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ON THE GENERALIZED RAO–RUBIN CONDITION AND SOME VARIANTS

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Summary

This paper gives a characterization of some members of the compound Poisson family of distributions based on the generalized Rao–Rubin condition. By considering some variants of this condition and using power series arguments, characterizations of the Poisson distribution are also obtained.

Key words: Compound Poisson family of distributions; generalized Rao–Rubin condition; power series distributions.

1. Introduction

Rao & Rubin (1964) obtained the following characterization of the Poisson distribution: let (X, Y) be a random vector of $\mathbb{N}_+$-valued components such that $\Pr(Y \leq X) = 1$. If $\Pr(X = 0) < 1$ and the conditional distribution $Y \mid X$ (termed the survival distribution) is given by

$$S(r \mid n) \equiv \Pr(Y = r \mid X = n) = \binom{n}{r} p^r (1-p)^{n-r} \quad (r = 0, \ldots, n;\ n = 0, 1, \ldots),$$

where p is a fixed number in (0,1), then

$$\Pr(Y = r) = \Pr(Y = r \mid X = Y) \qquad (r = 0, 1, \ldots) \tag{1}$$

if and only if $X \sim$ Poisson. (For an elementary proof of this result we refer to Shanbhag, 1974.)

The above result is known in the literature as the Rao–Rubin characterization of the Poisson distribution and the condition (1) as the Rao–Rubin (RR) condition. By working on data from toxicological experiments, Talwalker (1975) considered the following interesting variant of the RR condition: let (X_1, Y_1) and (X_2, Y_2) be two-vector random variables with $\mathbb{N}_+$-valued components such that

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X_1 and X_2 have the same distribution $g_n = \Pr(X = n)$ with $g_0 < 1$, and Y_1 and Y_2 are such that for each n with $g_n > 0$

$$S_1(r \mid n) \equiv \Pr(Y_1 = r \mid X_1 = n) = \binom{n}{r} p_1^r (1-p_1)^{n-r} \quad (r = 0, \ldots, n), \quad (2)$$

$$S_2(r \mid n) \equiv \Pr(Y_2 = r \mid X_2 = n) = \binom{n}{r} p_2^r (1-p_2)^{n-r} \quad (r = 0, \ldots, n), \quad (3)$$

where $0 < p_1, p_2 < 1$ are fixed. She observed that for suitable choices of p_1 and p_2, the condition

$$\Pr(Y_1 = r) = \Pr(Y_2 = r \mid X_2 = Y_2) \qquad (r = 0, 1, \ldots), \quad (4)$$

called the generalized Rao–Rubin (GRR) condition, holds when X_1 (and so X_2) has a Poisson, or a binomial, or a negative binomial distribution. Later, Talwalker (1980) and Rao *et al.* (1980) solved the converse problem of characterizing the distribution of X_1 by the GRR condition. They proved that when (X_1, Y_1) and (X_2, Y_2) are two-vector random variables as defined above, then (4) holds if and only if either

(i) $0 < p_1 = p_2 < 1$ and X_1 has a Poisson distribution, or
(ii) $0 < p_2 < p_1 < 1$ and X_1 has a binomial distribution with an arbitrary index and success parameter $p = (p_1 - p_2)/p_1(1 - p_2)$, or
(iii) $0 < p_1 < p_2 < 1$ and X_1 has a probability generating function (pgf) of the form

$$G(s) = \int_0^\infty e^{-(s_0 - s)t}\, d\mu(t) \qquad (|s| < s_0),$$

where $s_0 = p_2(1 - p_1)/(p_2 - p_1)$ and $\mu(t)$ is any possibly infinite measure such that

$$d\mu(t) = G(p_2)\, d\mu\left(\frac{p_2 t}{p_1}\right) \qquad (0 < t < \infty).$$

In case (iii), Talwalker (1980) and Rao *et al.* (1980) identified the family of distributions $\{g_n\}$ for which (4) holds as a certain family of mixed Poisson distributions with the mixing distribution itself satisfying a certain functional equation. Recently, Alzaid *et al.* (1987) gave an explicit expression for the family of distributions $\{g_n\}$ and showed that the stationary measure corresponding to a discrete branching process is linked with this distribution. They proved that when (X_1, Y_1) and (X_2, Y_2) are two-vector random variables as defined in the GRR set-up with $0 < p_1 < p_2 < 1$, then (4) is valid if and only if

$$g_j = K \sum_{n=-\infty}^{\infty} \int_{[0,1)} c^{n-t}\, e^{-(p_1/p_2)^{n-t}} \frac{(p_1/p_2)^{(n-t)j}}{j!} \left[\frac{p_2 - p_1}{p_2(1 - p_1)}\right]^j d\nu(t) \quad (5)$$

($j = 0, 1, \ldots$), where ν is a probability measure on $[0, 1)$, c is a real number lying in $(0, 1)$ and K is a normalizing constant.

