$

Similarly, for J_2, letting $k = m - l$, we have

(11.27)

$$J_2 = m^{4\beta} \sum_{1\le k\le m,\ l=m-k} \left(\left(1 - \frac{l}{m}\right)^{4\beta}\right) \exp(-2\alpha l)(1 - \exp(-2\alpha l)) \asymp m^{4\beta}.$$

By (11.26) and (11.27), we have

$$J_0 = J_1 - J_2 \sim J_1 \sim c_1 m^{4\beta+1}, \quad J_2 \asymp m^{4\beta}, \tag{11.28}$$

where $c_1 = 1/(4\beta + 1)$. Hence using (11.8), we get the the relations

(11.29)

$$r_\varepsilon \asymp m^{1/2} e^{-\alpha m},\ u_\varepsilon^2 \sim d_1 (r_\varepsilon/\varepsilon)^4 m^{-(4\beta+1)} \sim d_2 (r_\varepsilon/\varepsilon)^4 (\log r_\varepsilon^{-1})^{-(4\beta+1)},$$

where $d_1 = 1/(2c_1)$ and $d_2 = \gamma_1 \alpha^{4\beta+1}$. Hence, the value u_ε of the extreme problem (3.1) and the efficient dimensions $m = m_\varepsilon$ satisfy

$$u_\varepsilon^2 \asymp \varepsilon^{-4} \exp\left(-4\alpha m\right) m^{-(4\beta-1)}, \quad m \sim \log(r_\varepsilon^{-1/\alpha}). \tag{11.30}$$

Observe also that, for the extreme sequence determined by (11.24), one has

$$w_0 = \frac{\max_{1\le k\le m} \tilde{\eta}_k^2}{\sqrt{2\sum_{k=1}^m \tilde{\eta}_k^4}} \le \frac{B z_0^2 m^{2\beta}}{z_0^2 m^{2\beta+1/2}} \asymp m^{-1/2} \to 0.$$

Hence, the theorem follows on applying Theorem 4.1.

11.8. *Proof of Theorem 7.1.* In view of the embedding (4.8), we only need to consider the case $q = 2$. It was shown in the proof of Theorem 8.1 that $\alpha(\psi_{\varepsilon,H}) \to 0$ as $H \to \infty$. In order to evaluate the type II error probability, it suffices to consider only the case where $u_\varepsilon(r_\varepsilon(\alpha, \beta)) = o(\log(\varepsilon^{-1}))$, uniformly over $(\alpha, \beta) \in \Sigma$. Similar to the proof of Theorem 8.1, we have the relation (11.16), and it suffices to evaluate the quantity

$$h_\varepsilon(\alpha, \beta) = \inf_{\eta \in \Theta_{\varepsilon,\alpha,\beta}(r_\varepsilon(\alpha,\beta))} \frac{1}{\varepsilon^2\sqrt{2\tilde{m}}} \sum_{k=1}^{\tilde{m}} \eta_k^2.$$

Since $Dm_\varepsilon(\alpha, \beta) \geq \tilde{m} \geq m_\varepsilon(\alpha, \beta)$ and $D = C(1 + o(1))/c > 0$, we have

$$h_\varepsilon(\alpha, \beta) \geq dh_\varepsilon^*(\alpha, \beta), \quad d = D^{-1/2}(1 + o(1)),$$

and

$$h_\varepsilon^*(\alpha, \beta) = \frac{1}{\varepsilon^2\sqrt{2\tilde{m}(\alpha, \beta)}} \inf_{\eta \in \Theta_{\varepsilon,\alpha,\beta}(r_\varepsilon(\alpha,\beta))} \sum_{k=1}^{\tilde{m}(\alpha,\beta)} \eta_k^2,$$

with $\tilde{m}(\alpha, \beta) = [m_\varepsilon(\alpha, \beta)]$, where $[a]$ is the integral part of a. By (11.29), the assumptions of Proposition 11.1 are fulfilled uniformly over $(\alpha, \beta) \in \Sigma$. In particular one can take $a > 0$ such that (11.12) holds true for all $(\alpha, \beta) \in \Sigma$, as ε is small enough. Applying now Proposition 11.1, we have $h_\varepsilon^*(\alpha, \beta) \geq bu_{\varepsilon,\alpha,\beta}(r_\varepsilon(\alpha, \beta)) \geq bu_\varepsilon(\Sigma)$. Therefore, we get $h_\varepsilon(\alpha, \beta) \geq b_1 u_\varepsilon(\Sigma) \to \infty$, $b_1 = bd$. By (11.16), this implies that it suffices to take $H_\varepsilon \to \infty$, $H_\varepsilon < b_2 u_\varepsilon(\Sigma)$ with any $b_2 \in (0, b_1)$. The theorem now follows.

11.9. *Proof of Theorem 4.6.* Before we prove the theorem we need the following result.

Recall, that the extreme sequence (4.1) in the extreme problem (3.1) is of the form

$$\tilde{\eta}_k^2 = z_0^2 \sigma_k^2 (1 - Aa_k^2)_+, \quad k \in \mathbb{N}, \tag{11.31}$$

where the quantities $z_0 = z_{0,\varepsilon}$ and $A = A_\varepsilon$ are determined by the equations

$$\begin{cases} \sum_{k \in \mathbb{N}} a_k^2 \sigma_k^2 \tilde{\eta}_k^2 = 1, \\ \sum_{k \in \mathbb{N}} \sigma_k^2 \tilde{\eta}_k^2 = r_\varepsilon^2. \end{cases}$$

and, thus, the value of the extreme problem (3.1) takes the form

$$u_\varepsilon^2 = \frac{1}{2\varepsilon^4} \sum_{k \in \mathbb{N}} \tilde{\eta}_k^4.$$

Consider now the following "truncated" version of the above system of equations

$$\left\{\begin{array}{l}\sum_{k=1}^m a_k^2\sigma_k^2\tilde\eta_k^2=1,\\ \sum_{k=1}^m \sigma_k^2\tilde\eta_k^2=r_\varepsilon^2,\end{array}\right.\quad u_\varepsilon^2=\frac{1}{2\varepsilon^4}\sum_{k=1}^m\tilde\eta_k^4. \tag{11.32}$$

In order to solve the equations (11.31)-(11.32), let us define a function $r(A)$, $A\in(0,a_2^{-2})$ as follows. Take $m=m(A)\in\mathbb{N},\ m\ge 2$, such that $a_{m+1}^{-2}\le A\le a_m^{-2}$ and set

$$r(A)=\left(\frac{\sum_{k=1}^m\sigma_k^4(1-Aa_k^2)}{\sum_{k=1}^m\sigma_k^4a_k^2(1-Aa_k^2)}\right)^{1/2},\quad A\in(0,a_2^{-2}). \tag{11.33}$$

Then, for r_ε small enough, the quantity $A=A_\varepsilon$ in (11.31) is determined by the equation

$$r_\varepsilon=r(A_\varepsilon). \tag{11.34}$$

Note first that $r(A)$ is a positive continuous functions in $A\in(0,a_2^{-2})$. The following proposition ensures the existence of a unique solution in (11.34). (Note that its validity does not depend on the assumption (4.10).)

