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ON THE POSTERIOR MEDIAN ESTIMATORS OF POSSIBLY SPARSE SEQUENCES

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Abstract. We adopt the Bayesian paradigm and discuss certain properties of posterior median estimators of possibly sparse sequences. The prior distribution considered is a mixture of an atom of probability at zero and a symmetric unimodal distribution, and the noise distribution is taken as another symmetric unimodal distribution. We derive an explicit form of the corresponding posterior median and show that it is an antisymmetric function and, under some conditions, a shrinkage and a thresholding rule. Furthermore we show that, as long as the tails of the nonzero part of the prior distribution are heavier than the tails of the noise distribution, the posterior median, under some constraints on the involved parameters, has the bounded shrinkage property, extending thus recent results to larger families of prior and noise distributions. Expressions of posterior distributions and posterior medians in particular cases of interest are obtained. The asymptotes of the derived posterior medians, which provide valuable information of how the corresponding estimators treat large coefficients, are also given. These results could be particularly useful for studying frequentist optimality properties and developing statistical techniques of the resulting posterior median estimators of possibly sparse sequences for a wider set of prior and noise distributions.

Key words and phrases: Bayes model, sparse sequences, wavelets.

1. Introduction

Suppose that $\underline{y} = (y_1, y_2, \dots, y_n)'$ are observations satisfying

$$y_i = \theta_i + \xi_i, \qquad i = 1, 2, \dots, n, \tag{1.1}$$

where $\underline{\theta} = (\theta_1, \theta_2, \dots, \theta_n)'$ is the unknown vector of means and ξ_i $(i = 1, 2, \dots, n)$ is a sequence of independent and identically distributed random variables representing random noise. Clearly, in estimation problems, without some knowledge of the mean vector $\underline{\theta}$ it will not be possible to estimate it very effectively. Motivated by practical applications, in what follows, we consider the advantage that may be taken of possible sparsity in the sequence. For example, in astronomical and other imaging processing contexts, the y_i $(i = 1, 2, \dots, n)$ may be noisy observations of the pixels of an image, where it is known that a large number of the pixels may be zero. In nonparametric regression using wavelets, the true wavelet coefficients at each level form a possibly sparse sequence, and the discrete wavelet transform yields a sequence of raw coefficients, which are observations of those coefficients subject to error.

An empirical Bayes approach to the estimation of possibly sparse sequences observed in Gaussian white noise was recently investigated by Johnstone and Silverman (2004). The prior considered was a mixture of an atom of probability at zero and a heavy-tailed density (in particular, a standard Laplace density or a quasi-Cauchy density), with the mixing weight chosen by marginal maximum likelihood. Using the posterior median (the Bayes estimator under the L^1-loss function) and deriving probability bounds on the threshold chosen by the marginal maximum likelihood approach led to overall bounds on the risk of the method over the class of signal sequences of length n with normalized l_p-norm bounded by η, for $\eta > 0$ and $p \in (0, 2]$; estimation error was measured by mean q-th-loss, for $q \in (0, 2]$. Johnstone and Silverman (2004) showed that for all the classes considered and for all $q \in (0, 2]$, this method achieves the frequentist optimal estimation rate when $\eta = \eta_n \to 0$ and $n \to \infty$ at various rates, and in this sense the method adapts automatically to the sparseness or otherwise of the underlying signal. In addition, the risk is uniformly bounded over all signals and the same results hold if the posterior mean (the Bayes estimator under the L^2-loss function) is used as an estimator, provided that $q \in (1, 2]$. Frequentist optimality results of Bayesian estimators in the wavelet regression context have also been considered by Johnstone and Silverman (2002), Pensky (2003) and Abramovich *et al.* (2004).

In order to study the robustness of their results, Johnstone and Silverman (2004) relaxed the assumption of Gaussian errors in the sequence space model by assuming that the noise coefficients are independent and identically distributed from a symmetric Polya frequency PF_3 density. The tails of such a density cannot be heavier than exponential and examples of such densities include the Gaussian density, the Laplace density and the logistic density. It has been indicated that, under some conditions on the nonzero part of the prior and on the noise distribution, one may expect the qualitative features of the frequentist optimality results obtained by Johnstone and Silverman (2004) to remain true. We point out that the posterior mean, for the prior model considered by Johnstone and Silverman (2004), fails to have the thresholding property and hence produces estimates in which, essentially, all the coefficients are nonzero. On the other hand, the posterior median, in certain cases, is of thresholding type which is, in many cases, considered to be a very useful property (e.g., when good compression rates of signals under study are sought).

In this paper, we try to shed some more light on the estimation problem of possibly sparse sequences. In particular, we adopt the Bayesian paradigm and study the posterior median. In Section 2, we consider a more flexible prior distribution that is a mixture of an atom of probability at zero and a symmetric unimodal distribution, and we take the noise distribution as another symmetric unimodal distribution. First, we derive an explicit form of the corresponding posterior median. Then, we show that the posterior median is an antisymmetric function and, under some conditions, a shrinkage and a thresholding rule. Furthermore, we show that, as long as the tails of the nonzero part of the prior distribution are heavier than the tails of the noise distribution, the posterior median, under some constraints on the involved parameters, has the bounded shrinkage property, extending thus a recent result of Johnstone and Silverman (2004) to larger families of prior and noise distributions. In Section 3, expressions of posterior distributions and posterior medians in particular cases of interest are obtained. The asymptotes of the derived posterior medians, which provide valuable information of how the corresponding estimators treat large coefficients, are also given. Concluding remarks are provided in Section 4. Finally, in order to improve the readability, all proofs are deferred to Section 5.

2. A Bayesian formalism

2.1 *The prior model*

Since the errors ξ_i $(i = 1, 2, \ldots, n)$ are assumed to be independent and identically distributed random variables, we concentrate throughout the paper on the one-dimensional version of the abstract sequence model (1.1), which we call the one-dimensional observation model, i.e.,

$$y = \theta + \xi, \tag{2.1}$$

where ξ is a random variable (representing random noise) with a symmetric unimodal probability density function $\varphi(x)$ on $\mathbb{R} = (-\infty, +\infty)$. In the hope of adapting between sparse and dense sequences, we consider the following prior model on θ

$$\theta \sim (1-p)\delta_0(x) + ph(x), \tag{2.2}$$

where $p \in [0, 1]$, $\delta_0(x)$ is an atom of probability at zero, and $h(x)$ is a symmetric unimodal probability density function on $\mathbb{R}$. We also assume that both $h(x)$ and $\varphi(x)$ are positive for all $x \in \mathbb{R}$ and are finite at zero.

According to the prior model (2.2), θ is either zero with probability $(1-p)$ or with a probability p is distributed with the probability density function $h(x)$; the proportion p indicates whether a value is small or large and can be used to 'control' the trade-off between sparse and dense sequences. We have considered the probability density functions $h(x)$ and $\varphi(x)$ to never vanish on $\mathbb{R}$. Narrowing the support of $h(x)$ would imply that we ignored large values which, in some contexts, is inappropriate; for example in the wavelet context this means we ignore large wavelet coefficients which represent important characteristics of the (possibly) inhomogeneous signal of interest. Similarly, narrowing the support of $\varphi(x)$ means that we exclude noise distributions with heavy tails, like Laplace or Cauchy distributions. Finally, we assume that both $h(x)$ and $\varphi(x)$ are finite at zero for slightly different reasons. For $h(x)$, we assume that all zero mass is accounted for in the other part of the mixture otherwise the mixture would not be identifiable, whereas in $\varphi(x)$ we assume that there is no zero mass to exclude the 'pathological' case of observing data without errors. Note also that due to the unimodality assumption, both $h(x)$ and $\varphi(x)$ cannot have atom masses at any other points.

The Bayes model (2.1)–(2.2) can be seen as generalization of the sequence model studied recently by Johnstone and Silverman (2004), where $h(x)$ is considered to be a standard Laplace density or a quasi-Cauchy density while $\varphi(x)$ is chosen to be a Gaussian density or a symmetric Polya frequency PF_3 density. Versions of the one-dimensional observation model (2.1) in the wavelet domain, for specific choices of the probability density functions $h(x)$ and $\varphi(x)$, have also been considered by Abramovich *et al.* (1998), Clyde *et al.* (1998), Antoniadis *et al.* (2002), Johnstone and Silverman (2002), Averkamp and Houdré (2003) and Pensky (2003), in order to study wavelet shrinkage and wavelet thresholding in nonparametric regression for both Gaussian and non-Gaussian errors.

2.2 *The posterior distribution*

Combining the one-dimensional observation model (2.1) with the prior model (2.2), we are able to find the posterior distribution, i.e., the distribution of θ given the observed value y. We introduce the following auxiliary functions which we shall use for describing the posterior distribution and the posterior median,

$$\eta(y) = \int_{\mathbb{R}} \varphi(y-u)h(u)du,$$

$$\tilde{H}(\theta \mid y) = \frac{1}{\eta(y)} \int_{-\infty}^{\theta} \varphi(y-u)h(u)du.$$

We shall regard $\tilde{H}(\cdot \mid y)$ as a family of functions indexed by y. Straightforward calculations lead to the following proposition.

PROPOSITION 2.1. *Under the one-dimensional observation model* (2.1) *and the prior model* (2.2), *the cumulative distribution function of the posterior distribution is given by*

$$F(\theta \mid y) = \frac{\omega(y)}{1+\omega(y)} I_{[0,+\infty)}(\theta) + \frac{1}{1+\omega(y)} \tilde{H}(\theta \mid y), \tag{2.3}$$

where $\omega(y) = \frac{(1-p)\varphi(y)}{p\eta(y)}$ *is the posterior odds ratio for the component at zero.*

Note that the posterior distribution (2.3) is of the same form as the prior, i.e., it is a mixture of a probability mass at zero and some other distribution $\tilde{H}(\theta \mid y)$ which though ceases to be symmetric for $y \neq 0$. The proportion of nonzero θ's has also changed from p to $1/(1+\omega(y))$; it obviously depends on the parameter p and on the probability density functions $\varphi(x)$ and $h(x)$.

2.3 *The posterior median*

Since the posterior median can be used as a point estimate of θ, it is important to know its form, for both theoretical and practical reasons. The following proposition gives an explicit form for the median of the posterior distribution (2.3) that we will subsequently use to derive some useful properties. The following assumption is needed.

(A1) For each y, the function $\tilde{H}(\theta \mid y)$ is invertible.

PROPOSITION 2.2. *Assume that assumption* (A1) *holds. Then, under the one-dimensional observation model* (2.1) *and the prior model* (2.2), *the median of the posterior distribution* (2.3) *is given by*

$$m_\theta(y) = \max\{0, \operatorname{sign}[|\gamma(y)| - 1]\} \operatorname{sign}(y) \tilde{H}^{-1}\left(\frac{1-\min\{1,\omega(y)\}}{2} \,\middle|\, |y| \right), \tag{2.4}$$

where $\operatorname{sign}(\cdot)$ *is the signum function and* $\gamma(y) = \frac{2\tilde{H}(0|y)-1}{\omega(y)}$.

Remark 2.1. (i) Assumption (A1) can be relaxed by using the generalized inverse function $\tilde{H}^{-}(z \mid y) = \inf\{\theta : \tilde{H}(\theta \mid y) \geq z\}$ since $\tilde{H}(\theta \mid y)$ is an increasing, right-continuous function.

(ii) If $|\gamma(y)| \leq 1$ for all $y \in \mathbb{R}$, the posterior median is identically zero which is the case, in particular, if the posterior odds ratio $\omega(y) \geq 1$ for all $y \in \mathbb{R}$.

We now state a proposition that will help us to show that the posterior median (2.4), under some constraints on the involved parameters, has the bounded shrinkage property (see Definition 2.1 below). Before proceeding, it is convenient to set up some notation. We write $g_1(x) \asymp g_2(x)$ to denote $0 < \liminf(g_1(x)/g_2(x)) \leq \limsup(g_1(x)/g_2(x)) < +\infty$, as $x \to \infty$. Throughout the paper, $x \to \infty$ means both $x \to -\infty$ and $x \to +\infty$.

PROPOSITION 2.3. *Suppose that $h(x)/\varphi(x) \to +\infty$, as $x \to \infty$, i.e., the tail of $h(x)$ is heavier than the tail of $\varphi(x)$. Suppose also that there exists a probability density function $f(x)$ satisfying the following conditions*

(L1) *$f(x)$ is symmetric, unimodal, positive for all $x \in \mathbb{R}$ and finite at zero.*

(L2) *$\exists\, M > 0$ such that for each $x > M$, $h(x)/f(x)$ is increasing.*

(L3) *$\exists$ functions $Q_1(u), Q_2(u) > 0$ defined on $[0, +\infty)$ such that, for each $x > M$,*

$$\frac{f(x+u)}{f(x)} \geq Q_1(u) \quad \text{and} \quad \frac{f(x-u)}{f(x)} \leq Q_2(u), \qquad \text{for} \quad u \geq 0,$$

and

$$\int_0^{+\infty} Q_i(u)\varphi(u)du < +\infty, \qquad \text{for} \quad i = 1, 2.$$

Then, $\eta(x) \asymp h(x)$.