The present paper derives characterizations of some probability distributions based on the GRR condition and certain of its variants. Section 2 gives a characterization of some members of the compound Poisson family of distributions. Section 3 obtains one characterization of the Poisson distribution for the case when a variant of the GRR condition is valid only at the points 0 and 1 and assuming X_1 and X_2 to have the same power series distribution. We find another characterization of the Poisson distribution when moment conditions replace the GRR condition, again using power series arguments. Throughout the paper, $'$ denotes differentiation and $a^{(r)} = a(a-1)\ldots(a-r+1)$ $(r = 1, 2, \ldots;\ a^{(0)} = 1)$.

2. Characterization of the Compound Poisson Family of Distributions

Consider the family $\{I\}$ of compound Poisson distributions specified by its pgf as

$$\{I\} = \{G(s) = \exp\{\lambda[h(s) - 1]\} : \lambda \in (0, \infty),\ h \in \mathcal{H}\},$$

where $\mathcal{H}$ is the class of pgfs of positive integer-valued random variables (i.e. every $h \in \mathcal{H}$ has $h(0) = 0$).

If X is a $\mathbb{N}_+$-valued random variable, then its conditioning on positivity (CP) law is defined as $\Pr(X = j \mid X > 0)$ $(j = 1, 2, \ldots)$.

Theorem 2.1. *Let (X_1, Y_1) and (X_2, Y_2) be two-vector random variables with $\mathbb{N}_+$-valued components, and let S_1, S_2 be the survival distributions given by (2) and (3) respectively. Assume also that $G_1 \in \{I\}$ and $G_2(s) = [G_1(s)]^{\beta}$ with $\beta > 0$, where G_1 and G_2 are the pgfs of X_1 and X_2, respectively. Then (4) holds if and only if $\beta = p_1 h'(1)/p_2 h'(p_2)$ and one of the following holds:*

(i) *h is the CP Bernoulli pgf (i.e. h is degenerate at 1);*
(ii) *$0 < p_1 = p_2 < 1$ and h is the CP Poisson pgf;*
(iii) *$0 < p_2 < p_1 < 1$ and h is the CP binomial pgf with index ≥ 2 and success parameter $p = (p_1 - p_2)/p_1(1 - p_2)$;*
(iv) *$0 < p_1 < p_2 < 1$ and h is the CP pgf of the distribution of the form (5).*

Proof. It can easily be seen that (4) is equivalent to

$$G_1(1 - p_1 + p_1 s) = \frac{G_2(p_2 s)}{G_2(p_2)} \qquad (|s| \leq 1),$$

where $0 < p_1, p_2 < 1$ are fixed. Under the assumptions of the theorem and after some simple calculations, the latter equation can be seen to be equivalent to

$$h(1 - p_1 + p_1 s) - 1 = \beta[h(p_2 s) - h(p_2)] \qquad (|s| \leq 1), \tag{6}$$

which, in turn, is equivalent to the assertion that $h'(\cdot)$ exists at least on $[-1, 1]$ and satisfies

$$h'(1 - p_1 + p_1 s) = \frac{\beta p_2}{p_1} h'(p_2 s) \qquad (|s| \leq 1). \tag{7}$$

We can also express (7) as

$$H(1-p_1+p_1s) = cH(p_2s) \quad (|s| \le 1), \quad \text{where } H(s) = \frac{h'(s)}{h'(1)},\ c = \frac{\beta p_2}{p_1}. \tag{8}$$