Proposition 11.2. *The function $r(A)$ defined in (11.33) is strictly increasing in $A\in(0,a_2^{-2})$.*

Proof. Let $a_{m+1}^{-2}\le A<a_m^{-2},\ m\ge 2$. Introduce a probability measure $P=\{p_i\}_{i\in I}$ on the set $I=\{1,2,\dots,m\}$ such that $p_i=\sigma_i^4/\sum_{k=1}^m\sigma_k^4,\ i\in I$. Set

$$H_m(A)=\left(\sum_{k=1}^m\sigma_k^4a_k^2(1-Aa_k^2)\right)^2,\quad m\ge 2.$$

We consider $a=\{a_i\}_{i\in I}$ as random variable on the set I. Then, we have

$$\begin{aligned}(r^2(A))'_A &= \frac{(\sum_{k=1}^m\sigma_k^4)(\sum_{k=1}^m\sigma_k^4a_k^4)-(\sum_{k=1}^m\sigma_k^4a_k^2)^2}{H_m(A)}\\ &= \frac{\left(\sum_{k=1}^m a_k^4p_k-(\sum_{k=1}^m a_k^2p_k)^2\right)(\sum_{k=1}^m\sigma_k^4)^2}{H_m(A)}\\ &= \frac{\left(E_P(a^4)-(E_P(a^2))^2\right)(\sum_{k=1}^m\sigma_k^4)^2}{H_m(A)}\\ &= \frac{\mathrm{Var}_P(a^2)(\sum_{k=1}^m\sigma_k^4)^2}{H_m(A)}>0,\end{aligned}$$

where $(\cdot)'_A$ denotes differentiation with respect to A. The proposition now follows.

We are now ready to prove part (a) of the theorem. Let $A = A_\varepsilon$ be the solution of (11.34). It then follows from (4.10) that

$$\begin{cases} \sum_{k=1}^{m-2} a_k^2\sigma_k^2\tilde{\eta}_k^2 = \tau_1 a_{m-1}^2\sigma_{m-1}^2\tilde{\eta}_{m-1}^2, \\ \sum_{k=1}^{m-2} \sigma_k^2\tilde{\eta}_k^2 = \tau_2\sigma_{m-1}^2\tilde{\eta}_{m-1}^2, \end{cases} \qquad \sum_{k=1}^{m-2}\tilde{\eta}_k^4 = \tau_0\tilde{\eta}_{m-1}^4, \tag{11.35}$$

where $\tau_i = \tau_{m,i}(A),\ i = 0,1,2$, are such that

$$\tau_1 \sim \frac{\sigma_{m-2}^4 a_{m-2}^2(1-Aa_{m-2}^2)}{\sigma_{m-1}^4 a_{m-1}^2(1-Aa_{m-1}^2)} = o(1), \tag{11.36}$$

$$\tau_2 \sim \frac{\sigma_{m-2}^4(1-Aa_{m-2}^2)}{\sigma_{m-1}^4(1-Aa_{m-1}^2)} = o(1), \quad \tau_0 = o(1). \tag{11.37}$$

Therefore, we can rewrite the equations (11.32) in the form

$$\begin{cases} \theta_1 a_{m-1}^2\sigma_{m-1}^2\tilde{\eta}_{m-1}^2 + a_m^2\sigma_m^2\tilde{\eta}_m^2 = 1, \\ \theta_2\sigma_{m-1}^2\tilde{\eta}_{m-1}^2 + \sigma_m^2\tilde{\eta}_m^2 = r_\varepsilon^2, \end{cases} \qquad u_\varepsilon^2 = \varepsilon^{-4}(\theta_0\tilde{\eta}_{m-1}^4 + \tilde{\eta}_m^4)/2, \tag{11.38}$$

with $\theta_i = \theta_{m,i}(A) = 1 + \tau_{m,i}(A) \sim 1,\ i = 0,1,2$. Setting $z_1 = \tilde{\eta}_{m-1}^2,\ z_2 = \tilde{\eta}_m^2$ we find $z = (z_1, z_2)$ from (11.38):

$$z_1 = \frac{a_m^2 r_\varepsilon^2 - 1}{(\theta_2 a_m^2 - \theta_1 a_{m-1}^2)\sigma_{m-1}^2}, \quad z_2 = \frac{\theta_2 - a_{m-1}^2 r_\varepsilon^2\theta_1}{(\theta_2 a_m^2 - \theta_1 a_{m-1}^2)\sigma_m^2}; \quad u_\varepsilon^2 \sim \frac{\|z\|^2}{2\varepsilon^4}.$$

We have $\tilde{\eta}_m = 0$ (this corresponds to $A = a_m^{-2}$) as $r_\varepsilon^2 = r_{m-1}^2 := a_{m-1}^{-2}\theta_{m-1}$, where by (11.36), (11.37), $\theta_{m-1} = \theta_{m,2}(a_m^{-2})/\theta_{m,1}(a_m^{-2}) > 1,\ \theta_{m-1} \sim 1$. The conditions $z_1 > 0,\ z_2 \geq 0$ correspond to

$$a_m^{-2} < r_\varepsilon^2 \leq a_{m-1}^{-2}\theta_{m-1}. \tag{11.39}$$

By (11.36), (11.37), and the definition of r_m we have, as $m \to \infty$,

$$r_m^2 \sim \frac{1}{a_m^2}\left(1 + \frac{\sigma_{m-1}^4}{\sigma_m^4}\cdot\frac{(1 - a_{m-1}^2/a_m^2)(1 - a_{m-1}^2/a_{m+1}^2)}{(1 - a_m^2/a_{m+1}^2)}\right), \quad r_m^2 > \frac{1}{a_m^2}.$$

Recalling the monotonicity of $r(A)$, we see that if $a_{m+1}^{-2} \leq A_\varepsilon \leq a_m^{-2}$, then $r_\varepsilon = r(A_\varepsilon) \in \Delta_m = [r_m, r_{m-1}] = [a_m^{-1}(1+o(1)),\, a_{m-1}^{-1}(1+o(1))]$, where $r_i > a_i^{-1},\ i = m-1,\, m$.

Let $u_{\varepsilon,\min} = \min_{r_\varepsilon \in \Delta_m} u_\varepsilon(r_\varepsilon)$. Thus, we get

$$u_\varepsilon(r_\varepsilon) \geq u_{\varepsilon,\min} \asymp (\varepsilon^2 a_m^2 \sigma_m^2)^{-1} \quad \text{as} \quad r_\varepsilon \in \Delta_m. \tag{11.40}$$

Let us now consider the interval $\Delta_m^* = [r_{m,1}, r_{m-1,1}]$, $r_{l,1} = 1/a_l$. For $r_\varepsilon \in \Delta_m^*$, we set $\tilde{z} = (\tilde{z}_1, \tilde{z}_2)$, $z^* = (z_1^*, z_2^*)$,

$$\begin{aligned}
\tilde{z}_1 &= \frac{a_m^2 r_\varepsilon^2 - 1}{(\theta_2 a_m^2 - \theta_1 a_{m-1}^2)\sigma_{m-1}^2}, \quad \tilde{z}_2 = \frac{1 - a_{m-1}^2 r_\varepsilon^2}{(\theta_2 a_m^2 - \theta_1 a_{m-1}^2)\sigma_m^2};\\
z_1^* &= \frac{a_m^2 r_\varepsilon^2 - 1}{(a_m^2 - a_{m-1}^2)\sigma_{m-1}^2}, \quad z_2^* = \frac{1 - a_{m-1}^2 r_\varepsilon^2}{(a_m^2 - a_{m-1}^2)\sigma_m^2};\\
\tilde{u}_\varepsilon &= \tilde{u}_\varepsilon(r_\varepsilon) = \frac{\|\tilde{z}\|}{\sqrt{2}\varepsilon^2}, \quad u_\varepsilon^* = u_\varepsilon^*(r_\varepsilon) = \frac{\|z^*\|}{\sqrt{2}\varepsilon^2}.
\end{aligned}$$