Proposition 2.3 shows that the convolution $\eta(x)$ behaves asymptotically as $h(x)$, the density with the heavier tail between the two convolving densities. It is actually a generalization of the first part of Lemma 1 in Johnstone and Silverman (2004), since this lemma does not allow the tails of $\varphi(x)$ to be heavier than exponential, and therefore important heavy-tailed distributions, like those with polynomial decay (e.g. t distributions), are excluded. It is clearly illustrated in the following corollary. We denote by $ep_{\alpha,\tau}(x)$ the probability density function of an Exponential-Power distribution with parameters $\alpha, \tau > 0$, i.e., $ep_{\alpha,\tau}(x) = \frac{\alpha}{2\tau\Gamma(1/\alpha)} \exp\{-(|x|/\tau)^\alpha\}$, for $x \in \mathbb{R}$, and by t_ν the probability density function of a t distribution with $\nu \geq 1$ degrees of freedom, i.e., $t_\nu(x) = \frac{\Gamma((n+1)/2)}{(\nu\pi)^{1/2}\Gamma(\nu/2)}(1 + x^2/\nu)^{-(\nu+1)/2}$, for $x \in \mathbb{R}$, where $\Gamma(a) = \int_0^\infty x^{a-1}e^{-x}dx$, $a > 0$, is the Gamma function. (Recall that t_1 is the standard Cauchy distribution.)

COROLLARY 2.1. *Suppose that $h(x)$ and $\varphi(x)$ are such that*

(1) *$h(x) = ep_{\alpha,\tau}(x)$ and $\varphi(x) = ep_{\beta,\sigma}(x)$ with $\alpha < \beta$ or with $\alpha = \beta$ and $\tau > \sigma$. Then, $\eta(x) \asymp h(x)$ if and only if $\alpha \leq 1$.*

(2) *$h(x) = t_{\nu_1}(x)$ and $\varphi(x) = t_{\nu_2}(x)$ with $\nu_2 > \nu_1 \geq 1$. Then, $\eta(x) \asymp h(x)$ if and only if $\nu_2 - \nu_1 > 1$.*

Now we give some definitions, following closely those given in Johnstone and Silverman (2004), also leading to the main properties of the posterior median.

DEFINITION 2.1. Let $\delta(x,t)$ be a function defined on $\mathbb{R} \times [0, +\infty)$ with values in $\mathbb{R}$. Then

(D1) $\delta(x,t)$ is *antisymmetric* in x if, for each $t \geq 0$,

$$\delta(-x,t) = -\delta(x,t) \quad \text{for all} \quad x \geq 0.$$

(D2) $\delta(x,t)$ is a *shrinkage* rule if and only if $\delta(x,t)$ is antisymmetric in x and increasing on $\mathbb{R}$ for each $t \geq 0$, and

$$0 \leq \delta(x,t) \leq x \quad \text{for all} \quad x \geq 0.$$

(D3) The shrinkage rule $\delta(x,t)$ is a *thresholding* rule with threshold $t > 0$ if and only if

$$\delta(x,t) = 0 \quad \text{for all} \quad |x| \leq t.$$

(D4) The shrinkage rule $\delta(x,t)$ has the *bounded shrinkage property* (relative to the threshold $t > 0$) if, for some constant b,

$$|x - \delta(x,t)| \leq t + b \quad \text{for all} \quad x \text{ and } t.$$

While (D1), (D2) and (D3) in Definition 2.1 are well understood, let us spend a minute to explain the meaning of (D4), the bounded shrinkage property. It essentially means that rare large observations are more or less reliably assigned to sparse signals rather than noise in the Bayesian model (2.1)–(2.2) considered above. Therefore an estimation rule (shrinkage or thresholding) satisfying this property ensures that if θ is large then y is not shrunk severely in the estimation of θ. Some examples demonstrating this behavior will be given in Section 3.

Now we are in position to conclude this section by stating some important properties for the posterior median. The following assumption is needed for the posterior median to be an increasing function.

(A2) The function $Q(y) = \int_0^{+\infty}[\varphi(y-x) - \varphi(y+x)]h(x)dx$ is increasing for all $y > 0$.

THEOREM 2.1. *Suppose that assumption* (A1) *holds. Then, under the one-dimensional observation model* (2.1) *and the prior model* (2.2), *posterior median* (2.4) *has the following properties*

(P1) *it is an antisymmetric function*;

(P2) *it is a shrinkage rule, provided that assumption* (A2) *holds*;

(P3) *it is a thresholding rule, provided that assumption* (A2) *holds. Furthermore, it can be expressed as*

$$m_\theta(y) = \begin{cases} 0, & \textit{if} \quad |y| \leq \lambda, \\ \operatorname{sign}(y)\tilde{H}^{-1}(\frac{1-\omega(y)}{2} \mid y), & \textit{if} \quad |y| > \lambda, \end{cases} \tag{2.5}$$

where the threshold $\lambda > 0$ *is defined by the equation* $\gamma(\lambda) = -1$;

(P4) *it has the bounded shrinkage property, provided that* $h(x)/\varphi(x) \to +\infty$, *as* $x \to \infty$, *and that the assumption* (L3) *in Proposition* 2.3 *holds with* $f(x) = h(x)$ *for all* $x \in \mathbb{R}$.

From the proof of the theorem it follows that very basic properties of the densities $\varphi(x)$ and $h(x)$, such as symmetry, continuity and support on $\mathbb{R}$, guarantee some important properties of the posterior median, in particular, antisymmetry, having a symmetric neighborhood of zero where the median is zero, having values between 0 and the observed value y. For other properties of the posterior median, i.e., monotonicity and bounded shrinkage property, further assumptions on the densities $\varphi(x)$ and $h(x)$ are required.

Remark 2.2. (i) Assumption (A1) can be relaxed by using the generalized inverse function $\tilde{H}^{-}(z \mid y) = \inf\{\theta : \tilde{H}(\theta \mid y) \geq z\}$ since $\tilde{H}(\theta \mid y)$ is an increasing, right-continuous function.

(ii) It is not difficult to check that assumption (A2) can also be relaxed by narrowing the interval where the integral increases to $(\lambda, +\infty)$, where $\lambda > 0$ is the threshold.

(iii) If, for each $x > 0$, the function $G_x(y) = \varphi(y - x) - \varphi(y + x)$ is increasing for all $y > 0$, then assumption (A2) is satisfied. For certain distributions, this condition is easier to check than assumption (A2).

Condition (iii) (and thus assumption (A2)) in Remark 2.2 holds for $\varphi(x)$ being a Gaussian density but not for Laplace or t densities. It immediately implies that the posterior median is a thresholding rule in the cases where $\varphi(x)$ is a Gaussian density and $h(x)$ is a Gaussian density or a Laplace density. In the considered cases with $\varphi(x)$ being a Laplace density, it implies directly from the formula for the posterior median that it increases (and thus is a shrinkage rule) and it is a thresholding rule. In case of combinations of t distributions, $\gamma(y)$ is not monotonic in a neighborhood of zero, and thus several areas where the posterior median is zero may exist. Nevertheless, in the case where $\varphi(x) = t_3(x)$ and $h(x) = t_1(x)$, for certain values of the mixing parameter p, assumption (A2) holds for $|y| > \lambda$, implying that the corresponding posterior median is a shrinkage and a thresholding rule. In the case where $\varphi(x) = \frac{1}{\sigma}t_1(\frac{x}{\sigma})$, $\sigma > 0$, and $h(x) = \frac{1}{\tau}t_1(\frac{x}{\tau})$, $\tau > 0$, the posterior median increases for certain values of the mixing parameter p and thus it is a shrinkage and a thresholding rule. More details on these and on the bounded shrinkage property will be given in Section 3 below.

3. Some special cases of interest

In this section, we derive expressions of posterior distributions and posterior medians in some particular cases of interest. We also study the asymptotes of the derived posterior medians, which provide valuable information of how the corresponding estimators treat large coefficients. The cases we consider, in the form of "nonzero part of the prior distribution"-"noise distribution", are: Gaussian-Gaussian, Laplace-Gaussian, Gaussian-Laplace, Laplace-Laplace, t_1-t_3 and (scaled) t_1-t_1.

Before proceeding, it is convenient to set up some further notation. We use $\phi(x)$ and $\Phi(x)$ for the probability density and cumulative distribution functions, respectively, of a standard Gaussian random variable, and $\phi_\alpha(x)$ denotes the probability density function of a Gaussian random variable with mean zero and variance α^2. We also denote by $l_\rho(x)$ the probability density function of a Laplace distribution with scale parameter $0 < \rho < \infty$, i.e., $l_\rho(x) = \frac{\rho}{2}\exp\{-\rho|x|\}$ for $x \in \mathbb{R}$. (Note that $l_\rho(x) = ep_{1,1/\rho}(x)$.)

3.1 *The Gaussian-Gaussian case*

This case has been discussed, in the wavelet regression context, by Abramovich *et al.* (1998) which we quote here for completeness. However, we note that (3.3) and (3.4) given below are explicitly appeared for first time in this paper.

THEOREM 3.1. *Consider the one-dimensional observation model* (2.1) *with* $\varphi(x) = \phi_\sigma(x)$ *and take the prior model* (2.2) *with* $h(x) = \phi_\tau(x)$. *Then, the following hold*

(1) *The cumulative distribution function of the posterior distribution is given by*

$$F(\theta \mid y) = \frac{\omega(y)}{1+\omega(y)} I_{[0,+\infty)}(\theta) + \frac{1}{1+\omega(y)} \Phi\left(\frac{\theta\sqrt{\tau^2+\sigma^2}}{\tau\sigma} - \frac{y\tau}{\sigma\sqrt{\tau^2+\sigma^2}}\right), \tag{3.1}$$

where

$$\omega(y) = \frac{1-p}{p}\frac{\sqrt{\sigma^2+\tau^2}}{\sigma} \exp\left\{-\frac{\tau^2}{\sigma^2(\sigma^2+\tau^2)}\frac{y^2}{2}\right\}.$$

(2) *The median of the posterior distribution* (3.1) *is given by*

$$m_\theta(y) = \begin{cases} 0, & \text{if } |y| \le \lambda, \\ \operatorname{sign}(y)[\frac{\tau^2}{\sigma^2+\tau^2}|y| + \frac{\sigma\tau}{\sqrt{\sigma^2+\tau^2}}\Phi^{-1}(\frac{1-\min\{1,\omega(y)\}}{2})], & \text{if } |y| > \lambda, \end{cases} \tag{3.2}$$

where the threshold $\lambda > 0$ *is defined as a solution of the equation*

$$\frac{p}{1-p}\frac{\sigma}{\sqrt{\sigma^2+\tau^2}}\left[2\Phi\left(\frac{\tau\lambda}{\sigma\sqrt{\tau^2+\sigma^2}}\right) - 1\right] = \exp\left\{-\frac{\tau^2}{\sigma^2(\sigma^2+\tau^2)}\frac{\lambda^2}{2}\right\}. \tag{3.3}$$

(3) *The asymptotes of the posterior median* (3.2), *as* $y \to \infty$, *are given by*

$$m_\theta(y) = \operatorname{sign}(y)\left[\frac{\tau^2}{\sigma^2+\tau^2}|y| + \frac{(1-p)\tau}{2p\phi(0)}\exp\left\{-\frac{\tau^2}{\sigma^2(\sigma^2+\tau^2)}\frac{y^2}{2}\right\}(1+o(1))\right]. \tag{3.4}$$

Since both densities $\varphi(x)$ and $h(x)$ are continuous, symmetric and defined on $\mathbb{R}$, the posterior median (3.2) is antisymmetric. Since also assumption (A2) is satisfied for any $h(x)$ given that $\varphi(x)$ is Gaussian (due to Remark 2.2(iii)), the posterior median is a thresholding rule with threshold parameter λ satisfying (3.3). However, according to Corollary 2.1, it does not possess the bounded shrinkage property. Furthermore, as $y \to \infty$, it tends to a linear asymptote, $\frac{\tau^2}{\tau^2+\sigma^2}|y|$, with an exponential rate, $e^{-\tau^2/\sigma^2(\sigma^2+\tau^2)y^2/2}$. Figure 1 plots the posterior median (3.2) and its asymptote (3.4) for a particular choice of parameters.