In view of the conditions on h, H is clearly a well-defined pgf. Note that (8) implies that $\beta = p_1/p_2 H(p_2)$ and H is a pgf satisfying

$$H(1-p_1+p_1s) = \frac{H(p_2s)}{H(p_2)} \qquad (|s| \le 1). \tag{9}$$

If $H(s) \equiv 1$, then (9) is trivially met, the condition $\beta = p_1/p_2 H(p_2)$ reduces to $\beta = p_1/p_2$, and case (i) of the assertion holds. Consider now the case when $H(s) \not\equiv 1$. In this case, $H(p_2) < 1$ and (9) is precisely the functional equation solved by Talwalker (1980), Rao *et al.* (1980) and Alzaid *et al.* (1987). In other words, (9) holds if and only if either $0 < p_1 = p_2 < 1$ and H is the Poisson pgf, or $0 < p_2 < p_1 < 1$ and H is the binomial pgf with an arbitrary index and success parameter $p = (p_1 - p_2)/p_1(1-p_2)$, or $0 < p_1 < p_2 < 1$ and H is the pgf of the distribution of the form (5). This ensures that the solution of (6) obtained by (9) is a pgf, and conversely, any pgf solution of (6) gives a pgf solution of (9). This gives cases (ii), (iii) and (iv) of the assertion, and hence the theorem.

Remark 2.1. We obtain the following characterizations:

Case (ii). Since

$$h(s) = \frac{e^{\alpha s}-1}{e^{\alpha}-1} \qquad (|s| \le 1),$$

then

$$G_1(s) = \exp\{\lambda[h(s)-1]\} = \exp\left\{\frac{\lambda e^{\alpha}}{e^{\alpha}-1}(e^{\alpha(s-1)}-1)\right\} \qquad |s| \le 1,$$

which represents the pgf of a Poisson (λ_1)–Poisson (α) distribution, where $\lambda_1 = \lambda e^{\alpha}/(e^{\alpha}-1) > 0$ and $\alpha = (1-p_1)^{-1}\log\beta > 0$.

Case (iii). We have

$$h(s) = \frac{[(p_1-p_2)s + p_2(1-p_1)]^{n+1} - [p_2(1-p_1)]^{n+1}}{[p_1(1-p_2)]^{n+1} - [p_2(1-p_1)]^{n+1}} \qquad (|s| \le 1);$$

therefore, for $|s| \le 1$,

$$G_1(s) = \exp\{\lambda[h(s)-1)]\} = \exp\left\{\lambda_2\left[\left(\frac{(p_1-p_2)s + p_2(1-p_1)}{p_1(1-p_2)}\right)^{n+1} - 1\right]\right\},$$

which represents the pgf of a Poisson (λ_2)-Binomial (N,p) distribution, where

$$\lambda_2 = \frac{\lambda[p_1(1-p_2)]^{n+1}}{[p_1(1-p_2)]^{n+1} - [p_2(1-p_1)]^{n+1}} > 0,$$

$$N = n+1 = \frac{\log\beta}{\log(p_1/p_2)} > 0 \quad \text{is an integer, and} \quad p = \frac{p_1 - p_2}{p_1(1-p_2)}.$$

Case (iv). In this case, if the measure ν in (5) is taken as Lebesgue measure on $[0,1)$ then, as Alzaid *et al.* (1987) observed, the distribution $\{g_j\}$ in question reduces to a negative binomial distribution with some index $\alpha > 0$ and parameter $\pi = p_1(1-p_2)/p_2(1-p_1)$. Thus, in this case,

$$h(s) = \frac{(1-\phi s)^{-(\alpha-1)} - 1}{(1-\phi)^{-(\alpha-1)} - 1} \qquad (|s| \le 1,\ \phi = 1-\pi),$$

which, for $|s| \le 1$, is equivalent to

$$G_1(s) = \exp\{\lambda[h(s)-1]\} = \exp\left\{\frac{\lambda}{1-(1-\phi)^{\alpha-1}}\left[\left(\frac{1-\phi}{1-\phi s}\right)^{\alpha-1} - 1\right]\right\}.$$

This represents the pgf of a Poisson (λ_3)-Negative Binomial ($\alpha^\star, \pi$) distribution, where $\lambda_3 = \lambda/[1-(1-\phi)^{\alpha-1}] > 0$ and $\alpha^\star = \alpha - 1 = \log\beta/\log(p_2/p_1) > 0$.

3. Characterizations of the Poisson Distribution

This section gives characterizations of the Poisson distribution based on some variants of the GRR condition. For the first characterization of the Poisson distribution, consider a variant of the GRR condition to be valid only at the points 0 and 1 and assume that X_1 and X_2 have the same power series distribution.