Note that, for some $B > 0$,

$$|u_\varepsilon^*(r_2) - u_\varepsilon^*(r_1)| \leq B(r_2^2 - r_1^2)/\varepsilon^2 \sigma_{m-1}^2, \ \text{as } r_{m,1} \leq r_1 < r_2 \leq r_{m-1,1}, \tag{11.41}$$

and it is easily seen that

$$\tilde{u}_\varepsilon(r_\varepsilon) \sim u_\varepsilon^*(r_\varepsilon) \quad \text{as } r_\varepsilon \to 0; \quad u_\varepsilon^*(r) \geq u_\varepsilon(r) \quad \forall\ r > 0.$$

Also, for $\delta = \tilde{z} - z$ and for $r_\varepsilon \in \Delta_m \bigcap \Delta_m^* = [r_m, r_{m-1,1}]$, we have

$$\|\delta\| = o((a_{m-1}^2 r_\varepsilon^2 + 1)/a_m^2 \sigma_m^2) = \varepsilon^2 o(u_{\varepsilon,\min}).$$

These yields, as $r_\varepsilon \in [r_m, r_{m-1,1}]$,

$$\begin{aligned}
&u_\varepsilon(r_\varepsilon) \sim u_\varepsilon^*(r_\varepsilon)\\
&= \frac{1}{\sqrt{2}\varepsilon^2 (a_m^2 - a_{m-1}^2)} \left(\frac{(a_m^2 r_\varepsilon^2 - 1)^2}{\sigma_{m-1}^4} + \frac{(1 - a_{m-1}^2 r_\varepsilon^2)^2}{\sigma_m^4} \right)^{1/2}.
\end{aligned} \tag{11.42}$$

Let $r_\varepsilon \in [r_{m-1,1}, r_{m-1}] \subset \Delta_{m-1}^*$. Observe that

$$\begin{aligned}
&0 \leq u_\varepsilon^*(r_\varepsilon) - u_\varepsilon(r_\varepsilon)\\
&\leq u_\varepsilon^*(r_{m-1,1}) - u_\varepsilon(r_{m-1,1}) + |u_\varepsilon^*(r_\varepsilon) - u_\varepsilon^*(r_{m-1,1})| = \xi_1 + \xi_2,
\end{aligned}$$

where

$$\xi_1 = u_\varepsilon^*(r_{m-1,1}) - u_\varepsilon(r_{m-1,1}), \quad \xi_2 = |u_\varepsilon^*(r_\varepsilon) - u_\varepsilon^*(r_{m-1,1})|.$$

By (11.42),

$$\xi_1 = o(u_\varepsilon(r_{m-1,1})) = o(u_\varepsilon(r_\varepsilon)).$$

Applying (11.41) for the interval Δ^*_{m-1} we get

$$\xi_2 \le B\frac{r^2_{m-1} - r^2_{m-1,1}}{\varepsilon^2\sigma^2_{m-2}}.$$

Since $r^2_{m-1} - r^2_{m-1,1} = (\theta_{m-1} - 1)/a^2_{m-1}$, using (11.36), (11.37), we have $\theta_{m-1} - 1 = O(\sigma^4_{m-2}/\sigma^4_{m-1})$. By (11.40), these yield

$$\xi_2 = O\left(\frac{\sigma^2_{m-2}}{\varepsilon^2 a^2_{m-1}\sigma^4_{m-1}}\right) = o(u_\varepsilon(r_\varepsilon)),$$

as $r_\varepsilon \in \Delta^*_{m-1}$. This completes part (a) of the theorem.

We now prove part (b) of the theorem. For $r \in \Delta^*_m$, $m \in \mathbb{N}$, $m \ge 2$, consider the piecewise linear (in r^2) function $u^{lin}_\varepsilon(r)$ defined in (4.12). We then have, at the break points,

$$u^{lin}_\varepsilon(1/a_m) = \frac{1}{\varepsilon^2 a^2_m \sigma^2_m}, \quad u^{lin}_\varepsilon(1/a_{m-1}) = \frac{1}{\varepsilon^2 a^2_{m-1}\sigma^2_{m-1}}. \tag{11.43}$$

Using the standard inequalities

$$(x+y)/\sqrt{2} \le \sqrt{x^2+y^2} \le x+y, \quad x \ge 0,\ y \ge 0,$$

we get, for $r > 0$ small enough,

$$u^{lin}_\varepsilon(r)/2 \le u^*_\varepsilon(r) \le u^{lin}_\varepsilon(r)/\sqrt{2}. \tag{11.44}$$

Part (b) of the theorem follows by (11.44) and part (a) of the theorem.

Parts (c) and (d) of the theorem follow immediately by combining Theorem 4.1 and part (b) of the theorem. The theorem now follows.

11.10. *Proof of Theorem 9.1.* For type I error probability, we have

$$\alpha(\psi_{\varepsilon,\alpha}) = P_{\varepsilon,0}(\overline{\mathcal{Y}}_{\varepsilon,\alpha}) \le \sum_{k=1}^m P_{\varepsilon,0}(|y_k| \ge T_{m,k}\varepsilon) \le 2\sum_{k=1}^m \Phi(-T_{m,k}) = \alpha.$$

In order to evaluate type II error probability, observe that

$$\begin{aligned}\beta(\eta,\psi_{\varepsilon,\alpha}) &= P_{\varepsilon,\eta}(\mathcal{Y}_{\varepsilon,\alpha}) \le \min_{1\le k\le m} P_{\varepsilon,\eta}(|y_k| < T_{m,k}\varepsilon)\\ &\le \Phi(\min_{1\le k\le m}(T_{m,k} - \varepsilon^{-1}\eta_k)),\end{aligned}$$

and it suffices to check that

(11.45)
$$\inf_{\eta\in\Theta(r_\varepsilon)}\left(\max_{1\le k\le m}(\varepsilon^{-1}\eta_k - T_{m,k})\right)\to\infty \quad \text{as} \quad u_\varepsilon^{lin}(r_\varepsilon)\to\infty, \quad r_\varepsilon\in\Delta_m^*.$$

The following proposition is useful to our goal.

Proposition 11.3. *Let assume (4.10) holds true. Let $r_\varepsilon \in \Delta_m^*$, consider the collection $H_{m,k},\ 1\le k\le m$ satisfying $0 < H_{m,k} \le B_1(m-k+1)^{B_2}$ for some $B_l > 0,\ l = 1,2$ if $1 \le k \le m-2$ and $H_{m,m} = H_{m,m-1} = 1$. Then*

$$\inf_{\eta\in\Theta(r_\varepsilon)}\max_{1\le k\le m}\varepsilon^{-2}H_{m,k}^{-1}\eta_k^2 \ge u_\varepsilon^{lin}(r_\varepsilon)(1/(2\sqrt{2})+o(1)).$$

Proof. Let $\eta\in\Theta(r_\varepsilon)$, take

$$r_\varepsilon^2 = \frac{1-t}{a_m^2}+\frac{t}{a_{m-1}^2}, \quad t\in[0,1], \tag{11.46}$$

and suppose that

$$\max_{1\le k\le m-2}\varepsilon^{-2}H_{m,k}^{-1}\eta_k^2 \le u^{lin}(r_\varepsilon).$$