3.2 *The Laplace-Gaussian case*

In order to describe this case better we need to define a modified Gaussian distribution. We consider the positive and negative truncated Gaussian distributions

$$TG_-(u;z) = \frac{\Phi(u+z)}{\Phi(z)}, \qquad u \le 0,$$

$$TG_+(u;z) = 1 - TG_-(-u;z) = \frac{\Phi(u-z) - \Phi(-z)}{\Phi(z)}, \qquad u \ge 0.$$

Then, we glue them together into the Combined Truncated Gaussian (CTG) distribution,

$$CTG(u \mid p_0, p_1) = (1-\alpha)TG_-(u;p_0) + \alpha TG_+(u;p_1) \tag{3.5}$$

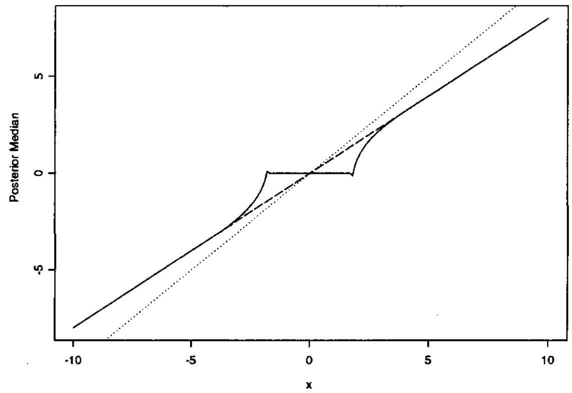

Fig. 1. Posterior median for Gaussian-Gaussian distributions with parameters $\tau = 2$, $\sigma = 1$, $p = 0.4$. Line $y = x$ is plotted as a dotted line, and the asymptote is plotted as a dashed line.

with parameters p_0, p_1, while the mixing parameter $\alpha = \alpha(p_0, p_1) \in [0,1]$ is chosen such that the derived probability density function is continuous. It is not difficult to check that the tails of the CTG distribution decrease faster or slower than the tail of the standard Gaussian distribution depending on the sign of the parameters p_0 and p_1. We are now in the position to state the following result.

THEOREM 3.2. *Consider the one-dimensional observation model* (2.1) *with* $\varphi(x) = \phi_\sigma(x)$ *and take the prior model* (2.2) *with* $h(x) = l_a(x)$. *Then, the following hold*

(1) *The cumulative distribution function of the posterior distribution is given by*

$$F(\theta \mid y) = \frac{\omega(y)}{1+\omega(y)} I_{[0,+\infty)}(\theta) + \frac{1}{1+\omega(y)} CTG(\theta/\sigma \mid -y/\sigma - a\sigma, y/\sigma - a\sigma), \tag{3.6}$$

where

$$\omega(y) = \frac{2(1-p)\phi(y/\sigma)}{ap\sigma e^{(a\sigma)^2/2}\nu(y)},$$
$$\alpha(y) = \frac{e^{-ay}\Phi(y/\sigma - a\sigma)}{\nu(y)},$$
$$\nu(y) = e^{ay}\Phi(-y/\sigma - a\sigma) + e^{-ay}\Phi(y/\sigma - a\sigma).$$

(2) *The median of the posterior distribution* (3.6) *is given by*

$$m_\theta(y) = \begin{cases} 0, & \text{if } |y| \le \lambda, \\ \operatorname{sign}(y)[|y| - a\sigma^2 - \sigma\Phi^{-1}(\frac{(1+\omega(y))\nu(y)e^{a|y|}}{2})], & \text{if } |y| > \lambda, \end{cases} \tag{3.7}$$

where the threshold $\lambda > 0$ *is defined as a solution of the equation*

$$\frac{\Phi(\lambda/\sigma - a\sigma)}{\phi(\lambda/\sigma - a\sigma)} - \frac{\Phi(-\lambda/\sigma - a\sigma)}{\phi(-\lambda/\sigma - a\sigma)} = \frac{2(1-p)}{a\sigma p}. \tag{3.8}$$

(3) *The asymptotes of the posterior median* (3.7), *as* $y \to \infty$, *are given by*

$$m_\theta(y) = \operatorname{sign}(y)\left[|y| - a\sigma^2 - \frac{1-p}{ap}\exp\left\{-\frac{(|y| - a\sigma^2)^2}{2\sigma^2}\right\}(1 + o(1))\right]. \tag{3.9}$$

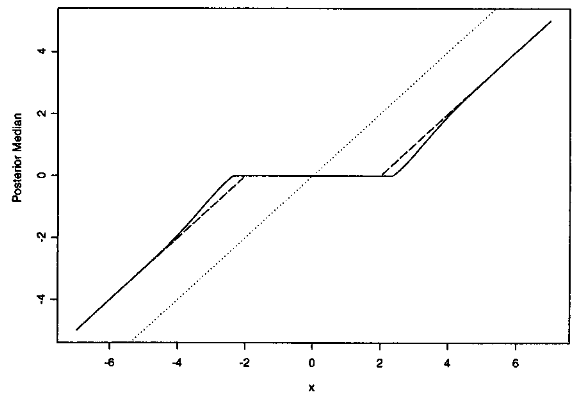

Fig. 2. Posterior median for Laplace-Gaussian distributions with parameters $a = 2$, $\sigma = 1$, $p = 0.4$. Line $y = x$ is plotted as a dotted line, and the asymptotes are plotted as dashed lines.

Since both densities $\varphi(x)$ and $h(x)$ are continuous, symmetric and defined on $\mathbb{R}$, the posterior median is antisymmetric. Also, it is easy to see from the formula for the posterior median (3.7) that it increases as a function of y, and thus it is a thresholding rule with threshold parameter λ defined by (3.8). According to Corollary 2.1 and Theorem 2.1, the posterior median possesses the bounded shrinkage property. Furthermore, as $y \to \infty$, it tends to a linear asymptote, $\text{sign}(y)[|y| - a\sigma^2]$, with an exponential rate, $\exp\{-(|y| - a\sigma^2)^2/2\sigma^2\}$. Figure 2 plots the posterior median (3.7) and its asymptotes (3.9) for a particular case. We point out that the special case $a = \sigma = 1$ has also been discussed in Johnstone and Silverman (2004).

3.3 *The Gaussian-Laplace case*

To describe the posterior distribution in this case we need to define another modification of the Gaussian distribution, $CTN(x \mid z, \mu, \sigma)$. It is a Combined Truncated Normal distribution which is a combination of Gaussian distributions with variance σ truncated at point z to the left semiline with mean μ_1 ($TN_-(x \mid z, \mu_1, \sigma)$) and to the right semiline with mean μ_2 ($TN_+(x \mid z, \mu_2, \sigma)$) with mixing parameter $\beta = \beta(z, \mu_1, \mu_2, \sigma) \in [0, 1]$ in such a way that the density function is continuous, i.e.,

$$CTN(x \mid z, \mu_1, \mu_2, \sigma) = \beta TN_-(x \mid z, \mu_1, \sigma) + (1 - \beta) TN_+(x \mid z, \mu_2, \sigma);$$
$$TN_-(x \mid z, \mu, \sigma) = \frac{\Phi((x - \mu)/\sigma)}{\Phi((z - \mu)/\sigma)} \mathbf{1}_{(-\infty, z)}(x) + \mathbf{1}_{(z, +\infty)}(x)$$
$$TN_+(x \mid z, \mu, \sigma) = \frac{\Phi((x - \mu)/\sigma) - \Phi((z - \mu)/\sigma)}{1 - \Phi((z - \mu)/\sigma)} \mathbf{1}_{(z, +\infty)}(x),$$

where $\mathbf{1}_A$ is the indicator function of A. The CTN distribution is different from the CTG distribution used in the Laplace-Gaussian case studied in Subsection 3.2. The CTG distribution combines the same Gaussian distribution truncated to the left and to the right at different points, whereas the CTN distribution combines Gaussian distributions with different means truncated at the same point. Now we can formulate the main result.

THEOREM 3.3. *Consider the one-dimensional observation model* (2.1) *with* $\varphi(x) = l_a(x)$ *and take the prior model* (2.2) *with* $h(x) = \phi_\sigma(x)$. *Then, the following hold*

(1) *The cumulative distribution function of the posterior distribution is given by*

$$F(\theta \mid y) = \frac{\omega(y)}{1+\omega(y)} I_{[0,+\infty)}(\theta) + \frac{1}{1+\omega(y)} CTN(\theta \mid y, a\sigma^2, -a\sigma^2, \sigma), \tag{3.10}$$

where

$$\omega(y) = \frac{(1-p)e^{-a|y|}}{pe^{(a\sigma)^2/2}\nu(y)},$$
$$\beta(y) = \frac{e^{-ay}\Phi(y/\sigma - a\sigma)}{\nu(y)},$$
$$\nu(y) = e^{ay}\Phi(-y/\sigma - a\sigma) + e^{-ay}\Phi(y/\sigma - a\sigma).$$

(2) *The median of the posterior distribution* (3.10) *is given by*

$$m_\theta(y) = \begin{cases} 0, & if \quad |y| \le \lambda, \\ \mathrm{sign}(y)[a\sigma^2 + \sigma\Phi^{-1}(\frac{(1-\mathrm{sign}(y)\omega(y))\nu(y)e^{a|y|}}{2})], & if \quad |y| > \lambda, \end{cases} \tag{3.11}$$

where the threshold $\lambda > 0$ *is defined as a solution of the equation*

$$\Phi(\lambda/\sigma - a\sigma) + \Phi(-\lambda/\sigma - a\sigma)e^{2a\lambda} = 2\Phi(-a\sigma) + \frac{1-p}{p}e^{-(a\sigma)^2/2}. \tag{3.12}$$

The equation above has a finite solution if and only if $p > \frac{1}{1+(2\Phi(a\sigma)-1)e^{(a\sigma)^2/2}}$, *otherwise* $\lambda = +\infty$ *which implies that* $m_\theta(y) = 0$ *for all* $y \in \mathbb{R}$.

(3) *If* $p > \frac{1}{1+(2\Phi(a\sigma)-1)e^{(a\sigma)^2/2}}$, *the asymptotes of the posterior median* (3.11), *as* $y \to \infty$, *are given by*

$$m_\theta(y) = \mathrm{sign}(y)\left[a\sigma^2 - \sigma\Phi^{-1}\left(\frac{1}{2} + \frac{1-p}{2p}e^{-(a\sigma)^2/2}\right)(1+o(1))\right]. \tag{3.13}$$

Since both densities $\varphi(x)$ and $h(x)$ are continuous, symmetric and defined on $\mathbb{R}$, the posterior median is antisymmetric. Also, it is easy to see from the formula for the posterior median (3.11) that it increases as a function of y, and thus it is a thresholding rule with threshold parameter λ defined by (3.12) for $p > \frac{1}{1+(2\Phi(a\sigma)-1)e^{(a\sigma)^2/2}}$. For $p \le \frac{1}{1+(2\Phi(a\sigma)-1)e^{(a\sigma)^2/2}}$, the posterior median is zero for all $y \in \mathbb{R}$. Corollary 2.1 and Theorem 2.1 imply that the posterior median does not possess the bounded shrinkage property. Moreover, as $y \to \infty$, the posterior median tends to a constant, $\mathrm{sign}(y)[a\sigma^2 - \sigma\Phi^{-1}(\frac{1}{2} + \frac{1-p}{2p}e^{-(a\sigma)^2/2})]$. Figure 3 plots the posterior median (3.11) and its asymptotes (3.13) for a particular case.

