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CHARACTERISATIONS OF SOME INCOME DISTRIBUTIONS BASED ON MULTIPLICATIVE DAMAGE MODELS

E.B. FOSAM[1] AND T. SAPATINAS[2]

Sheffield Hallam University and Exeter University

Summary

This paper characterises the Pareto and scaled beta distributions within the context of multiplicative damage and generating models. The results obtained allow the destructive or generating mechanism to have more general distributions on (0,1) than the beta(ℓ,1) distribution considered by several authors, thus generalising some recent results. Errors in earlier work are mentioned.

Key words: Damage model; generating model; regression function; Pareto, beta and log-gamma distributions.

1. Introduction

In most practical situations, people recording an observation X are affected by some destructive or generative influences so that Y, the recorded value, is different from the true value X. Various examples have been cited in the literature including income distribution analysis (where people have the tendency to under-report their incomes for tax purposes) and insurance claim distribution analysis (where people have the tendency to over-report their true insurance claim).

Krishnaji (1970) envisaged, in the case of income under-reporting, that the recorded income Y is related to actual X in a multiplicative way

$$Y \stackrel{d}{=} RX, \tag{1}$$

where X and R ($R \in (0,1)$) are independent random variables and $\stackrel{d}{=}$ denotes equality in distribution; we refer to (1) as the multiplicative damage model. Instead of (1) we can view some recorded insurance Y as being related to actual insurance X by

$$Y \stackrel{d}{=} X/R, \tag{2}$$

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[1]School of CMS, Sheffield Hallam University, Sheffield S11-8HD, UK.

[2]Dept of MSOR, Exeter University, Exeter EX4-4QE, UK.

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where X and R ($R \in (0,1)$) are independent random variables; we refer to (2) as the multiplicative generating model.

Krishnaji (1970), considering model (1) and taking $R \sim \text{beta}(\ell, 1)$, obtained two characterisations for the Pareto distribution. By his first result (K1) the distribution of Y truncated to the left at some point x_0 is the same as the distribution of X, assuming $\Pr\{X > x_0\} > 0$, if and only if $X \sim$ Pareto. His second result (K2) posits the existence of a random variable Z such that the regression $\mathrm{E}(Z \mid X = x)$ is linear. It proves, subject to a smoothness condition, that $\mathrm{E}(Z \mid Y = RX = y)$ is linear if and only if $X \sim$ Pareto. This connects with subsequent work of Dimaki & Xekalaki (1990) who claim that these regression functions determine the distribution of X. While that is true for the cases discussed by these authors, our Example 2.1 below shows it is not true in general. Chaubey & Srivastava (1991) have also obtained characterisation results based on model (1); however their main theory has a focus different from ours.

The present paper shows that K1 holds for any distribution $R \in (0,1)$ such that $\log R$ is non-arithmetic (Theorem 2.1 below), and extends K2 in a similar way to allow R to be beta(ℓ, m), or log-gamma(ℓ, m) distributed ($\ell > 0$, $m \in \mathbb{N} = \{1, 2, \ldots\}$) (Theorem 2.2 below). Several authors have commented implicitly on the general version of K1; for example see Rao (1983), Arnold (1983 pp.83–84) and Ramachandran & Lau (1991 p.47). Our Theorem 2.1 below states explicitly the general version and the proof is simple.

We note that if one considers the generating model (2), then one can obtain characterisations for a scaled beta distribution. These results are immediate deductions from Theorems 2.1 and 2.2 below, via the transformations $R \to R^{-1}$ and $X \to X^{-1}$.

2. Main Results

Theorem 2.1. *Let* $x_0 > 0$ *and* X *be a positive random variable such that* $\Pr\{X > x_0\} = 1$, *and let* R *be a random variable independent of* X *with* $R \in (0,1)$ *almost surely (a.s.) such that the distribution of* $\log R$ *is non-arithmetic. Assume also that* $\Pr\{RX > x_0\} > 0$. *Then*

$$\Pr\{RX > y \mid RX > x_0\} = \Pr\{X > y\} \qquad (y > x_0), \tag{3}$$

if and only if $X \sim$ Pareto *on* (x_0, ∞). *(The result also holds if '>' in (3) is replaced by '≥' everywhere, and simultaneously* $\Pr\{RX > x_0\} > 0$ *is replaced by* $\Pr\{RX \geq x_0\} > 0$.*)*

Remark 2.1. The 'if' part of Theorem 2.1 is valid for any arbitrary distribution $R \in (0,1)$, as observed by Krishnaji (1970).