On noting that u_ε^{lin}, in view of (11.46), takes the form

$$u_\varepsilon^{lin}(r) = \frac{1-t}{\varepsilon^2\sigma_m^2 a_m^2}+\frac{t}{\varepsilon^2\sigma_{m-1}^2 a_{m-1}^2}, \quad t\in[0,1],$$

we then get

$$\begin{aligned}\sum_{k=1}^{m-2}\sigma_k^2\eta_k^2 &\le \varepsilon^2 u^{lin}(r_\varepsilon)\sum_{k=1}^{m-2}\sigma_k^2 H_{m,k}\\ &\asymp \sigma_{m-2}^2\left(\frac{1-t}{\sigma_m^2 a_m^2}+\frac{t}{\sigma_{m-1}^2 a_{m-1}^2}\right) =: \delta = o(r_\varepsilon^2).\end{aligned}$$

Set $\tilde\eta = (0,\ldots,0,\eta_{m-1},\eta_m,\ldots)$. It follows from the estimation above that $\tilde\eta\in\Theta(\tilde r_\varepsilon),\ \tilde r_\varepsilon^2 = r_\varepsilon^2 - b\delta$ for some $b>0$, and

$$\begin{aligned}u_\varepsilon(\tilde r_\varepsilon) &\ge u_\varepsilon^{lin}(\tilde r_\varepsilon)(1/2+o(1)) \ge (1/2+o(1))\left(u_\varepsilon^{lin}(r_\varepsilon)-\frac{B\delta}{\sigma_{m-1}^2\varepsilon^2}\right)\\ &\sim \frac{1}{2}\left(1-\frac{B\sigma_{m-2}^2}{\sigma_{m-1}^2}\right)u_\varepsilon^{lin}(r_\varepsilon)\sim u_\varepsilon^{lin}(r_\varepsilon)/2 \ge u_\varepsilon(r_\varepsilon)(1/\sqrt{2}+o(1)).\end{aligned}$$

This implies

$$\varepsilon^{-4}\sum_{k\in\mathbb{N}}\eta_k^4 = \varepsilon^{-4}\sum_{k=m-1}^{\infty}\eta_k^4 \geq 2u_\varepsilon^2(\tilde{r}_\varepsilon) \geq u_\varepsilon^2(r_\varepsilon)(1+o(1)).$$

Since $\sum_{k\in\mathbb{N}} a_k^2\sigma_k^2\eta_k^2 \leq 1$, we have $\eta_k \leq (a_k\sigma_k)^{-1}$ and

$$\begin{aligned}\sum_{k=m+1}^{\infty}\eta_k^4 \leq \sum_{k=m+1}^{\infty}(a_k\sigma_k)^{-4} &\leq (a_{m+1}\sigma_{m+1})^{-4}\sum_{k=m+1}^{\infty}\frac{(a_{m+1}\sigma_{m+1})^4}{(a_k\sigma_k)^4}\\ &\sim (a_{m+1}\sigma_{m+1})^{-4} = o(\varepsilon^4 u_\varepsilon^2(r_\varepsilon)).\end{aligned}$$

Thus, for m large enough,

$$\varepsilon^{-4}\max(\eta_{m-1}^4,\eta_m^4) \geq \varepsilon^{-4}(\eta_{m-1}^4+\eta_m^4)/2 \geq u_\varepsilon^2(r_\varepsilon)(1/2+o(1)),$$

which yields

$$\varepsilon^{-2}\max(\eta_{m-1}^2,\eta_m^2) \geq u_\varepsilon(r_\varepsilon)(1/\sqrt{2}+o(1)) \geq u^{lin}(r_\varepsilon)(1/2\sqrt{2}+o(1)).$$

The proposition now follows.

We are now ready to complete the proof of the theorem. Note that $T_{m,k} \geq \Phi^{-1}(1-c\alpha)$ are bounded away from 0. The collection $H_{m,k} = (T_{m,k}/T_{m,m})^2$ satisfies the assumption of Proposition 11.3 since, as $m-k\to\infty$,

$$T_{m,k} = \Phi^{-1}\left(1-\frac{c\alpha}{(m-k-1)^2}\right) \sim \sqrt{2\log\left(\frac{(m-k-1)^2}{c\alpha}\right)}.$$

Applying now Proposition 11.3 to this collection we get that there exists $k,\ 1\leq k\leq m$ such that

$$\varepsilon^{-1}\eta_k \geq 8^{-1/4}(T_{m,k}/T_{m,m})\sqrt{u_\varepsilon^{lin}(r_\varepsilon)}(1+o(1)),$$

which yield

$$\max_{1\leq k\leq m}(\varepsilon^{-1}\eta_k - T_{m,k}) \to \infty \quad \text{as} \quad u_\varepsilon^{lin}(r_\varepsilon)\to\infty.$$

This implies (11.45). The theorem now follows.

11.11. *Proof of Theorem 5.1.* We first obtain the lower bounds. Take a collection κ_l such that

$$\phi(\kappa_l) = a + l\delta_\varepsilon,\ 1 \le l \le L = L_\varepsilon,\ \phi(\kappa_L) = b,\ \ \delta = \delta_\varepsilon = \frac{(b-a)}{L} \sim \frac{\log(3)}{\log(\varepsilon^{-1})}.$$

Assume, without loss of generality, that $u_\varepsilon(\kappa_l) \asymp \sqrt{\log\log(\varepsilon^{-1})}$ uniformly in $l = 1, 2, \ldots, L$. Observe that $\log(L) \sim \log\log(\varepsilon^{-1})$. Set

$$m_l \sim \left(\varepsilon\left(\log\log(\varepsilon^{-1})\right)^{1/4}\right)^{-\phi(\kappa_l)}. \tag{11.47}$$

By construction, we have

$$\begin{aligned} m_l - m_{l-1} &\sim m_{l-1}\left(\exp\left(\delta\log\left(\varepsilon^{-1}\left(\log\log(\varepsilon^{-1})\right)^{-1/4}\right)\right) - 1\right) \\ &= m_{l-1}(3(1+o(1))) - 1) \sim 2m_{l-1}. \end{aligned}$$

Set

$$\Delta_l = \{k \in \mathbb{N} : m_{l-1} < k \le m_l\}, \quad M_l = \#(\Delta_l)2 \sim m_{l-1}. \tag{11.48}$$

Take a collection $z_l > 0$ such that

$$z_l^2 M_l a_{m_l}^2(\kappa_l)\sigma_{m_l}^2(\kappa_l) = 1, \quad 1 \le l \le L. \tag{11.49}$$

By (11.10), (11.11), the relation (11.49) implies that, as the quantity d in Theorem 5.1 (a) is small enough (this corresponds to $r_\varepsilon(\kappa)$ small enough), one has

$$z_l^2 M_l \sigma_{m_{l-1}}^2(\kappa_l) \ge r_\varepsilon^2(\kappa_l), \quad 1 \le l \le L. \tag{11.50}$$

Set $u_l^2 = M_l z_l^4/(2\varepsilon^4)$. Observe that the relations (11.49), (11.47), (11.48) imply

$$u_l^2 \sim 3\log\log(\varepsilon^{-1})/4 \asymp u_\varepsilon^2(\kappa_l), \tag{11.51}$$