3.4 *The Laplace-Laplace case*

To describe the posterior distribution in this case we need to define a Truncated Laplace (TL) distribution with parameter $\lambda \in \mathbb{R}$ on the interval $[a, b]$ with probability density function

$$TL(x \mid \lambda, a, b) = \frac{\lambda}{2(L_\lambda(b) - L_\lambda(a))}e^{-\lambda|x|}\mathbf{1}(a \le x < b),$$

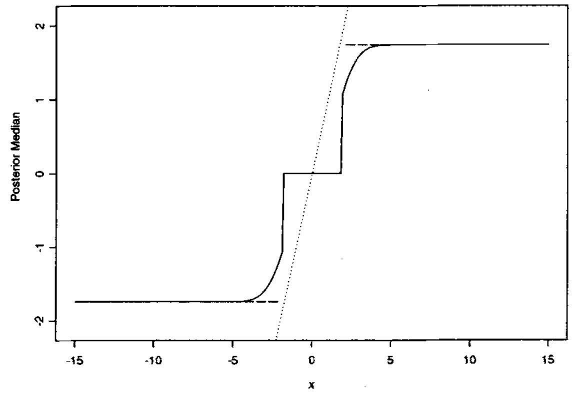

Fig. 3. Posterior median for Gaussian-Laplace distributions with parameters $a = 2$, $\sigma = 1$, $p = 0.4$. Line $y = x$ is plotted as a dotted line, and the asymptotes are plotted as dashed lines.

where $-\infty \le a < b \le +\infty$ and $L_\lambda(x) = \frac{1}{2}e^{\lambda x}\mathbf{1}_{(-\infty,0)}(x) + (1 - \frac{1}{2}e^{-\lambda x})\mathbf{1}_{[0,+\infty)}(x)$ is the cumulative distribution function of the Laplace distribution. If $\lambda \le 0$, both end points a and b have to be finite. In case $\lambda = 0$, this distribution coincides with the uniform distribution on $[a, b]$, and in case $\lambda > 0$, $a = 0$, $b = +\infty$ it coincides with the exponential distribution with parameter λ. Now, given two finite parameters, a and b, we divide the real line into three intervals, and glue together Truncated Laplace distributions on these intervals, such that parameters on the infinite intervals ($(-\infty, a)$ and $(b, +\infty)$) are the same making the right and left tails of the distribution of the same order, i.e.,

$$\begin{aligned} CTL(x \mid \lambda_1, \lambda_2, a, b) &= \alpha_1 TL(x \mid \lambda_1, -\infty, a) + \alpha_2 TL(x \mid \lambda_1, b, +\infty) \\ &\quad + (1 - \alpha_1 - \alpha_2) TL(x \mid \lambda_2, a, b), \end{aligned}$$

with parameters $-\infty < a < b < +\infty$, $\lambda_1 > 0$, $\lambda_2 \in \mathbb{R}$, while the mixing parameters $\alpha_1 = \alpha_1(\lambda_1, \lambda_2, a, b)$ and $\alpha_2 = \alpha_2(\lambda_1, \lambda_2, a, b) \in [0, 1]$ are chosen such that the probability density function of the CTL distribution is continuous. We are now in the position to state the following results, separating the cases different ($\nu \ne \mu$) and equal ($\nu = \mu$) parameters of prior and noise distributions.

THEOREM 3.4. *Consider the one-dimensional observation model* (2.1) *with* $\varphi(x) = l_\mu(x)$ *and take the prior model* (2.2) *with* $h(x) = l_\nu(x)$. *Then, providing* $\nu \ne \mu$, *the following hold*

(1) *The cumulative distribution function of the posterior distribution is given by*

$$\begin{aligned} F(\theta \mid y) &= \frac{\omega(y)}{1 + \omega(y)} I_{[0,+\infty)}(\theta) \\ &\quad + \frac{1}{1 + \omega(y)} CTL(\theta \mid \nu + \mu, (\nu - \mu)\,\mathrm{sign}(y), \min(0, y), \max(0, y)), \end{aligned} \tag{3.14}$$

where

$$\omega(y) = \frac{1 - p}{p} \frac{1 - (\mu/\nu)^2}{1 - (\mu/\nu)e^{-(\nu-\mu)|y|}},$$

$$\alpha_1(y) = \frac{(\nu-\mu)e^{[\mu \min(0,y)+\nu \min(0,-y)]}}{2(\nu e^{-\mu|y|} - \mu e^{-\nu|y|})},$$
$$\alpha_2(y) = \frac{(\nu-\mu)e^{[\nu \min(0,y)+\mu \min(0,-y)]}}{2(\nu e^{-\mu|y|} - \mu e^{-\nu|y|})}.$$

(2) *The median of the posterior distribution* (3.14) *is given by*

$$m_\theta(y) = \begin{cases} 0, & \text{if } |y| \le \lambda, \\ \text{sign}(y)[\frac{1}{\mu-\nu}\log(\frac{\nu+\mu e^{(\mu-\nu)|y|}}{\nu+\mu} + \frac{(1-p)(\nu-\mu)}{p\nu})], & \text{if } |y| > \lambda, \end{cases} \tag{3.15}$$

where the threshold $\lambda > 0$ *is defined by*

$$\lambda = \frac{1}{\mu-\nu}\log\left[\max\left(0, 1 + \frac{(1-p)(\mu^2-\nu^2)}{p\mu\nu}\right)\right]. \tag{3.16}$$

(3) *The asymptotes of the posterior median* (3.15), *as* $y \to \infty$, *are given, for* $\nu < \mu$, *by*

$$m_\theta(y) = \text{sign}(y)\left[|y| - \frac{1}{\mu-\nu}\log\left(\frac{\nu+\mu}{\mu}\right) + \frac{\nu e^{-(\mu-\nu)|y|}}{\mu(\mu-\nu)}\left(\frac{(1-p)(\nu^2-\mu^2)}{p\nu^2}+1\right)(1+o(1))\right], \tag{3.17}$$

and, for $\nu > \mu$, *by*

$$m_\theta(y) = \text{sign}(y)\left[\frac{1}{\mu-\nu}\log\left(\frac{\nu}{\mu+\nu} + \frac{(1-p)(\nu-\mu)}{p\nu}\right) + \frac{1}{\mu-\nu}\frac{(\mu/\nu)e^{-(\nu-\mu)|y|}}{\frac{1-p}{p}\frac{\nu^2-\mu^2}{\nu^2}+1}(1+o(1))\right], \tag{3.18}$$

provided that $p > \frac{\nu^2-\mu^2}{\nu\mu+\nu^2-\mu^2}$.

Since both densities $\varphi(x)$ and $h(x)$ are continuous, symmetric and defined on $\mathbb{R}$, the posterior median is antisymmetric. Also, it is easy to see from the formula for the posterior median (3.15) that it increases as a function of y, and thus it is a thresholding rule with threshold parameter λ defined by (3.16) for $p > \frac{\nu^2-\mu^2}{\nu\mu+\nu^2-\mu^2}$. For $\nu > \mu$ and $p \le \frac{\nu^2-\mu^2}{\nu\mu+\nu^2-\mu^2}$, the posterior median is zero for all $y \in \mathbb{R}$. According to Corollary 2.1 and Theorem 2.1, the posterior median (3.15) possesses the bounded shrinkage property only in case $\nu < \mu$. Furthermore, as $y \to \infty$, it tends to a linear asymptote, $\text{sign}(y)[|y| - \frac{1}{\mu-\nu}\log(\frac{\nu+\mu}{\mu})]$, with an exponential rate, $e^{-(\mu-\nu)|y|}$, when $\nu < \mu$, while it tends to a constant, $\text{sign}(y)[\frac{1}{\mu-\nu}\log(\frac{\nu}{\mu+\nu} + \frac{(1-p)(\nu-\mu)}{p\nu})]$, with an exponential rate, $e^{-(\nu-\mu)|y|}$, when $\nu > \mu$ and $p > \frac{\nu^2-\mu^2}{\nu\mu+\nu^2-\mu^2}$. Figure 4 plots the posterior median (3.15) and its asymptotes (3.17)–(3.18) for particular cases.

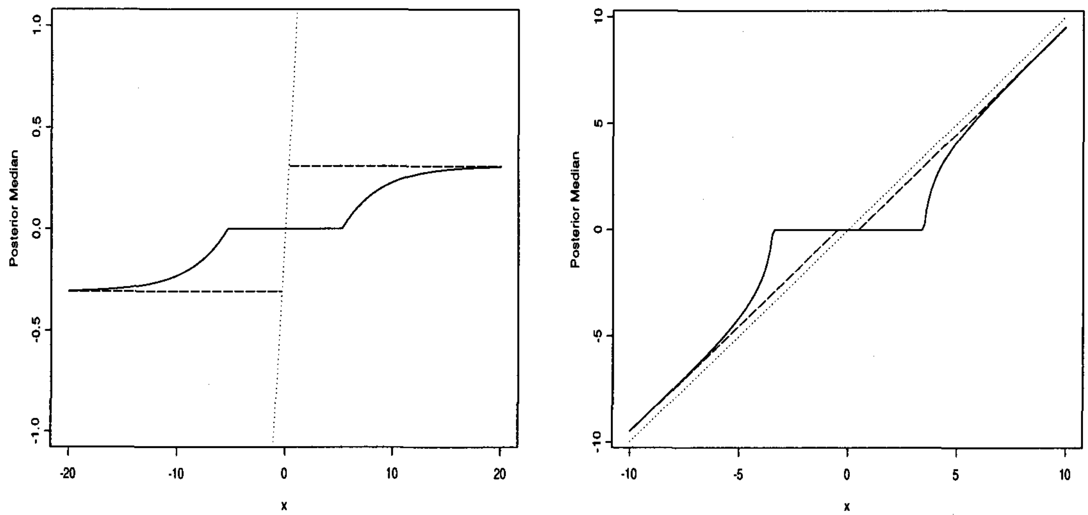

Fig. 4. Posterior median for Laplace-Laplace distributions with parameters: (left) $\mu = 1$, $\nu = 1.3$, $p = 0.4$; (right) $\mu = 1.5$, $\nu = 0.7$, $p = 0.1$. Line $y = x$ is plotted as a dotted line, and the asymptotes are plotted as dashed lines.

THEOREM 3.5. *Consider the one-dimensional observation model* (2.1) *with* $\varphi(x) = l_\nu(x)$ *and take the prior model* (2.2) *with* $h(x) = l_\nu(x)$. *Then, the following hold*

(1) *The cumulative distribution function of the posterior distribution is given by*

$$F(\theta \mid y) = \frac{\omega(y)}{1+\omega(y)} I_{[0,+\infty)}(\theta) + \frac{1}{1+\omega(y)} CTL(\theta \mid 2\nu, 0, \min(0,y), \max(0,y)), \tag{3.19}$$

where

$$\omega(y) = \frac{2(1-p)}{p(1+\nu|y|)},$$
$$\alpha_1(y) = \alpha_2(y) = \frac{1}{2(1+\nu|y|)}.$$

(2) *The median of the posterior distribution* (3.19) *is given by*

$$m_\theta(y) = \begin{cases} 0, & \text{if } |y| \le \lambda, \\ \operatorname{sign}(y)[\frac{|y|}{2} - \frac{1-p}{\nu p}], & \text{if } |y| > \lambda, \end{cases} \tag{3.20}$$

where the threshold $\lambda > 0$ *is defined by*

$$\lambda = \frac{2(1-p)}{\nu p}. \tag{3.21}$$

Since both densities $\varphi(x)$ and $h(x)$ are continuous, symmetric and defined on $\mathbb{R}$, the posterior median is antisymmetric. Also, it is easy to see from the formula for the posterior median (3.20) that it increases as a function of y, and thus it is a thresholding rule with threshold parameter λ defined by (3.21). The formula for the posterior median also implies that the median does not possess the bounded shrinkage property. Figure 5 plots the posterior median (3.20) for a particular case.

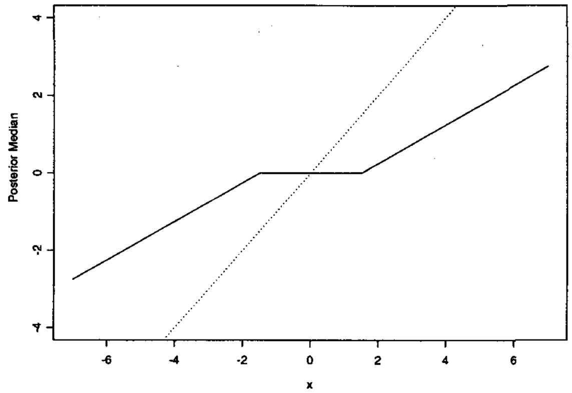

Fig. 5. Posterior median for Laplace-Laplace distributions with parameters $\nu = \mu = 2$, $p = 0.4$. Line $y = x$ is plotted as a dotted line.

3.5 *The t_1-t_3 case*

THEOREM 3.6. *Consider the one-dimensional observation model (2.1) with $\varphi(x) = t_3(x)$ and take the prior model (2.2) with $h(x) = t_1(x)$. Then, the following hold*

(1) *The cumulative distribution function of the posterior distribution is given by*

$$F(\theta \mid y) = \frac{\omega(y)}{1+\omega(y)} I_{[0,+\infty)}(\theta) + \frac{1}{1+\omega(y)} \tilde{H}(\theta \mid y), \tag{3.22}$$

where

$$\begin{aligned}\tilde{H}(\theta \mid y) = \frac{1}{\eta(y)\pi^2\sqrt{3}} \Bigg[& 3C(y)\log\left(\frac{\theta^2+1}{(\theta-y)^2+3}\right) + B(y)(\pi + 2\arctan(\theta)) \\ & + \sqrt{3}(2D(y)+G(y))\left(\frac{\pi}{2} + \arctan\left(\frac{\theta-y}{\sqrt{3}}\right)\right) \\ & + 3F(y)\frac{1}{1+(y-\theta)^2/3} + G(y)\frac{\theta-y}{1+(y-\theta)^2/3}\Bigg],\end{aligned}$$

$$\omega(y) = \frac{1-p}{p} \frac{1}{(B(y)+\sqrt{3}(D(y)+G(y)/2))(1+y^2/3)^2},$$

$$\eta(y) = \frac{2}{\pi\sqrt{3}}(B(y)+\sqrt{3}(D(y)+G(y)/2)),$$

and

$$\begin{aligned} B(y) &= 9(y^4+4)/\Delta, \\ C(y) &= 12y(y^2+2)/\Delta, \\ D(y) &= 3(3y^4+8y^2-4)/\Delta, \\ F(y) &= 2y(y^4+8y^2+4)/\Delta, \\ G(y) &= (y^6+6y^4-12y^2-8)/\Delta, \end{aligned}$$

where $\Delta = 9y^8 + 48y^6 + 38y^4 - 52y^2 + 48$.