Proof. Write equation (3) as

$$\Pr\left\{R\Big(\frac{X}{x_0}\Big) > y^* \;\Big|\; R\Big(\frac{X}{x_0}\Big) > 1\right\} = \Pr\left\{\frac{X}{x_0} > y^*\right\} \qquad \Big(y^* = \frac{y}{x_0} > 1\Big),$$

which is equivalent to

$$\Pr\{X^* > R^* + x \mid X^* > R^*\} = \Pr\{X^* > x\} \qquad (x > 0),$$

where $X^* = \log(X/x_0)$, $R^* = -\log R$ and $x = \log y^*$. Hence, the conclusion follows immediately from the 'strong lack of memory property' of the exponential distribution (see, e.g. Ramachandran & Lau, 1991 p.40) on noting that when $Z \sim \text{Exp}(\lambda)$ for some $\lambda > 0$ then, for some $x_0 > 0$, $W = x_0 e^Z \sim \text{Pareto}(\lambda)$ on (x_0, ∞).

Remark 2.2. (i) If (3) is replaced by the condition

$$\text{E}(Y - x \mid Y > x) = \text{E}(X - x \mid X > x) \qquad (x > x_0,\ x_0 > 0),$$

with $\text{E}(X^+) < \infty$, where $X^+ = \max\{0, X\}$, the conclusion of Theorem 2.1 remains unchanged. Kotz & Shanbhag (1980) have shown each side of this condition to be sufficient to determine the distribution of the random variable.

(ii) Talwalker (1980) has given two characterisation theorems on the Pareto distribution based on model (1). Her Theorem 1 holds as a corollary to the 'strong lack of memory property' of the exponential distribution without her assumption that the distribution function of the random variable X is continuous. Her Theorem 2 is not correct, because her proof assumes $-F'(x)/(1 - F(x))$ to be a non-decreasing function of x, whenever $F(x)$ is a concave function of x. However, her result holds even if the assumption that F is concave is replaced by an assumption that F is log-concave.

Remark 2.3. In their Theorem 2.1, Dimaki & Xekalaki (1990) claim to have extended K2. Roughly speaking, they proved that, under some conditions, arbitrary regressions $h(x) = \text{E}(Z \mid X = x)$ and $\lambda(y) = \text{E}(Z \mid Y = RX = y)$ uniquely determine the distribution of X. Their proof uses the unstated assumptions that $\lambda(y)$ is differentiable and that $h(y) < \lambda(y)$, although these are satisfied in their applications. Also, their assumption that $h(x)$ is non-constant is not sufficient to yield the result, as shown by the following counter-example.

Example 2.1. Let R, X and V be independent integrable random variables with $R \sim \text{beta}(\ell, 1)$, $\text{E}(V)$ given, and the part of the distribution of X for $X > c$ known, where c is such that $\Pr\{RX > c\} > 0$. Define

$$Z = \begin{cases} V & \text{if } RX \le c, \\ V + RX & \text{if } RX > c. \end{cases}$$

Then we have,

$$h(x) = \text{E}(V) + x\text{E}\big(RI(R > c/x)\big) \quad \text{and} \quad \lambda(y) = \begin{cases} \text{E}(V) & \text{if } y \le c, \\ \text{E}(V) + y & \text{if } y > c. \end{cases}$$

Consequently, $h(x)$ and $\lambda(y)$ can be computed without any knowledge of the distribution of X except that its right extremity is greater than c.

Now we consider (1) with some regression conditions to characterise the Pareto distribution. In the case where the distribution of R on (0,1) has a more general form than that of a beta$(\ell, 1)$, and the regression functions h, λ are more general than linear, the problem of unique determination of the random variable X becomes more complicated. In the sequel, however, we extend K2 by assuming R to be beta(ℓ, m) or log-gamma(ℓ, m) distributed $(\ell > 0,\ m \in \mathbb{N})$, and taking power-type regression functions.