Therefore the relations $z_l^4 = 2\varepsilon^4 u_l^2/M_l$, (11.47), (11.51) and (11.48) imply

$$z_l = o(\varepsilon). \tag{11.52}$$

Consider the priors

$$\pi_l = \prod_{k\in\Delta_l} (\delta_{z_l e_k} + \delta_{-z_l e_k})/2, \quad \pi = \frac{1}{L}\sum_{l=1}^{L} \pi_l,$$

where $\{e_k\}_{k\in\mathbb{N}}$ is the standard basis in l^2 and δ_η is the Dirac mass at the point $\eta \in l^2$. The relations (11.49), (11.50) imply, for $d = d(\Sigma)$ small enough, $\pi_l(\Theta_{\kappa_l}(r_\varepsilon(\kappa_l))) = 1$, $\pi(\Theta(\Sigma)) = 1$. Let $P_{\pi_l} = E_{\pi_l} P_{\varepsilon,\eta}$ and $P_\pi = E_\pi P_{\varepsilon,\eta}$ be the mixtures over the priors. It suffices to check that

$$E_{\varepsilon,0}\left((dP_\pi/dP_{\varepsilon,0} - 1)^2\right) = o(1). \tag{11.53}$$

Using evaluations similar to [11] (see formulae (3.64)–(3.69)), we have

$$\begin{aligned} E_{\varepsilon,0}\left((dP_\pi/dP_{\varepsilon,0} - 1)^2\right) &= \frac{1}{L^2}\sum_{l=1}^{L} E_{\varepsilon,0}\left((dP_{\pi_l}/dP_{\varepsilon,0} - 1)^2\right) \\ &= \frac{1}{L^2}\sum_{l=1}^{L}\left(E_{\varepsilon,0}\left(dP_{\pi_l}/dP_{\varepsilon,0}\right)^2 - 1\right) \\ &\le \frac{1}{L^2}\sum_{l=1}^{L}\left(e^{\tilde{u}_l^2} - 1\right), \end{aligned}$$

where $\tilde{u}_l^2 = 2M_l \sinh^2(z_l^2/(2\varepsilon^2)) \sim u_l^2$ by (11.52). By (11.51) one has

$$\frac{\max_l u_l^2}{\log(L)} \sim 3/4 < 1. \tag{11.54}$$

This yields (11.53) and completes part (a) of the theorem.

We now obtain the upper bounds. Recall that we have, in Theorem 5.1 (b), $L_\varepsilon = o(\log(\varepsilon^{-1}))$, $L_\varepsilon \to \infty$. It follows from the exponential inequality for χ^2-statistics that

$$\log(P_{\varepsilon,0}(t_m > H)) \le -H^2/2(1+o(1)) \quad \text{as } H = o(\sqrt{m}), \quad H \to \infty, \tag{11.55}$$

see, e.g., (5.22) in Ingster & Suslina (2004). This implies that, for the type I error probability,

$$\alpha(\psi_\varepsilon) \le \sum_{l=L_\varepsilon}^{\infty} P_{\varepsilon,0}(t_{m_l} > H_l) \le \sum_{l=L_\varepsilon}^{\infty} l^{-C/2+o(1)} \to 0 \quad \text{as} \quad L_\varepsilon \to \infty.$$

Let us evaluate the type II error probability. It suffices to consider the case $u_\varepsilon = D\sqrt{\log\log(\varepsilon^{-1})}$ with D larger enough. Observe that (see (11.16))

$$\beta_\varepsilon(\eta,\psi_\varepsilon) \le \min_{l\ge L_\varepsilon} P_{\varepsilon,\eta}(t_{m_l} \le H_l) = \min_{l\ge L_\varepsilon} \Phi(H_l - h_{m_l}(\eta)) + o(1),$$

where $h_m(\eta)$ is determined by (11.13). Therefore uniformly over $\kappa\in\Sigma$,

$$\beta_\varepsilon(\Theta_{\varepsilon,\kappa},\psi_\varepsilon) \le \Phi(\sqrt{C\log L} - \max_{l\ge L_\varepsilon} h_{m_l}(\kappa)) + o(1),\ \ h_{m_l}(\kappa) = \inf_{\eta\in\Theta_{\varepsilon,\kappa}} h_{m_l}(\eta).$$

For $\kappa\in\Sigma$, let us set $m_\varepsilon(\kappa) = (\varepsilon^{-4}\log\log(\varepsilon^{-1}))^{1/(4\alpha+4\beta+1)}$ and take l such that $m_{l-1}\le m_\varepsilon(\kappa) < m_l$, i.e.,

$$m_l = cm_\varepsilon(\kappa),\ \ c\in(1,2],\quad l\sim \frac{4\log(\varepsilon^{-1})}{(4\alpha+4\beta+1)\log(2)} > L_\varepsilon.$$

It follows from (11.10), (8.2) that, for $D = D_{max}(\Sigma)$ larger enough, $r_\varepsilon(\kappa)a_{m_l+1}(\kappa)\ge B+o(1)$, with $B=B(\Sigma)>1$ that could be taken common for all $\kappa\in\Sigma$. It follows from (8.2) that the assumptions of Proposition 11.1 are fulfilled for $\tilde m = m_l$ with some $a(\Sigma)=\sup_{\kappa\in\Sigma} a(\kappa)>0$, uniformly over $\kappa\in\Sigma$. Applying Proposition 11.1 one can take $b=b(\Sigma)$ such that, uniformly over $\kappa\in\Sigma$, $h_{m_l}(\kappa)\ge bu_\varepsilon(\kappa)$. Thus, it suffices take $D(\Sigma) > \max(D_{max}(\Sigma), C/b(\Sigma))$. This completes part (b) of the theorem.

Part (c) of the theorem follows immediately in view of parts (a) and (b) of the theorem and (11.10). The theorem now follows.

11.12. *Proof of Theorem 5.2.* We first obtain the lower bounds. Take a collection κ_l such that $\phi(\kappa_l) = a_\varepsilon + l\delta_\varepsilon,\ 1\le l\le L = L_\varepsilon,\ \phi(\kappa_L) = b_\varepsilon$, where $a < a_\varepsilon < b_\varepsilon < b,\ a_\varepsilon = a+o(1),\ b_\varepsilon = b+o(1)$ and take L such that

$$\delta_\varepsilon = \frac{b_\varepsilon - a_\varepsilon}{L} \sim \frac{2}{2\log(\varepsilon^{-1}) - \log\log\log(\varepsilon^{-1})},$$

$$m_l = [\phi(\kappa_l)(2\log(\varepsilon^{-1}) - \log\log\log(\varepsilon^{-1}) - \log(c))]\in\mathbb{N},\ \ c<\exp(-1/2),$$

where $[a]$ is the integral part of a. By construction, $m_l - m_{l-1}\sim 2$.

Applying (11.18), we see that, if $u_\varepsilon(\kappa) < d\log\log(\varepsilon^{-1})$ for all $\kappa\in\Sigma$ and some $d>0$, then $u^*_\varepsilon(\kappa) = \varepsilon^{-2}r_\varepsilon^{2(\alpha(\kappa)+\beta(\kappa))/\alpha(\kappa)} < d_1\log\log(\varepsilon^{-1})$ for all $\kappa\in\Sigma$ and some d_1. Observe that, for any $c>0$ from the definition of m_l above, one can take d small enough (this corresponds to $r_\varepsilon(\kappa)$ small enough) such that $d_1 \le c$. This yields

$$\exp(-\alpha_l m_l) \ge r_\varepsilon(\kappa_l). \tag{11.56}$$

For $\kappa_l \in \Sigma$, let us take $z_l = \eta_l e_{m_l}$, where $\eta_l = \exp(-(\alpha_l + \beta_l) m_l)$ and $\{e_l\}_{l\in\mathbb{N}}$ is the standard basis in l^2. By (11.56) this yields $\eta_l \in \Theta_\varepsilon(\kappa_l, r_\varepsilon(\kappa_l))$ for any $q = q_l > 0$. Let us consider the prior

$$\pi = \frac{1}{L}\sum_{l=1}^{L} \delta_{z_l}$$

and the mixture P_π over π. Since $\pi(\Theta_\varepsilon(\Sigma)) = 1$, it suffices to verify that (see [11], Section 2.5.2, Propositions 2.11, 2.12)

$$E_{\varepsilon,0}(dP_\pi/dP_{\varepsilon,0} - 1)^2 \to 0. \tag{11.57}$$

One has

$$\begin{aligned} E_{\varepsilon,0}(dP_\pi/dP_{\varepsilon,0} - 1)^2 &= \frac{1}{L^2}\sum_{l=1}^{L} E_{\varepsilon,0}(dP_{\varepsilon,z_l}/dP_{\varepsilon,0} - 1)^2 \\ &= \frac{1}{L^2}\sum_{l=1}^{L}(e^{\eta_l^2/\varepsilon^2} - 1). \end{aligned} \tag{11.58}$$