(2) *For $p > p_0 \approx 0.401$, the median of the posterior distribution* (3.22) *is a thresholding rule, with the value of the threshold $\lambda > 0$ defined as a solution of the equation*

$$3C(\lambda)\log(\lambda^2+3)+\sqrt{3}(2D(\lambda)+G(\lambda))\arctan\left(\frac{\lambda}{\sqrt{3}}\right)+\frac{\lambda G(\lambda)-3F(\lambda)}{1+\lambda^2/3} = \frac{\pi(1-p)}{p(1+\lambda^2/3)^2}. \tag{3.23}$$

For $y > \lambda$, the posterior median, $m_\theta(y)$, is given implicitly by the equation

$$\begin{aligned} 3C(y)\log\left(\frac{m_\theta(y)^2+1}{(m_\theta(y)-y)^2+3}\right) &+ 2B(y)\arctan m_\theta(y) \\ &+\sqrt{3}(2D(y)+G(y))\arctan\left(\frac{m_\theta(y)-y}{\sqrt{3}}\right) \\ &+3F(y)\frac{1}{1+(y-m_\theta(y))^2/3} \\ &+G(y)\frac{m_\theta(y)-y}{1+(y-m_\theta(y))^2/3} = -\frac{1-p}{p}\frac{\pi}{(1+y^2/3)^2}. \end{aligned} \tag{3.24}$$

(3) *The asymptotes of the posterior median defined in* (3.24), *as $y \to \infty$, are given by*

$$m_\theta(y) = \mathrm{sign}(y)\left(|y| - \frac{3}{|y|}(1+o(1))\right). \tag{3.25}$$

The behavior of function $\gamma(y) = \frac{2\tilde{H}(0|y)-1}{\omega(y)}$ determines such important properties of the posterior median as its being an increasing function, being a thresholding rule, and the value of the threshold. It depends on the non-zero part of the prior distribution, on the noise distribution, and also on the proportion p of the non-zero part in the prior distribution. We split function $\gamma(y)$ in the product of two functions, i.e., $\gamma(y) = \frac{p}{1-p}\tilde{\gamma}(y)$, where $\tilde{\gamma}(y)$ is independent of the parameter p. The latter function is uniquely defined by the noise distribution and the non-zero part in the prior distribution. We shall call function $\tilde{\gamma}(y)$ as the *characterizing* function of the posterior median. Plot of the characterizing function for the case t_1-t_3 distributions is given in Fig. 6(left).

Hence, the equation for the threshold becomes $\tilde{\gamma}(\lambda) = -\frac{1-p}{p}$, and we can see that for the prior odds ratio below $\frac{1-p_0}{p_0} \approx 0.671$ (corresponding to $p \le p_0 \approx 0.401$) there are two solutions for this equation, and the posterior median is not an increasing function, as it has two intervals on $(0, +\infty)$ where it is zero. This implies that if the proportion of the nonzero distribution in the mixture is too low and the tails of both distributions (nonzero part of the prior and noise) are both heavy, for small observed values the posterior median is not as well-behaved as in the case of lighter-tailed noise distributions (see Subsections 3.1–3.4). Also, the tradeoff for the good asymptotic properties is that the model cannot allow for too much sparsity since the posterior median is well-behaved only for $p > p_0$. However, for a sufficiently high prior probability of the value to be non-zero, the posterior median possesses the same basic properties of a posterior median which make it as an estimator, i.e., it is an antisymmetric function, a shrinkage rule, and a thresholding rule with threshold parameter λ satisfying (3.23). It also possesses the

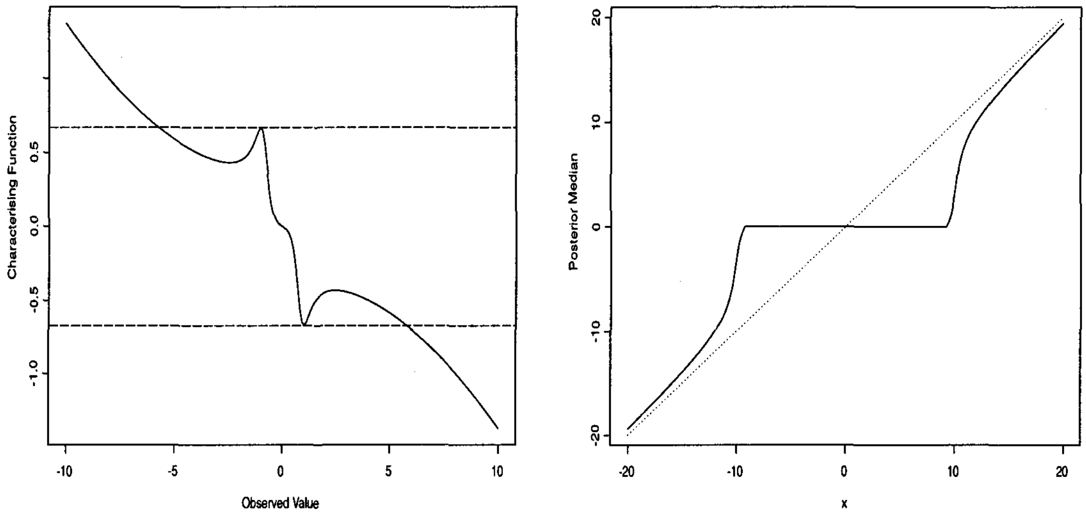

Fig. 6. (Left) Characterizing function for t_1-t_3 distributions. The dashed line corresponds to the critical value of the prior odds ratio $(1 - p_0)/p_0$; (right) posterior median for t_1-t_3 distributions with parameter $p = 0.45$. Line $y = x$ (which is also the asymptote) is plotted as a dotted line.

bounded shrinkage property and moreover, it shrinks large observations by a very small value, $\frac{3}{y}$. Figure 6(right) plots the posterior median defined in (3.24) and its asymptotes (3.25) for a particular case.

3.6 *The (scaled) t_1-t_1 case*

THEOREM 3.7. *Consider the one-dimensional observation model (2.1) with $\varphi(x) = \frac{1}{\sigma}t_1(\frac{x}{\sigma})$ and take the prior model (2.2) with $h(x) = \frac{1}{\tau}t_1(\frac{x}{\tau})$. Then, the following hold*

(1) *The cumulative distribution function of the posterior distribution is given by*

$$F(\theta \mid y) = \frac{\omega(y)}{1+\omega(y)} I_{[0,+\infty)}(\theta) + \frac{1}{1+\omega(y)} \tilde{H}(\theta \mid y), \tag{3.26}$$

where

$$\begin{aligned}\tilde{H}(\theta \mid y) = \frac{1}{2} + \frac{1}{2\pi(B(y)\tau + D(y)\sigma)} \bigg[&\sigma\tau C(y) \log\left(\frac{\theta^2+\tau^2}{(\theta-y)^2+\sigma^2}\right) \\ &+ 2\tau B(y) \arctan\left(\frac{\theta - y}{\sigma}\right) \\ &+ 2\sigma D(y) \arctan\left(\frac{\theta}{\tau}\right)\bigg],\end{aligned}$$

$$\omega(y) = \frac{1-p}{p} \frac{\sigma}{(y^2+\sigma^2)(\tau B(y) + \sigma D(y))},$$
$$\eta(y) = \frac{1}{\pi}(B(y)\tau + D(y)\sigma),$$

and

$$B(y) = (y^2 + \tau^2 - \sigma^2)/\Delta',$$

$$C(y) = 2y/\Delta',$$
$$D(y) = (y^2 - \tau^2 + \sigma^2)/\Delta',$$

where $\Delta' = (y^2 - \tau^2 + \sigma^2)^2 + 4\tau^2 y^2$.

(2) *For* $y > \lambda$, *the median* $m_\theta(y)$ *of the posterior distribution* (3.26) *is given implicitly by the equation*

$$(3.27) \qquad 2\tau B(y) \arctan\left(\frac{m_\theta(y) - y}{\sigma}\right) + 2\sigma D(y) \arctan\left(\frac{m_\theta(y)}{\tau}\right) + \sigma\tau C(y) \log\left(\frac{m_\theta(y)^2 + \tau^2}{(y - m_\theta(y))^2 + \sigma^2}\right) = -\frac{(1-p)\pi\sigma}{p(y^2 + \sigma^2)},$$

where the threshold $\lambda > 0$ *is defined as a solution of the equation*

$$(3.28) \qquad \frac{p(\lambda^2 + \sigma^2)}{(1-p)\pi\sigma}\left[\sigma\tau C(\lambda) \log\left(\frac{\lambda^2 + \sigma^2}{\tau^2}\right) + 2\tau B(\lambda) \arctan\left(\frac{\lambda}{\sigma}\right)\right] = 1.$$

The equation above has a finite solution if and only if $p > \frac{\sigma}{\sigma+\tau}$, *otherwise* $\lambda = +\infty$ *which implies that* $m_\theta(y) = 0$ *for all* $y \in \mathbb{R}$. *Furthermore, the posterior median increases if and only if* $\sigma < \tau$ *and* $p \geq \frac{\sigma}{\tau}$, *implying that it is, in this case, a shrinkage and a thresholding rule.*

(3) *The asymptotes of the posterior median defined in* (3.28), *as* $y \to \infty$, *are given, for* $p > \frac{\sigma}{\tau}$, *by*

$$(3.29) \qquad m_\theta(y) = \text{sign}\left[|y| - \sigma \tan\left(\frac{\pi\sigma}{2\tau p}\right)(1 + o(1))\right],$$

for $p = \frac{\sigma}{\tau}$, *by*

$$(3.30) \qquad m_\theta(y) = \text{sign}\left[\frac{|y|}{2} - \frac{\pi}{32}\frac{(\tau - \sigma)(\tau - 2\sigma)}{\sigma}\right] + o(1),$$

and, for $\frac{\sigma}{\sigma+\tau} < p < \frac{\sigma}{\tau}$, *by*

$$(3.31) \qquad m_\theta(y) = \text{sign}(y)\tau \tan\left(\frac{\pi}{2}\left[\frac{\tau}{\sigma} - \frac{1-p}{p}\right]\right)(1 + o(1)).$$

The posterior median is an antisymmetric function, and it is a shrinkage and a thresholding rule if and only if $\sigma < \tau$ and $p \geq \frac{\sigma}{\tau}$. In case $\sigma < \tau$ and $p > \frac{\sigma}{\tau}$, it also possesses the bounded shrinkage property. If $p < \frac{\sigma}{\tau}$, the posterior median is not an increasing function and thus not a thresholding rule. Furthermore, if $p \leq \frac{\sigma}{\sigma+\tau}$, the posterior median is identically zero. It is interesting to see that in the Laplace-Laplace case (see Subsection 3.4), the bound between the posterior median being bounded shrinkage and asymptotic constant is the case of equal parameters $\mu = \nu$ where the posterior median behaves as $y/2$, whereas in the (scaled) t_1-t_1 case, this bound is the case where $\sigma < \tau$ and $p = \frac{\sigma}{\tau}$.

In the case of equal scaling parameters, $\tau = \sigma$, the posterior median is not a monotonic function, as we can see in Fig. 7. For positive observed values, it decreases very slowly to its limit (increases for negative observed values), as does the function $Q(y)$

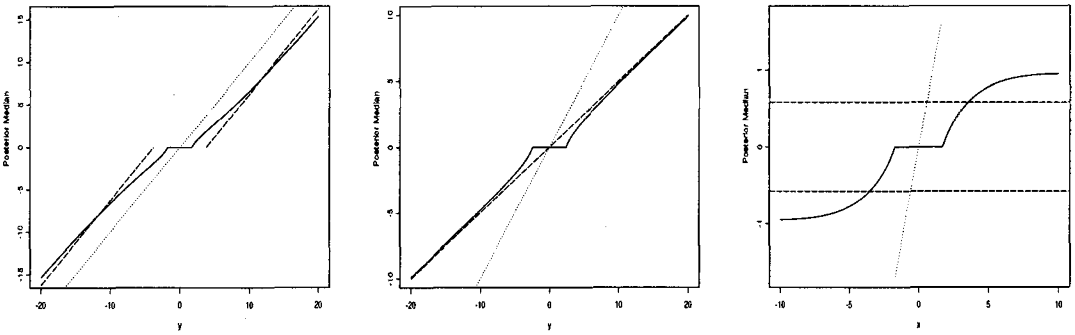

Fig. 7. Posterior median for (scaled) t_1-t_1 distributions with parameters: (left) $\tau = 2$, $\sigma = 1$, $p = 0.6$; (center) $\tau = 2$, $\sigma = 1$, $p = 0.5$; (right) $\tau = 1$, $\sigma = 1$, $p = 0.6$. Line $y = x$ is plotted as a dotted line, and asymptotes are plotted as dashed lines.