Theorem 3.1. *Let (X_1, Y_1) and (X_2, Y_2) be bivariate random variables with $\mathbb{N}_+$-valued components, and let S_1, S_2 be the survival distributions given by (2) and (3) respectively. Also, assume that X_1 and X_2 have the same power series distribution with probability mass function (pmf) given by*

$$g_n = \frac{a_n\theta^n}{A(\theta)} \qquad (n = 0, 1, \ldots,),$$

with at least one $a_n > 0$ for $n \ge k$, where k is a fixed non-negative integer, and $\theta \in (a,b)$ $(0 \le a < b)$. Then

$$\Pr(Y_1 = r) = \Pr(Y_2 = r \mid X_2 = Y_2 + k) \qquad (r = 0, 1;\ \text{all } \theta \in (a,b)), \qquad (10)$$

if and only if X_1 (and hence X_2) has a Poisson distribution and $p_1 = p_2$.

Proof. The 'if' part of the assertion is trivial. We establish the 'only if' part of the assertion. Under the assumptions of the theorem, (10) is equivalent to

$$\frac{A[\theta(1-p_1)]}{A(\theta)} = \frac{a_k}{A^\star(\theta p_2)}, \tag{11}$$

$$\frac{A'[\theta(1-p_1]}{A(\theta)} = \frac{a_{k+1}(k+1)p_2}{p_1 A^\star(\theta p_2)}, \tag{12}$$

where $A^\star(\theta p_2) = \sum_{n=0}^{\infty} a_{n+k}\binom{n+k}{n}(\theta p_2)^n$. From (11) and (12),

$$\frac{A'[\theta(1-p_1)]}{A[\theta(1-p_1)]} = \frac{a_{k+1}(k+1)p_2}{a_k p_1},$$

which implies that

$$A(\theta) = k^\star e^{c\theta} \qquad \left(k^\star > 0,\ c = \frac{a_{k+1}(k+1)p_2}{a_k p_1}\right).$$

From the above, we easily conclude that X_1 (and hence X_2) has a Poisson distribution. Without loss of generality we can take $k^\star = 1$. Then, (11) can be written as $e^{-c\theta p_1} = e^{-c\theta p_2}$ which implies that $p_1 = p_2$. This completes the proof of the 'only if' part and hence of the theorem.

Remark 3.1. Theorem 3.1 does not remain valid if (10) is assumed to be valid just at the point 0, as shown by the following counter-example.

Example 3.1. Take $k = 0$ and $A(\theta) = e^{c\theta^\beta}$ with c and β as positive integers. By choosing appropriate β, p_1 and p_2 such that $p_1 \neq p_2$ for which $p_2^\beta + (1-p_1)^\beta = 1$, it is not difficult to see that the condition

$$\Pr(Y_1 = 0) = \Pr(Y_2 = 0 \mid X_2 = Y_2) \qquad (\text{all } \theta)$$

is met.

Finally, assuming that X_1 and X_2 have the same power series distribution and replacing the GRR condition with some conditional and marginal moments, leads to another characterization of the Poisson distribution. In Theorem 3.2, S_1 and S_2 are general survival distributions and not restricted to the parametric forms given by (2) and (3).

Theorem 3.2. *Let (X_1,Y_1) and (X_2,Y_2) be bivariate random variables with $\mathbb{N}_+$-valued components such that X_1 and X_2 have the same power series distribution with pmf given by*

$$g_n = \frac{a_n\theta^n}{A(\theta)} \qquad (n = 0, 1, \ldots,)$$

with at least one $a_n > 0$ for $n \geq k$, where k is a fixed non-negative integer, and $\theta \in (a,b)$ $(0 \leq a < b)$. For some fixed s and $0 < p_1 < 1$ independent of θ, suppose the moments of order s, $s+1$, $s+2$ of the survival distribution $\{S_1(r \mid n) : r = 0,1,\ldots,n\}$ have the form $n^{(s)}p_1^s$, $n^{(s+1)}p_1^{s+1}$ and $n^{(s+2)}p_1^{s+2}$, respectively; assume also that $S_2(n \mid n+k)$ is independent of θ. If $\mathrm{E}(Y_1^{(s)}) > 0$ and $\Pr(X_2 = Y_2 + k) > 0$, then

$$\mathrm{E}(Y_1^{(r)}) = \mathrm{E}(Y_2^{(r)} \mid X_2 = Y_2 + k) \qquad (r = s, s+1, s+2; \text{all } \theta), \tag{13}$$

if and only if X_1 (and hence X_2) has a Poisson distribution and

$$S_2(n \mid n+k) = \binom{n+k}{n} p_1^n S_2(0 \mid k).$$

Proof. The 'if' part follows easily on substitution. For the 'only if' part, equation (13) is equivalent to

$$\frac{p_1^r \sum_n n^{(r)} a_n \theta^{n-r}}{A(\theta)} = \frac{\sum_n n^{(r)} a_{n+k} S_2(n \mid n+k)\theta^{n-r}}{\sum_n a_{n+k} S_2(n \mid n+k)\theta^n} \quad (r = s, s+1, s+2). \tag{14}$$