The relation (11.57) holds true as $L \asymp \log(\varepsilon^{-1})$ and for c small enough

(11.59)
$$\max_{1\le l\le L} \eta_l^2/\varepsilon^2 \le c\log\log(\varepsilon^{-1}) \sup_{\kappa\in\Sigma} \exp(2(\alpha_l + \beta_l)) = c_1 \log\log(\varepsilon^{-1}),\ c_1 < 1.$$

Thus (11.59) holds true under the assumption of the theorem for d small enough. This completes part (a) of the theorem.

In order to obtain the upper bounds, we need the following (general) proposition and its corollary.

Proposition 11.4. *Let $b = \{b_i\}_{i\in\mathbb{N}}$ and $c = \{c_i\}_{i\in\mathbb{N}}$ be positive sequences, $b = \{b_i\}_{i\in\mathbb{N}}$ be an increasing sequence, $b_i \to \infty$ and $c_i b_i \to \infty$ as*

$i \to \infty$. Let also $r > 0$ be a small enough quantity and let $X = \{x | x = \{x_i\}_{i\in\mathbb{N}}\}$ be a set of sequences $x = \{x_i\}_{i\in\mathbb{N}}$ that are determined by the constraints

$$\sum_{i\in\mathbb{N}} b_i c_i x_i \le 1, \quad \sum_{i\in\mathbb{N}} c_i x_i \ge r, \quad x_i \ge 0 \quad \forall\, i \in \mathbb{N}.$$

Consider the extreme problem

$$w = w(r) = \inf_{x\in X} \phi(x), \quad \phi(x) = \sup_{i\in\mathbb{N}} x_i.$$

Then, the extreme sequences $x^* = \{x_i^*\}_{i\in\mathbb{N}}$ such that $\phi(x^*) = w$ is of the form:

$$x_1^* = w,\ i = 1, 2, \ldots, m-1, \quad x_m^* = w_0, \quad x_i = 0 \quad \text{as} \quad i > m,$$

where the quantities w and w_0, $0 \le w_0 \le w$, are of the form

$$w = \frac{rb_m - 1}{\sum_{i=1}^{m-1} c_i(b_m - b_i)}, \quad w_0 = \frac{\sum_{i=1}^{m-1} c_i(1 - rb_i)}{c_m \sum_{i=1}^{m-1} c_i(b_m - b_i)},$$

and the integer m is determined by the inequalities

$$B_m \le r \le B_{m-1}, \quad B_k = \frac{\sum_{i=1}^{k} c_i}{\sum_{i=1}^{k} b_i c_i}, \quad k = 1, 2, \ldots, m. \tag{11.60}$$

One further obtains the inequalities

$$C_m \le w \le C_{m-1}, \quad C_k = \frac{1}{\sum_{i=1}^{k} b_i c_i}, \quad k = 1, 2, \ldots, m. \tag{11.61}$$

Proof. In order to find a minimum of a convex function defined on a convex set X, we use the methods of sub-differentials (see Tikhomirov (1976)). Consider X and ϕ as in the statement of the proposition, and let $x \in X$. Then, the structure of X implies that $\lim_{i\to\infty} x_i = 0$ and there exists $i \in \mathbb{N}$ such that $x_i > 0$.

Let as consider the sets $I(x)$ consisting of the indices $i \in \mathbb{N}$ such that $x_i = \sup_{i\in\mathbb{N}} x_i$. Then $I(x) \ne \emptyset,\ x \in X$, and for $i \in I(x)$ we have $x_i > 0$. The sub-differential of the convex function $\phi(x) = \sup_i x_i$ consists of sequences $d = \{d_i\}_{i\in\mathbb{N}}$ such that $d_i \ge 0$, $i \in \mathbb{N}$, $d_i = 0$ for $i \notin I(x)$, and $\sum_{i\in\mathbb{N}} d_i = 1$

(see Lemma 1 in Section 1.4.1 of Tikhomirov (1976)). We get the following relations for the extreme sequence x^*:

$$d_i = \lambda c_i - \mu c_i b_i + \varepsilon_i, \quad i \in \mathbb{N},$$

where $\lambda \geq 0, \mu \geq 0$ and d_i, ε_i, $i \in \mathbb{N}$, are non-negative quantities such that: if $\lambda > 0$, then $\sum_{i\in\mathbb{N}} c_i x_i^* = r$; if $\mu > 0$, then $\sum_{i\in\mathbb{N}} b_i c_i x_i^* = 1$; if $i \notin I(x^*)$, then $d_i = 0$ and $x_i^* \varepsilon_i = 0,\ i \in \mathbb{N}$, $\sum_{i\in\mathbb{N}} d_i = 1$. These relations are possible if $\lambda > 0, \mu > 0$ only, and it can be rewritten in the form

$$d_i = \lambda c_i (1 - b_i/B) + \varepsilon_i, \quad i \in \mathbb{N}, \quad B > 0.$$

Since $b_i > 0$ increases in $i \in \mathbb{N}$, and $b_i \to \infty$, as $i \to \infty$, then $d_i > 0$, $\varepsilon_i = 0$, $i \in \mathbb{N}$, $x_i^* = \sup_{i\in\mathbb{N}} x_i := w > 0$ as $i \leq m-1$, where $m = m(B) = \max\{i : b_i \leq B\}$ and $x_i^* = 0$ as $i > m$. The quantities B and $x_m^* := w_0$ are taken such that $b_m = B,\ d_m = \varepsilon_m \geq 0$,

$$w \sum_{i=1}^{m-1} b_i c_i + w_0 b_m c_m = 1, \quad w \sum_{i=1}^{m-1} c_i + w_0 c_m = r, \quad 0 \leq w_0 \leq w.$$

The proposition now follows.

Corollary 11.1. *Let $a_k = \exp(\alpha k)$ and $\sigma_k = \exp(\beta k)$, $k \in \mathbb{N}$, $\alpha > 0$ and $\beta > 0$. Let $r_\varepsilon > 0$, $r_\varepsilon \to 0$. Set $m = -(\log r_\varepsilon)/\alpha + O(1)$. Then, for $m_1 = m + c$ and $c > 0$ large enough, one has*

$$\inf_{\eta\in\Theta(r_\varepsilon)} \max_{1\leq i\leq m_1} \eta_i^2 \asymp \exp(-2m(\alpha+\beta)) \asymp \varepsilon^2 u_\varepsilon.$$

Proof. We apply Proposition 11.4 to $i = k \in \mathbb{N}$, $b_i = a_i^2$, $c_i = \sigma_i^2$, $x_i = \eta_i^2$, $X = \Theta(r_\varepsilon)$ and $r = r_\varepsilon^2$. It then follows from (11.60), (11.61) that

$$\inf_{\eta\in\Theta(r_\varepsilon)} \sup_{i\in\mathbb{N}} \eta_i^2 \asymp \exp(-2m(\alpha+\beta)), \quad m = -\frac{\log r_\varepsilon}{\alpha} + O(1).$$

Therefore and by (11.18) we have $\exp(-2m(\alpha+\beta)) \asymp r_\varepsilon^{2(\alpha+\beta)/\beta} \asymp \varepsilon^2 u_\varepsilon$. It suffices now to check that we can replace $\sup_{i\in\mathbb{N}}$ by $\max_{i\leq m_1}$ for $m_1 = m+c$ and $c > 0$ large enough. This follows immediately from the inequalities $a_i^2 \sigma_i^2 \eta_i^2 \leq 1$, $i \in \mathbb{N}$. This completes the proof of the corollary.