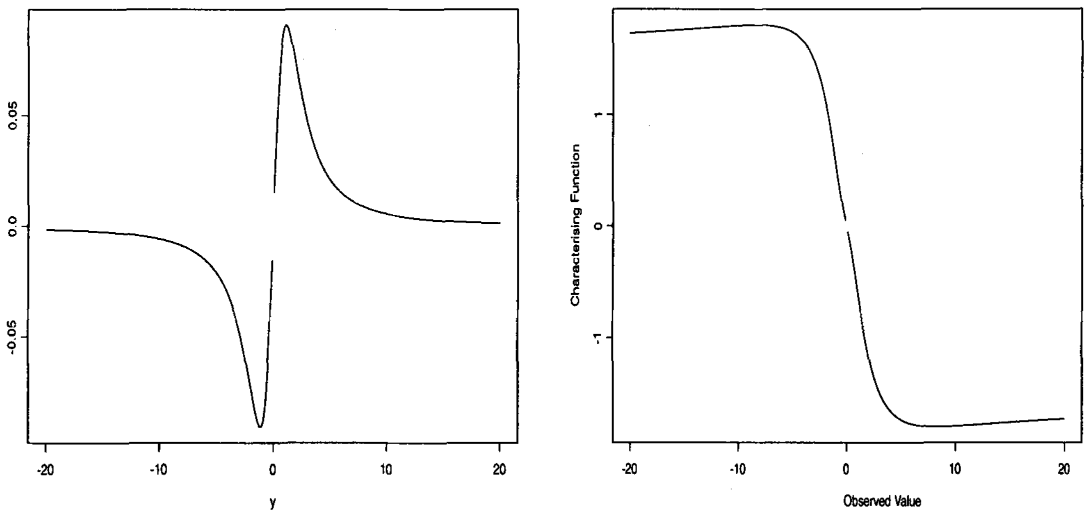

Fig. 8. (Scaled) t_1-t_1 distributions with equal scaling parameters, $\tau = \sigma = 1$: (left) function $Q(y)$ from assumption (A2); (right) characterizing function.

which, according to assumption (A2), should be increasing for the posterior median to increase (see Fig. 8). We also plot the characterizing function, as we did in the t_1-t_3 case (see Subsection 3.5), to illustrate that the posterior median in this case changes its behavior from increasing to decreasing at the same points as the characterizing function. Furthermore, as $y \to \infty$, it tends to a constant, $\tau \tan(\frac{\pi(2p-1)}{2p})$ and, hence, it does not possess the bounded shrinkage property. Figure 7 plots the posterior median defined in (3.28) for particular cases.

4. Concluding remarks

In this paper we studied the posterior median in a Bayesian framework which is applicable to objects which can be modelled as possibly sparse sequences. Certain properties of the posterior median, discussed recently in Johnstone and Silverman (2004), were extended to a more general framework. Specific case studies were considered which characterize the behavior of the posterior median and allow one to compare types of the asymptotic behavior and values of thresholds for different prior and noise distributions. It has been observed that if the tail of the noise distribution is heavier than the tail of the nonzero part of the prior distribution (see Subsections 3.3, 3.4, 3.6), the posterior median tends to a constant for large observations thus severely underestimating large observed values. On the other hand, as long as the tails of the nonzero part of the prior distribution are heavier than the tails of the noise distribution (see Subsections 3.2, 3.4, 3.5), the posterior median, under some constraints on the involved parameters, has the bounded shrinkage property.

According to the recent results of Johnstone and Silverman (2004), this latter property should assist us in studying frequentist optimality properties of the resulting Bayesian estimates to a wider set of prior and noise distributions. One expects that, under some mild conditions, posterior median estimation methods obtained by *certain* combinations of prior and noise distributions will achieve the frequentist optimal estimation rate for a wide range of classes of signal sequences of length n with normalized l_p-norm bounded by η, for $\eta > 0$ and $p \in (0,2]$ at various rates, simultaneously for a wide range of mean q-th-losses with $q \in (0,2]$.

Furthermore, since in nonparametric wavelet regression the true wavelet coefficients at each level form a possibly sparse sequence, and the discrete wavelet transform approximates them subject to error, these posterior median estimation methods should also be proved useful in wavelet regression. It is hoped that, under some mild conditions, posterior median estimation methods obtained by *certain* combinations of prior and noise distributions will also achieve the best possible minimax rate over a wide range of Besov spaces, $B^s_{p,\infty}$, for any value of the parameter $p \in [0,+\infty)$, simultaneously for a wide range of L_q-losses with $q \in (0,2]$, extending thus the results of Johnstone and Silverman (2002) to larger families of prior and noise distributions. A direct comparison is then possible with the frequenstist optimality results obtained by Pensky (2003) for Bayesian wavelet shrinkage methods (for both Gaussian and non-Gaussian noise distributions) based on the posterior mean.

However, all these are beyond the scope of this paper but present the direction which we hope will be addressed in the future.

5. Proofs

5.1 *Proof of Proposition* 2.2

Given the value y, let $m_\theta(y)$ be the median of the posterior distribution, i.e., $m_\theta(y)$ be such that $F(m_\theta(y) \mid y) = 0.5$. By Proposition 2.1, the former equation can be rewritten as

$$\frac{\omega(y)}{1+\omega(y)} I_{[0,+\infty)}(m_\theta(y)) + \frac{1}{1+\omega(y)}\tilde{H}(m_\theta(y) \mid y) = 0.5.$$

If $\omega(y) \geq 1$ then obviously the posterior median is zero since y is zero with probability $\frac{\omega(y)}{1+\omega(y)} \geq 0.5$. Moreover, the posterior median is zero if and only if $\frac{1}{1+\omega(y)}\tilde{H}(0 \mid y) \leq 0.5$ and $\frac{\omega(y)}{1+\omega(y)} + \frac{1}{1+\omega(y)}\tilde{H}(0 \mid y) \geq 0.5$, i.e., if and only if $0.5(1-\omega(y)) \leq \tilde{H}(0 \mid y) \leq 0.5(1+\omega(y))$.

If we denote $\gamma(y) = \frac{2\tilde{H}(0|y)-1}{\omega(y)}$, then the latter condition is equivalent to $-1 \leq \gamma(y) \leq 1$. In case $|\gamma(y)| > 1$, we can split the equation for the posterior median into two cases for $m_\theta(y)$ being positive or negative, i.e., $m_\theta(y) > 0$ and $\tilde{H}(m_\theta(y) \mid y) = \frac{1-\omega(y)}{2}$, or $m_\theta(y) < 0$ and $\tilde{H}(m_\theta(y) \mid y) = \frac{1+\omega(y)}{2}$. It is easy to show that this system of equations has a strictly positive or negative solution for every y such that $|\gamma(y)| > 1$. Thus we can separate cases of positive, negative and zero posterior median, i.e., $|\gamma(y)| \leq 1$ if and only if $m_\theta(y) = 0$, $\gamma(y) < -1$ if and only if $m_\theta(y) > 0$ and $\gamma(y) > 1$ if and only if $m_\theta(y) < 0$. Therefore, it is easily seen that the equation for the posterior median is expressed as

$$m_\theta(y) = \begin{cases} 0, & |\gamma(y)| \leq 1, \\ \tilde{H}^{-1}(\frac{1-\omega(y)}{2} \mid y), & \gamma(y) < -1, \\ -\tilde{H}^{-1}(\frac{1-\omega(y)}{2} \mid -y), & \gamma(y) > 1, \end{cases}$$

which, since $\operatorname{sign}(\gamma(y)) = -\operatorname{sign}(y)$, is equivalent to

$$m_\theta(y) = \max\{0, \operatorname{sign}[|\gamma(y)| - 1]\} \operatorname{sign}(y)\tilde{H}^{-1}\left(\frac{1-\min\{1,\omega(y)\}}{2} \,\middle|\, |y|\right),$$

completing the proof of Proposition 2.2.

5.2 *Proof of Proposition* 2.3

From the definition of f, $\frac{h(x)}{f(x)}$ is an increasing function for $x > M$, and thus for any $x > y > M$, we have $(h(x)/f(x)) \geq (h(y)/f(y))$. Under the assumptions on h and f, it is easily seen that for all $x, y \in [0, M]$ we have $(h(x)/f(x)) \leq C(h(y)/f(y))$ for some constant $C > 1$. By symmetry of h and f, this inequality holds for all $x, y \in [-M, M]$. Combining these two inequalities, we have, in the case $y < M$ and $x > M$, that $(h(x)/f(x)) \geq (h(M)/f(M)) \geq C^{-1}(h(y)/f(y))$. The function $\eta(y)$ can be written as

$$\eta(y) = \int_0^{+\infty} [h(y-u) + h(y+u)]\varphi(u)du,$$

and the following bounds hold for all $y > M$

$$\begin{aligned}\eta(y) &> \int_0^{+\infty} h(y+u)\varphi(u)du \geq C^{-1}h(y)\int_0^{\infty} \frac{f(x+u)}{f(x)}\varphi(u)du \\ &\geq C^{-1}\int_0^{+\infty} h(y)Q_1(u)\varphi(u)du \\ &= C^{-1}C_1 h(y),\end{aligned}$$

and

$$\begin{aligned}\eta(y) &\leq \int_0^{+\infty} \left[Ch(y)\frac{f(y-u)}{f(y)} + h(y)\right]\varphi(u)du \leq h(y)\int_0^{+\infty} [CQ_2(u)+1]\varphi(u) \\ &= h(y)[1/2 + CC_2].\end{aligned}$$

Therefore, we immediately get $\eta(x) \asymp h(x)$, completing the proof of Proposition 2.3.

5.3 *Proof of Corollary* 2.1

(i) Exponential-Power distributions. Suppose that both $h(x)$ and $\varphi(x)$ have Exponential-Power distributions with parameters $\alpha, \tau > 0$ and $\beta, \sigma > 0$ respectively, where $\beta \geq \alpha$ and $\tau > \sigma$ if $\alpha = \beta$, i.e., the tail of density $h(x)$ is heavier than the tail of $\varphi(x)$. We need to show that both parts of condition (L3) hold. Since $h(x)$ satisfies assumptions for $f(x)$ we take $f(x) = h(x)$. For the second part of condition (L3), we find the maximum of the following function on $(0, +\infty)$ with fixed $u > 0$

$$\frac{h(x-u)}{h(x)} = \exp\{-|x-u|^\alpha/\tau^\alpha + x^\alpha/\tau^\alpha\}.$$

The derivative of $F(x) = -|x-u|^\alpha + x^\alpha$ is $F'(x) = \alpha(x^{\alpha-1} - (x-u)^{\alpha-1})$ for $x \geq u$ and $F'(x) = \alpha(x^{\alpha-1} + (u-x)^{\alpha-1})$ for $x < u$. The derivative is always positive on $0 < x < u$, and on $x \geq u$ it is positive for $\alpha > 1$ and negative for $\alpha \leq 1$. In case $\alpha \leq 1$ the maximum is achieved at point $x = u$, therefore the upper bound of the ratio is

$$Q_2(u) = \exp\{u^\alpha/\tau^\alpha\},$$

which is integrable with $\varphi(u)$ since the integral of $\exp\{u^\alpha/\tau^\alpha - u^\beta/\sigma^\beta\}$ is finite under the specified constraints on the parameters. If the derivative is positive on $x \geq u$ it implies that weighed integral of $\frac{h(x-u)}{h(x)}$ increases as a function of x, and it is unbounded as $x \to +\infty$. The weighed integral is also an unbounded function of x and therefore $C_2 = +\infty$. Straightforward calculations show that the first part of condition (L3) also holds with $Q_1(u) = \exp\{-u^\alpha/\tau^\alpha\}$, for $\alpha \leq 1$. Thus, condition (L3) is satisfied if and only if $\alpha \leq 1$, completing the proof of this case.