Define $A_r(\theta) = \sum_n n^{(r)} a_n \theta^{n-r}$ and $A_r^*(\theta) = \sum_n n^{(r)} a_{n+k} S_2(n \mid n+k)\theta^{n-r}$, and note that

$$A_r'(\theta) = A_{r+1}(\theta) \qquad \text{and} \qquad A_r^{*\prime}(\theta) = A_{r+1}^*(\theta), \tag{15}$$

for all values of $r \geq 0$ and in particular for $r = s$, $s+1$. A simple division in (14) gives

$$p_1 \frac{A_{r+1}(\theta)}{A_r(\theta)} = \frac{A_{r+1}^*(\theta)}{A_r^*(\theta)} \qquad (r = s, s+1) \tag{16}$$

whence

$$p_1\{\log[A_r(\theta)]\}' = \{\log[A_r^*(\theta)]\}' \qquad (r = s, s+1)$$

or equivalently

$$A_r^*(\theta) = c_r [A_r(\theta)]^{p_1} \qquad (r = s, s+1), \tag{17}$$

where c_r are positive constants. In view of (17), equation (16) implies

$$p_1 \frac{A_{s+1}(\theta)}{A_s(\theta)} = \frac{c_{s+1}}{c_s}\left(\frac{A_{s+1}(\theta)}{A_s(\theta)}\right)^{p_1},$$

whence $A_{s+1}(\theta)/A_s(\theta)$ is a positive constant (say ℓ_1), or equivalently (using 15) that $A_s(\theta) = e^{\ell_0 + \ell_1 \theta}$ with $\ell_0 > 0$. Then, in view of (17) and (14) for $r = s$,

$$A_s^*(\theta) = c_s e^{\ell_0 p_1 + \ell_1 p_1 \theta},$$

$$A(\theta) \equiv A_0(\theta) = \ell_1^{-s} A_s(\theta) + p_s(\theta), \qquad \text{and}$$

$$A_0^*(\theta) \equiv \sum_n a_{n+k} S_2(n \mid n+k)\theta^n = (\ell_1 p_1)^{-s} A_s^*(\theta) + p_s^*(\theta),$$

for appropriate polynomials $p_s(\theta)$ and $p_s^*(\theta)$ in θ of degree at most $s-1$. (When $s=0$, $p_s(\theta)$ and $p_s^*(\theta)$ are defined to be constant.) The reciprocal of (14) for $r=s$ then implies

$$(\ell_1 p_1)^{-s} + p_1^{-s} p_s(\theta) e^{-\ell_0 - \ell_1 \theta} = (\ell_1 p_1)^{-s} + c_s^{-1} p_s^*(\theta) e^{-\ell_0 p_1 - \ell_1 p_1 \theta},$$

whence

$$c_s^{-1} p_s^*(\theta) e^{\ell_0(1-p_1)+\ell_1(1-p_1)\theta} = p_1^{-s} p_s(\theta). \tag{18}$$

Since (18) is valid in (a,b), it holds for all $\theta > 0$. But letting $\theta \to \infty$ yields a contradiction, except when $p_s(\theta) \equiv p_s^*(\theta) \equiv 0$. Consequently, the expressions for $A(\theta)$ and $A_0^*(\theta)$ imply that

$$X \sim \text{Poisson} \quad \text{and} \quad S_2(n \mid n+k) = \binom{n+k}{n} p_1^n S_2(0 \mid k).$$

This concludes the proof of the theorem.

Remark 3.2. When $s=0$ in Theorem 3.2, a characterization of the Poisson distribution based on the first two factorial moments is obtained. This result can be thought of as a variant of an earlier characterization given by Shanbhag & Clark (1972) and Srivastava & Singh (1975). Also, the conclusion of Theorem 3.2 remains valid for $k=0$ with S_2 satisfying $S_2(n \mid n) = p_1^n$.

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