We are now ready to obtain the upper bounds. One has

$$\begin{aligned}\alpha(\psi_\varepsilon) &\le \sum_{l=1}^{\infty} P_{\varepsilon,0}(|y_l|/\varepsilon > H_l) = 2\sum_{l=1}^{\infty}\Phi(-H_l)\\ &\asymp \frac{1}{\sqrt{\log(L)}} + \sum_{l=L}^{\infty}\frac{1}{l^{C/2}\sqrt{\log(l)}} \to 0.\end{aligned}$$

Let us now evaluate the type II error probability. In view of the embedding (4.8), it suffices to consider the case $q = 2$. We have

$$\beta_\varepsilon(\psi_\varepsilon,\eta) \le \min_{l\ge L} P_{\varepsilon,\eta}(|y_l|/\varepsilon \le H_l) \le \min_{l\ge L}\Phi(H_l - |\eta_l|/\varepsilon).$$

It suffices to verify that, uniformly over $\kappa \in \Sigma$,

$$\inf_{\eta\in\Theta_\varepsilon(\kappa,r_\varepsilon(\kappa))} \max_l(\eta_l^2/\varepsilon^2 - H_l^2) \to \infty. \tag{11.62}$$

We apply Corollary 11.1. Since

$$m = \frac{2\log(\varepsilon^{-1}) - \log(u_\varepsilon) + O(1)}{2(\alpha+\beta)} = O(\log(\varepsilon^{-1}))$$

and, as $L < l \le m_1 = m + c,\ c = O(1)$,

$$H_l^2 = C\log(l) \le C\log(m_1) \le C\log\log(\varepsilon^{-1})) + O(1),$$

it follows from Corollary 11.1 that

$$\begin{aligned}\inf_{\eta\in\Theta_\varepsilon(\kappa)}\max_l(\eta_l^2/\varepsilon^2 - H_l^2) &\ge \inf_{\eta\in\Theta_\varepsilon(\kappa)}\max_{l\le m_1}(\eta_l^2/\varepsilon^2 - H_l^2)\\ &\ge bu_\varepsilon - C\log\log(\varepsilon^{-1}) \to \infty,\end{aligned}$$

as $\liminf u_\varepsilon/\log\log(\varepsilon^{-1}) > D$, for D large enough. This completes part (b) of the theorem.

Part (c) of the theorem follows immediately in view of parts (a) and (b) of the theorem and (11.18). The theorem now follows.

11.13. *Proof of Theorem 5.3.* We first obtain the lower bounds. Set $H = (\varepsilon^2\log^{2\alpha}(\varepsilon^{-1})\log\log(\varepsilon^{-1}))^{-1}$. Take a collection $\kappa_l = (\alpha,\beta_l) \in \Sigma$ such that

$$\frac{1}{\beta_l} = 2a_\varepsilon + \frac{2l}{\log(H)},\ 1 \le l \le L = L_\varepsilon,\ \frac{1}{\beta_L} = 2b_\varepsilon,$$

where $L \asymp \log(H) \sim 2\log(\varepsilon^{-1})$, and $a < a_\varepsilon < b_\varepsilon < b$, $a_\varepsilon = a+o(1)$, $b_\varepsilon = b+o(1)$ are taken in such way that $m_l = \log(Ha^{-2\alpha})/2\beta_l \in \mathbb{N}$. By construction, we have $m_l - m_{l-1} = 1$ and

(11.63)
$$m_l^{-2\alpha}\exp(-2\beta_l m_l) \sim (a\beta_l)^{2\alpha}\varepsilon^2 \log\log(\varepsilon^{-1}) \leq 2^{-2\alpha}\varepsilon^2\log\log(\varepsilon^{-1})(1+o(1)).$$

Assume, without loss of generality, that $u_\varepsilon(\kappa_l) \asymp \log\log(\varepsilon^{-1})$, uniformly in $l = 1, 2, \ldots, L$. Taking into account (11.21), (11.22) and (11.23), we can assume that, for d small enough (this corresponds to $r_\varepsilon(\kappa)$ small enough),

$$m_l^{-\alpha} \geq r_\varepsilon(\kappa_l). \tag{11.64}$$

For $l = 1, 2, \ldots, L$, let us take $\eta_l = z_l e_{m_l}$, where $z_l = m_l^{-\alpha}\exp(-\beta_l m_l)$ and $\{e_l\}_{l\in\mathbb{N}}$ be the standard basis in l^2. By (11.64), this yields $\eta_l \in \Theta_\varepsilon(\kappa_l)$ for any $q = q_l > 0$. The following steps are along the lines of the proof of part (a) of Theorem 5.2. We consider the prior

$$\pi = \frac{1}{L}\sum_{l=1}^{L}\delta_{\eta_l}$$

and the mixture P_π over the prior π. Since $\pi(\Theta(\Sigma)) = 1$, it suffices to verify (11.57). By (11.58), this relation holds true as

$$\limsup \frac{\max_{1\leq l\leq L} z_l^2/\varepsilon^2}{\log(L)} < 1. \tag{11.65}$$

By construction, we have $\log(L) \sim \log\log(\varepsilon^{-1})$, and by (11.63), $z_l^2/\varepsilon^2 \leq 2^{-2\alpha}\log\log(\varepsilon^{-1})(1+o(1))$. This implies (11.65). This completes part (a) of the theorem.

In order to obtain the upper bounds, we need the following corollary.

Corollary 11.2. *Let $a_k = k^\alpha$ and $\sigma_k = \exp(\beta k)$, $k \in \mathbb{N}$, $\alpha > 0$, $\beta > 0$. Let $r_\varepsilon > 0$, $r_\varepsilon \to 0$. Set $m = r_\varepsilon^{-\alpha} + O(1)$. Then, for $m_1 = m + c$ and $c > 0$ large enough, one has*

$$\inf_{\eta\in\Theta(r_\varepsilon)} \max_{1\leq i\leq m_1} \eta_i^2 \asymp m^{-2\alpha}\exp(-2m\beta)) \asymp \varepsilon^2 u_\varepsilon, \quad \textit{as} \quad m_1 > m.$$

Proof. The first rate relation follows from Proposition 11.4 and is similar to the proof of Corollary 11.1, the second one follows from (11.22), (11.23). This completes the proof of the corollary.

We now obtain the upper bounds. In view of the embeddings (4.8) it suffices to consider the case $q = 2$. We work along the lines of the proof of part (b) of Theorem 5.2 and apply Corollary 11.2, (11.21), (11.22) and (11.23). This completes part (b) of the theorem.

Part (c) of the theorem follows immediately in view of parts (a) and (b) of the theorem and (11.21). The theorem now follows.