(ii) t distributions. Assume that both $h(x)$ and $\varphi(x)$ are t distributions with parameters $\nu_1 \geq 1$ and $\nu_2 \geq 1$ respectively. The tail of $h(x)$ is heavier than the tail of $\varphi(x)$ if $\nu_1 < \nu_2$. We show that both parts of condition (L3) hold for $f(x) = h(x)$. For the first part of condition (L3), we need to find a lower bound, independent of x, for the following ratio

$$\frac{h(x+u)}{h(x)} = \left(\frac{1}{1 + \dfrac{u^2 + 2xu}{x^2 + \nu_1}}\right)^{(\nu_1+1)/2},$$

where x and u are positive. It is easy to show that the following function $Q_1(u) = \frac{1}{5 + \frac{u^2}{\nu_1}}$ limits the above ratio from below, and it is integrable with $\varphi(u)$, therefore the first part of condition (L3) is satisfied. For the second part of condition (L3), straightforward calculations lead us to the following upper bound for the ratio $\frac{h(x-u)}{h(x)}$ which is independent of x

$$\frac{h(x-u)}{h(x)} = \left(\frac{1}{1 + \dfrac{u^2 - 2xu}{x^2 + \nu_1}}\right)^{(\nu_1+1)2} \geq \left(1 + \frac{u^2}{2\nu_1} + \frac{u}{\sqrt{\nu_1}}\sqrt{1 + \frac{u^2}{4\nu_1}}\right)^{(\nu_1+1)2} = Q_2(u),$$

which is of order u^{ν_1+1}. Since function $\varphi(u)$ is of order $u^{-\nu_2-1}$, the integral $\int_0^\infty Q_2(u)\varphi(u)du$ converges if and only if $\nu_2 - \nu_1 > 1$. Thus, condition (L3) is satisfied if and only if $\nu_2 - \nu_1 > 1$, completing the proof of this case.

Hence, the proof of Corollary 2.1 is completed.

5.4 *Proof of Theorem* 2.1

In order to prove Theorem 2.1 we need the following lemma that can be justified by straightforward calculations.

LEMMA 5.1. *The following properties hold*: (i) $\eta(-y) = \eta(y)$; (ii) $\omega(-y) = \omega(y)$, (iii) $\gamma(-y) = -\gamma(y)$; (iv) $\tilde{H}(x \mid -y) = 1 - \tilde{H}(-x \mid y)$; *and* (v) $\tilde{H}^{-1}(x \mid -y) = -\tilde{H}^{-1}(1 - x \mid y)$.

Now we prove Theorem 2.1.

(P1) Follows from the formula for the posterior median stated in Proposition 2.2 and the symmetric properties of the functions stated in Lemma 5.1, completing the proof for this case.

(P2) We need to show that $0 \le m_\theta(y) \le y$ for $y > 0$ and that it is an increasing function of y. First we show the former. As it was shown in the proof of Proposition 2.2, the posterior median is positive or zero for $y > 0$. So we need to show that $m_\theta(y) \le y$ for $y > 0$. Since $\tilde{H}$ is a strictly increasing function of its first argument, this statement is equivalent to $\tilde{H}(y \mid y) \ge \tilde{H}(m_\theta(y) \mid y) = \frac{1-\omega(y)}{2}$. Thus we need to show that $\tilde{H}(y \mid y) \ge \frac{1-\omega(y)}{2}$ for all $y > 0$. Since $\omega(y) > 0$ it is sufficient to show that $\tilde{H}(y \mid y) \ge \frac{1}{2}$

$$2\tilde{H}(y \mid y) = \frac{2\int_{-\infty}^{y} \varphi(y-u)h(u)du}{\int_{-\infty}^{+\infty} \varphi(y-u)h(u)du} = \frac{\int_{-\infty}^{y} \varphi(y-u)h(u)du + \int_0^{+\infty} \varphi(v)h(y-v)dv}{\int_{-\infty}^{y} \varphi(y-u)h(u)du + \int_0^{+\infty} \varphi(v)h(y+v)dv}.$$

To show that $2\tilde{H}(y \mid y) \ge 1$ it is sufficient to show that $\int_0^{+\infty} \varphi(v)h(y-v)dv \ge \int_0^{+\infty} \varphi(v)h(y+v)dv$. The difference between these two integrals is

$$\int_0^{+\infty} \varphi(v)[h(y-v) - h(y+v)]dv = \int_0^{y} \varphi(v)[h(y-v) - h(y+v)]dv + \int_y^{+\infty} \varphi(v)[h(v-y) - h(y+v)]dv.$$

Since $h(x)$ is unimodal, $h(x)$ decreases for $x > 0$ implying that $h(|v-y|) - h(y+v) \ge 0$ for $v, y \ge 0$. Thus both integrals are positive and $\tilde{H}(m_\theta(y) \mid y) \ge \frac{1}{2}$ which implies that $m_\theta(y) \le y$. Now we need to show that the posterior median $m_\theta(y)$ is an increasing function of y for $y > \lambda$ which by antisymmetry and thresholding property can then be extended to all real y. The equation for the posterior median, after some straightforward calculations, can be written as

$$2\int_0^{m_\theta(y)} \varphi(y-x)h(x)dx = \int_0^{+\infty} [\varphi(y-x) - \varphi(y+x)]h(x)dx - \frac{1-p}{p}\varphi(y).$$

To show that the posterior median increases, we show that the difference between the implicit equations for the median above at some points $y_2 > y_1 \ge \lambda$ is positive, i.e., that

$$2\int_{m_\theta(y_1)}^{m_\theta(y_2)} \varphi(y_2-x)h(x)dx = \frac{1-p}{p}[\varphi(y_1) - \varphi(y_2)] + 2\int_0^{m_\theta(y_1)} [\varphi(y_1-x) - \varphi(y_2-x)]h(x)dx$$

$$+\int_0^{+\infty}[\varphi(y_2-x)-\varphi(y_2+x)]h(x)dx$$
$$-\int_0^{+\infty}[\varphi(y_1-x)-\varphi(y_1+x)]h(x)dx>0.$$

Since φ is a unimodal density, it decreases on $(\lambda,+\infty)$, and since $y_2>y_1\geq m_\theta(y_1)$, it is easily seen that the first two summands on the right hand side are positive. Therefore the posterior median is non-negative if the function $Q(y)=\int_0^{+\infty}[\varphi(y-x)-\varphi(y+x)]h(x)dx$ increases on $(\lambda,+\infty)$, which is exactly assumption (A2), completing the proof for this case.

(P3) The statement "the posterior median is a thresholding rule if and only if there exists $\lambda>0$ such that for every $y\in[-\lambda,\lambda]$ we have $|\gamma(y)|=\frac{|2\tilde{H}(0|y)1|}{\omega(y)}\leq 1$" follows from the formula for the posterior median (2.4) since we saw in the proof of Proposition 2.2 that the posterior median is zero if and only if $|\gamma(y)|\leq 1$. Note that assumption (A2) is the same as the assumption that function $f(y)=-\gamma(y)\varphi(y)$ increases on $(0,+\infty)$. Thus, $-\gamma(y)=f(y)/\varphi(y)$ is a product of two increasing positive functions and therefore is also increasing, implying that $\gamma(y)$ decreases. Therefore, if assumption (A2) is satisfied, $\{y:|\gamma(y)|\leq 1\}$ is an interval and thus the condition of the statement above is satisfied. The compact form (2.5) for the posterior median is now easily seen, completing the proof for this case.

(P4) If we show that $\exists\ a,\ M^*>0$ such that for all $y>M^*$

$$P\{\mu>y-a\mid Y=y\}>\frac{1}{2},$$

then the posterior median possesses the bounded shrinkage property. Since probability $P\{\mu>y-a\mid Y=y,\mu=0\}$ is zero for $y>a$, the probability above can be presented as a product of the following probabilities

$$P\{\mu>y-a\mid Y=y\}=P\{\mu>y-a\mid Y=y,\mu\neq 0\}P\{\mu\neq 0\mid Y=y\}.$$

Next we shall find lower bounds for each of these probabilities so that their product is greater than $\frac{1}{2}$ using corresponding posterior odds. Consider the following odds ratio for some $c>0$ and such y that $y-c>0$

$$\frac{P\{\mu>y-c\mid Y=y,\mu\neq 0\}}{P\{\mu\leq y-c\mid Y=y,\mu\neq 0\}}=\frac{\int_c^{+\infty}\varphi(u)h(y+u)du}{\int_c^{+\infty}\varphi(u)h(y-u)du}.$$

According to Proposition 2.3, for $y>M$

$$h(y-u)\geq Q_1(-u)h(y)\quad \text{for}\quad u\leq 0,$$
$$h(y-u)\leq Q_2(u)h(y)\quad \text{for}\quad u\geq 0.$$

To estimate the integral in the numerator, we split it in two parts: integral on $u\in(-\infty,0]$ and integral on $(0,c)$. For $u\in(-\infty,0]$, $h(y-u)\geq h(y)Q_1(-u)$, and for $u\in(0,c)$ $h(y-u)\geq h(y)$ since $y>y-u>y-c>0$ and h is a decreasing function on the positive semiline. Combining these two estimates, for $y>M$ we have

$$\int_{-\infty}^{c}\varphi(u)h(y-u)du\geq h(y)\int_{-\infty}^{0}Q_1(-u)\varphi(u)du+h(y)\int_0^c\varphi(u)du\geq h(y)C_1>0,$$

for any $c > 0$, since, according to condition (L3) of Proposition 2.3, the first integral is a positive constant independent of y. The integral in the denominator can be estimated from below by

$$\int_c^{+\infty} \varphi(u)h(y-u)du \geq h(y) \int_c^{+\infty} Q_2(u)\varphi(u)du$$

for $y > M$ since $u > c > 0$. Thus, for $y > M$ the posterior odds can be estimated by

$$\frac{P\{\mu > y-c \mid Y=y, \mu \neq 0\}}{P\{\mu \leq y-c \mid Y=y, \mu \neq 0\}} \geq \frac{C_1}{\int_c^{+\infty} Q_2(u)\varphi(u)du}.$$

Since the integral $\int_c^{+\infty} Q(u)\varphi(u)du$ is finite for any $c > 0$ and decreases to zero as c grows to infinity, there exists a constant $a > 0$ such that

$$\int_a^{+\infty} Q_2(u)\varphi(u)du > C_1/3,$$

which, in its turn, implies that the posterior odds are greater than 3 and the studied probability is greater than 3/4 for $y > M + a$

$$P\{\mu > y-a \mid Y=y, \mu \neq 0\} > \frac{3}{4}.$$

The second odds ratio is

$$\frac{P\{\mu \neq 0 \mid Y=y\}}{P\{\mu = 0 \mid Y=y\}} = \frac{1}{\omega(y)} = \frac{p}{1-p}\frac{\eta(y)}{\varphi(y)}.$$

Since $\eta(y) \asymp h(y)$ and tail of h is heavier than tail of φ, this odds ratio tends to $+\infty$ as $y \to +\infty$, i.e.,

$$\frac{p}{1-p}\frac{\eta(y)}{\varphi(y)} \geq A\frac{p}{1-p}\frac{h(y)}{\varphi(y)} \to +\infty,$$

where $A = \liminf_{x\to+\infty} \eta(x)/h(x) > 0$. Since $h(x)$ and $\varphi(x)$ both decrease for $x > 0$, their ratio can either increase or decrease on $(0, +\infty)$. Since $h(x)/\varphi(x) \to +\infty$ as $x \to +\infty$, it implies that $h(x)/\varphi(x)$ increases to infinity. This, in its turn, implies that there exists $y^* = y^*(p)$ such that for any $y > y^*$

$$\frac{h(y)}{\varphi(y)} > 2\frac{1-p}{Ap}$$

and then the odds ratio

$$\frac{P\{\mu \neq 0 \mid Y=y\}}{P\{\mu = 0 \mid Y=y\}} > 2.$$

Therefore, for $y > y^*$ the probability $P\{\mu \neq 0 \mid Y = y\} > 2/3$. Using the obtained lower bounds for each of the probabilities, we can show that for $y > \max\{y^*, M+a\}$ the probability of interest is greater than 1/2, i.e.,

$$\begin{aligned} P\{\mu > y-a \mid Y=y\} &= P\{\mu > y-a \mid Y=y, \mu \neq 0\}P\{\mu \neq 0 \mid Y=y\} \\ &> \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}, \end{aligned}$$

completing the proof for this case.

Hence, the proof of Theorem 2.1 is completed.