Theorem 2.2. *Let $x_0 > 0$ and X be an absolutely continuous positive random variable such that* $\Pr\{X > x_0\} = 1$, *and let Z be another random variable such that*

$$\mathrm{E}(Z \mid X = x) = \delta + \beta x^{\alpha} \qquad (x > x_0), \tag{4}$$

for some $\delta \in \mathbb{R}$ and $\alpha, \beta \in \mathbb{R}\backslash\{0\}$. Furthermore, let R be a random variable independent of Z and X with a beta density function given by

$$f_R(r) = \frac{r^{\ell-1}(1-r)^{m-1}}{B(\ell, m)} \qquad (0 < r < 1,\ \ell > 0,\ m \in \mathbb{N}), \tag{5}$$

where $B(\cdot,\cdot)$ denotes the beta function. Then, the restriction of the regression of Z on RX to (x_0, ∞) is given by

$$\mathrm{E}(Z \mid Y = RX = y) = \delta + \gamma y^{\alpha} \qquad (y > x_0), \tag{6}$$

for some $\gamma \in \mathbb{R}\backslash\{0\}$, if and only if $X \sim$ Pareto(θ) on (x_0, ∞), with $\ell + \theta - \alpha > 0$ and $\gamma = \beta\big(B(\ell+\theta-\alpha, m)/B(\ell+\theta, m)\big)$ for some $\theta > 0$.

Proof. The 'if' part of the assertion can be verified easily by direct calculations. We establish the 'only if' part of the assertion by considering the following two cases.

(i) Assume that $\alpha \in (0, \infty)$. By taking into account (4) and (5), we can express equation (6) for $y > x_0$ as

$$y^{-\alpha}\int_y^{\infty} f_X(x)x^{\alpha+1-\ell-m}(x-y)^{m-1}\,dx = \frac{\gamma}{\beta}\int_y^{\infty} f_X(x)x^{1-\ell-m}(x-y)^{m-1}\,dx. \tag{7}$$

Differentiating both sides of (7) m times with respect to y, we obtain

$$\sum_{r=0}^{m}\left[\binom{m}{r}\frac{\alpha_{(r)}(m-1)_{(m-r)}}{y^{\alpha+r}}\int_y^{\infty}\frac{f_X(x)}{x^{m+\ell-\alpha-1}}(x-y)^{r-1}\,dx\right] = \frac{\gamma}{\beta}(m-1)!\,\frac{f_X(y)}{y^{m+\ell-1}}, \tag{8}$$

almost everywhere with respect to the Lebesgue measure, where $a_{(0)} = 1$, $a_{(r)} = a(a+1)\ldots(a+r-1)$, $(m-1)_{(m)} = (m-1)!$, and the integral in the last expression is taken as $f_X(y)y^{\alpha+1-\ell-m}$ if $r = 0$. By taking $x = y/z$ in the integral and after some simplification, the above equation (8) can be expressed as

$$f_X(y) = \int_0^1 f_X(y/z)\, d\mu(z) \qquad (y > x_0), \tag{9}$$

where the measure μ is a linear combination of certain measures with at least one of them being absolutely continuous (with respect to Lebesgue measure). By applying the Lau–Rao theorem (see, e.g. Ramachandran & Lau, 1991) we conclude from (9) that

$$f_X(y) \propto y^{-\eta} \quad \text{for some } \eta > 1 \qquad (y > x_0)$$

which, in view of the relation $\int_{x_0}^{\infty} f_X(y)\, dy = 1$, implies that $X \sim$ Pareto distribution on (x_0, ∞).

(ii) Assume now that $\alpha \in (-\infty, 0)$. Then, by transferring $y^{-\alpha}$ to the right hand side of (7) and working as above, we arrive at a functional equation similar to that of (9).

Remark 2.4. If we replace (5) in the above theorem by a log-gamma distribution, cited in Schultz (1975), given by

$$f_R(r) = \frac{\ell^m r^{\ell-1}(-\log r)^{m-1}}{\Gamma(m)} \qquad (0 < r < 1,\ \ell > 0,\ m \in \mathbb{N}),$$

where $\Gamma(\cdot)$ denote the gamma function, then the conclusion of Theorem 2.2 remains valid with the parameter θ now being replaced by $\theta^* > 0$ such that $\ell + \theta^* - \alpha > 0$ and $\gamma = \beta\big((\ell + \theta^*)/(\ell + \theta^* - \alpha)\big)^m$.

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