11.14. *Proof of Theorem 5.4.* We first obtain the lower bounds. By making $r_\varepsilon(\kappa)$ larger, we can assume, without loss of generality, that $C = 1$, i.e., for all $\kappa \in \Sigma$,

$$u_\varepsilon^{lin}(\kappa, r_\varepsilon(\kappa)) = \sup_{\kappa\in\Sigma} u_\varepsilon^{lin}(\kappa, r_\varepsilon(\kappa)) = u_\varepsilon^{lin}(\Sigma),$$

and, some $d > 0$, $u_\varepsilon^{lin}(\Sigma)/\log\log(\varepsilon^{-1}) = d$. Taking $A_\varepsilon = (\varepsilon\sqrt{u_\varepsilon^{lin}(\Sigma)})^{-1}$, find a collection $\kappa_l,\ 1 \le l \le M = M_\varepsilon \asymp M(A_\varepsilon, \Sigma)$ such that, for $m(A_\varepsilon, \kappa_l) = m_l$, one has

$$|m_l - m_k| > 1,\ \forall\ k, l = 1, ..., M,\ k \ne l; \quad r_\varepsilon(\kappa_l) \in \Delta^*_{m_l}.$$

Observe that $\log\log(A_\varepsilon) \sim \log\log(\varepsilon^{-1})$ and that, by (5.14),

$$\log(M(A_\varepsilon, \Sigma)) \sim \log(M), \quad \liminf \log(M)/\log\log(\varepsilon^{-1}) = b > 0.$$

For each $l = 1, 2, \ldots, M$, take $t_l \in [0, 1]$ such that

$$r_\varepsilon^2(\kappa_l) = \frac{1 - t_l}{a_{m_l}^2(\kappa_l)} + \frac{t_l}{a_{m_l-1}^2(\kappa_l)}.$$

Let us now consider a collections of vectors $\eta^l = (0, 0, \ldots, 0, \eta^l_{m_l-1}, \eta^l_{m_l}, 0, 0, \ldots)$ with

$$\eta^l_{m_l-1} = \frac{\sqrt{t_l}}{a_{m_l-1}(\kappa_l)\sigma_{m_l-1}(\kappa_l)}, \quad \eta^l_{m_l} = \frac{\sqrt{1 - t_l}}{a_{m_l}(\kappa_l)\sigma_{m_l}(\kappa_l)}.$$

One can easily check that $\eta^l \in \Theta_{\kappa_l}(r_\varepsilon(\kappa_l))$ and

$$\varepsilon^{-2}\|\eta^l\|_2^2 = u^{lin}(r_\varepsilon(\kappa_l)) = u_\varepsilon^{lin}(\Sigma), \ \ (\eta^l, \eta^k) = 0, \forall\ k, l = 1, \dots, M,\ k \neq l. \tag{11.66}$$

We now work along similar lines of the proof of part (a) of Theorem 5.2. We consider the prior

$$\pi = \frac{1}{M}\sum_{l=1}^{M} \delta_{\eta^l}$$

and the mixture P_π over π. Since $\pi(\Theta_\varepsilon(\Sigma)) = 1$, it suffices to verify (11.57). Similarly to (11.58), one has, by (11.66),

$$E_{\varepsilon,0}(dP_\pi/dP_{\varepsilon,0} - 1)^2 = M^{-2}\sum_{l=1}^{M}(\exp(\|\eta^l\|_2^2/\varepsilon^2) - 1) = M^{-1}\exp(u_\varepsilon^{lin}(\Sigma)).$$

Therefore, the relation (11.57) holds true as

$$\limsup \frac{u_\varepsilon^{lin}(\Sigma)}{\log(M)} < 1.$$

By (5.14), it suffices to take $d \in (0, b)$. This completes part (a) of the theorem.

We now obtain the upper bounds. First, observe that the family $T_{\varepsilon,k}$ satisfies

$$\begin{aligned}\sum_{k\in\mathbb{N}} \Phi(-T_{\varepsilon,k}) &\asymp T_\varepsilon^{-3} + \sum_{k>2\exp(T_\varepsilon^2/2)T_\varepsilon^{-2}} \frac{e^{-T_{\varepsilon,k}^2/2}}{T_{\varepsilon,k}} \\ &\asymp T_\varepsilon^{-3} + \sum_{k>2\exp(T_\varepsilon^2/2)T_\varepsilon^{-2}} \frac{1}{k(\log(k))^{3/2}} = o(1).\end{aligned} \tag{11.67}$$

By (11.67) we have

$$\alpha(\psi_\varepsilon) \le \sum_{k=1}^{\infty} P_{\varepsilon,0}(|y_k|/\varepsilon \ge T_{\varepsilon,k}) = 2\sum_{k=1}^{\infty}\Phi(-T_{\varepsilon,k}) = o(1).$$

Next, let $\eta \in \Theta_{\varepsilon,\kappa}(r_\varepsilon(\kappa))$. We have

$$\beta_\varepsilon(\psi_\varepsilon, \eta) \le \inf_{k\in\mathbb{N}} P_{\varepsilon,\eta}(|y_k|/\varepsilon < T_{\varepsilon,k}) \le \inf_{k\in\mathbb{N}} \Phi(T_{\varepsilon,k} - \varepsilon^{-1}|\eta_k|),$$

and it suffices to check that, uniformly over $\kappa \in \Sigma$ and $\eta \in \Theta_{\varepsilon,\kappa}(r_\varepsilon(\kappa))$,

$$\sup_{k\in\mathbb{N}}(\varepsilon^{-1}|\eta_k| - T_{\varepsilon,k}) \to \infty. \tag{11.68}$$

Let $m = m(A_{\varepsilon,\kappa}, \kappa)$ where $A_{\varepsilon,\kappa} = (\varepsilon\sqrt{u_\varepsilon^{lin}(\kappa, r_\varepsilon(\kappa))})^{-1}$. We have $r_\varepsilon(\kappa) \in \Delta_m^*$. Since the sequence $T_{\varepsilon,k}^2 \sim 2\log(k)$ increases in k, the relation (11.68) follows from

$$\liminf \frac{\max_{1\le k\le m}\varepsilon^{-2}\eta_k^2}{\max(T_\varepsilon^2, 2\log(m))} > 2. \tag{11.69}$$

Applying Proposition 11.3 to the collection $H_{m,k} = 1,\ k = 1, 2, \ldots, m$, we have

$$\max_{1\le k\le m}\varepsilon^{-2}\eta_k^2 \ge u_\varepsilon^{lin}(\kappa, r_\varepsilon(\kappa))(1/(2\sqrt{2}) + o(1)) \ge u_\varepsilon^{lin}(\Sigma)(1/(2\sqrt{2}) + o(1)).$$

Also, since $m(A,\kappa)$ increases in A, and $A_{\varepsilon,\kappa} \le bA_\varepsilon$ where $A_\varepsilon = (\varepsilon\sqrt{\log\log(\varepsilon^{-1})})^{-1}$, we have

$$2\log(m) \le 2L(bA_\varepsilon, \Sigma) \le 2B\log\log(\varepsilon^{-1})(1 + o(1)), \quad T_\varepsilon^2 \le \log\log(\varepsilon^{-1}).$$

Therefore, the relation (11.69) holds true as $u_\varepsilon^{lin}(\Sigma) > D\log\log(\varepsilon^{-1})$, for $D > 2\sqrt{2}\max(2B, 1)$. This completes part (b) of the theorem.

Part (c) of the theorem follows immediately from parts (a) and (b) of the theorem and the definition $r_\varepsilon^{ad}(\kappa)$. The theorem now follows.

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