5.5 *Proof of Theorem* 3.2. (*The Laplace-Gaussian case*)

It is not difficult to see that, in the considered case,

$$\frac{2\tilde{H}(\theta \mid y)\eta(y)}{a} = I_{(-\infty,0]}(\theta)\int_{-\infty}^{\theta}\phi_\sigma(x-y-a\sigma^2)\exp\{ay+(a\sigma)^2/2\}dx$$
$$+ I_{(0,+\infty)}(\theta)\Bigg(\exp\{ay+(a\sigma)^2/2\}\Phi(-y/\sigma-a\sigma)$$
$$+\int_0^{\theta}\phi_\sigma(x-y+a\sigma^2)\exp\{-ay+(a\sigma)^2/2\}dx\Bigg)$$
$$= e^{(a\sigma)^2/2}\nu(y)CTG(\theta/\sigma \mid -y/\sigma-a\sigma, y/\sigma-a\sigma),$$

where $\nu(y) = e^{ay}\Phi(-y/\sigma - a\sigma) + e^{-ay}\Phi(y/\sigma - a\sigma)$ and the parameters of the CTG distribution are $p_0 = -y/\sigma - a\sigma$, $p_1 = y/\sigma - a\sigma$ and $\alpha(y) = \frac{e^{-ay}\Phi(y/\sigma-a\sigma)}{\nu(y)}$. Taking the limit $\theta \to +\infty$, it is not difficult to see that the marginal density is expressed as $\eta(y) = \frac{ae^{(a\sigma)^2/2}}{2}\nu(y)$. Therefore, by combining the above facts, it is easily seen that the nonzero part of the posterior distribution is $\tilde{H}(\theta \mid y) = CTG(\theta/\sigma \mid -y/\sigma - a\sigma, y/\sigma - a\sigma)$ while the posterior odds ratio for the component at zero in the mixture is easily seen to be $\omega(y) = \frac{2(1-p)\phi(y/\sigma)}{ap\sigma e^{(a\sigma)^2/2}\nu(y)}$, completing the proof of Part (1) of Theorem 3.2.

To find the explicit formula for the posterior median, we need to explore the function $\tilde{H}^{-1}(\theta \mid y)$ at the point $\theta^* = \frac{1-\text{sign}(y)\omega(y)}{2}$ for $y \in \mathbb{R}$ such that $|\gamma(y)| > 1$, i.e.,

$$\tilde{H}{-}1(\theta^* \mid y) = CTG^{-1}(\theta^* \mid p_0, p_1) = \begin{cases} -p_0 + \Phi^{-1}\{\frac{\theta^*}{1-\alpha(y)}\Phi(p_0)\}, & \theta^* < 1-\alpha(y) \\ p_1 - \Phi^{-1}\{\frac{1-\theta^*}{\alpha(y)}\Phi(p_1)\}, & \text{otherwise.} \end{cases}$$

If $\theta^* < 1 - \alpha(y)$ then $\tilde{H}^{-1}(\theta^* \mid y)$ is negative, and if $\theta^* > 1 - \alpha(y)$ it is positive because for $\beta \in [0,1]$ and $p \in \mathbb{R}$, $p - \Phi^{-1}(\beta\Phi(p)) \geq 0$, and hence $p - \Phi^{-1}\{\beta\Phi(p)\} \geq 0 \Leftrightarrow \Phi(p) \geq \beta\Phi(p) \Leftrightarrow 1 \geq \beta$.

It is easy to show that the condition $1 - \text{sign}(y)\omega(y) < 2(1 - \alpha(y))$ is equivalent to $\text{sign}[\gamma(y)] < \gamma(y)$. Since we are considering the case $|\gamma(y)| > 1$, the above condition is equivalent to $\text{sign}[\gamma(y)] = 1 \Leftrightarrow \text{sign}(y) = -1 \Leftrightarrow y < 0$. Therefore given $|\gamma(y)| > 1$, $1 - \text{sign}(y)\omega(y) < 2(1 - \alpha(y))$ implies $y < 0$.

Calculating the inverse function of the CTG distribution at the point of interest with parameters $p_0 = -y/\sigma - a\sigma$, $p_1 = y/\sigma - a\sigma$, we get

$$\sigma CTG^{-1}\left(\frac{1-\text{sign}(y)\omega(y)}{2} \,\middle|\, -\frac{y}{\sigma} - a\sigma, \frac{y}{\sigma} - a\sigma\right)$$
$$= \text{sign}(y)\left[|y| - a\sigma^2 - \sigma\Phi^{-1}\left(\frac{(1+\omega(y))\nu(y)e^{a|y|}}{2}\right)\right],$$

since $\omega(y)$ and $\nu(y)$ are even functions by Lemma 5.1.

As it was shown above, for $y \in \mathbb{R}$ such that $|\gamma(y)| > 1$, $|y| - a\sigma^2 - \sigma\Phi^{-1}(\frac{1+\omega(y)}{2}\nu(y)e^{a|y|})$ is non-negative. Thus, the required expression (3.7) for the posterior median is easily obtained. It was also shown that the posterior median remains zero for the values of $y \in \mathbb{R}$ such that $|\gamma(y)| < 1$. From Theorem 2.1, if $\gamma(y)\phi_\sigma(y)$ (which is a monotonic antisymmetric function) decreases then the posterior median is zero on a symmetric interval $[-\lambda, \lambda]$, where the value of the threshold $\lambda > 0$ is the positive solution of the equation $\gamma(\lambda) = -1$, i.e., the positive solution of

$$\frac{\Phi(\lambda/\sigma - a\sigma)}{\phi(\lambda/\sigma - a\sigma)} - \frac{\Phi(-\lambda/\sigma - a\sigma)}{\phi(-\lambda/\sigma - a\sigma)} = \frac{2(1-p)}{a\sigma p}.$$

To show that the function $\gamma(y)\phi_\sigma(y)$ decreases we use Remark 2.2(iii), i.e., we show that for each $x > 0$, $\phi_\sigma(y-x) - \phi_\sigma(y+x)$ increases for $y > 0$. Indeed,

$$\phi_\sigma(y-x) - \phi_\sigma(y+x) = \frac{\phi(0)}{\sigma}[e^{-(y-x)^2/2\sigma^2} - e^{-(y+x)^2/2\sigma^2}] = \frac{\phi((y-x)/\sigma)}{\sigma}[1 - e^{-2xy/\sigma^2}].$$

Thus, the posterior median is a thresholding rule with the threshold $\lambda > 0$ defined above, completing the proof of Part (2) of Theorem 3.2.

For $y > 0$, the posterior median is given by

$$m_\theta(y) = y - a\sigma^2 - \sigma\Phi^{-1}\left(\frac{(1+\omega(y))\nu(y)e^{ay}}{2}\right).$$

Using the well-known expressions $\Phi(x) \approx 1 - \frac{\phi(x)}{x}$, as $x \to +\infty$, $\Phi(x) \approx -\frac{\phi(x)}{x}$, as $x \to -\infty$ and $\Phi^{-1}(0.5 + x) \approx \frac{x}{\phi(0)}$, as $x \to 0$, the asymptote of the argument of Φ^{-1} given above is easily seen to be

$$\frac{(1+\omega(y))\nu(y)e^{ay}}{2} = 0.5 + \phi(y/\sigma - a\sigma)\frac{1-p}{ap\sigma}(1+o(1)), \quad \text{as} \quad y \to +\infty,$$

and therefore, by the antisymmetry property of the posterior median, its asymptotes are given by

$$m_\theta(y) = y - \operatorname{sign}(y)a\sigma^2 - \operatorname{sign}(y)\exp(-(|y|/\sigma - a\sigma)^2/2)\frac{1-p}{ap}(1+o(1)), \quad \text{as} \quad y \to \infty,$$

which completes the proof of Part (3) of Theorem 3.2, and hence Theorem 3.2 is proved.

5.6 *Proof of Theorem* 3.3. (*The Gaussian-Laplace case*)

The marginal density $\eta(y)$ in the considered case coincides with the marginal density in the Laplace-Gaussian case obtained in Subsection 5.5. Using this result, it is not difficult now to see that

$$\begin{aligned}\tilde{H}(\theta \mid y) &= \frac{1}{\eta(y)}\int_{y-\theta}^{+\infty} l_a(x)h(y-x)dx \\ &= \frac{ae^{(a\sigma)^2/2}}{2\eta(y)}((\Phi(-y/\sigma - a\sigma) - \Phi(-\theta/\sigma - a\sigma))e^{ay}\mathbf{1}_{(y,+\infty)}(\theta) \\ &\qquad + e^{-ay}(1 - \Phi(-\min(\theta, y)/\sigma + a\sigma))) \\ &= CTN(\theta \mid y, a\sigma^2, -a\sigma^2, \sigma),\end{aligned}$$

where $\beta(y) = \frac{e^{-ay}\Phi(y/\sigma - a\sigma)}{e^{ay}\Phi(-y/\sigma - a\sigma) + e^{-ay}\Phi(y/\sigma - a\sigma)}$. The posterior odds ratio for the component at zero in the mixture is easily seen to be $\omega(y) = \frac{(1-p)e^{-a|y|}}{pe^{(a\sigma)^2/2}\{e^{ay}\Phi(-y/\sigma - a\sigma) + e^{-ay}\Phi(y/\sigma - a\sigma)\}}$, completing the proof of Part (1) of Theorem 3.3.

From Proposition 2.2 we know that for $y > \lambda$ the posterior median is obtained by $m_\theta(y) = \tilde{H}^{-1}(\frac{1-\omega(y)}{2} \mid y)$, which in this case is expressed as $2\beta(y)TN_-(m_\theta(y)) = 1-\omega(y)$, since we know that $0 \le m_\theta(y) \le y$. The former equation is equivalent to

$$\Phi(m_\theta(y)/\sigma - a\sigma) = \frac{1}{2}\Phi(y/\sigma - a\sigma) + \frac{1}{2}e^{2ay}\Phi(-y/\sigma - a\sigma) - \frac{1-p}{p}e^{-(a\sigma)^2/2},$$

from which easily follows formula (3.11) taking into account the antisymmetry property of the posterior median. The equation for the threshold $\lambda > 0$ is $\frac{1}{\omega(y)}(2CTN(0 \mid \lambda, a\sigma^2, -a\sigma^2, \sigma) - 1) = -1$ which can be expressed as

$$\frac{1-p}{p}e^{-(a\sigma)^2/2} = e^{2a\lambda}\Phi(-\lambda/\sigma - a\sigma) + \Phi(\lambda/\sigma - a\sigma) - 2\Phi(-a\sigma).$$

If the parameters are such that $|\gamma(y)| \le 1$ for all $y \in \mathbb{R}$, the equation above does not have a final solution and the posterior median is identically zero. This condition is equivalent to $\frac{1-p}{p} \ge (2\Phi(a\sigma) - 1)e^{(a\sigma)^2/2}$. This completes the proof of Part (2) of Theorem 3.3.

The asymptotes (3.13) for the posterior median in this case are immediately seen, and hence Theorem 3.3 is proved.

5.7 *Proof of Theorems* 3.4 *and* 3.5. (*The Laplace-Laplace case*)

We start with case of unequal parameters. The marginal density $\eta(y)$ in this case is given by $\eta(y) = \varphi \star h(y) = \frac{\nu\mu}{4}\int_{-\infty}^{+\infty}\exp\{-\mu|y-u| - \nu|u|\}du$. Denote $p_0 = \min(0, y)$ and $p_1 = \max(0, y)$. Since the integrand has two points of discontinuity, 0 and y, calculating the integral separately on the three intervals $(-\infty, p_0)$, (p_0, p_1) and $(p_1, +\infty)$, and after some simple algebra, we get the expression $\eta(y) = \frac{\nu\mu}{2}\frac{\nu e^{-\mu|y|} - \mu e^{-\nu|y|}}{\nu^2-\mu^2}$.

Now we work out the posterior distribution function. It is not difficult to see that

$$\begin{aligned}
\tilde{H}(\theta \mid y)\eta(y) &= \frac{\nu\mu}{4}\int_{-\infty}^{\theta}\exp\{-\mu|y-u| - \nu|u|\}du \\
&= \frac{\nu\mu}{4}\bigg(\exp\{-\mu y\}\frac{\exp\{(\nu+\mu)\min(\theta, p_0)\}}{\nu+\mu} \\
&\quad + \mathbf{1}(\theta \ge p_1)\exp\{\mu y\}\frac{\exp\{-(\nu+\mu)p_1\} - \exp\{-(\nu+\mu)\theta\}}{\nu+\mu} \\
&\quad + \mathbf{1}(\theta \ge p_0)\exp\{-\operatorname{sign}(y)\mu y\} \\
&\quad \times \frac{e^{-\operatorname{sign}(y)(\nu-\mu)\min(\theta,p_1)} - e^{-\operatorname{sign}(y)(\nu-\mu)p_0}}{-\operatorname{sign}(y)(\nu-\mu)}\bigg).
\end{aligned}$$

Using the expression for $\eta(y)$ given above, we can now rewrite the posterior distribution function in terms of the Truncated Laplace distribution, i.e.,

$$\begin{aligned}
\tilde{H}(\theta \mid y) &= \frac{\nu-\mu}{2}\frac{\exp\{-\mu\max(0,y) + \nu\min(0,y)\}}{\nu e^{-\mu|y|} - \mu e^{-\nu|y|}}TL(\theta \mid \nu+\mu, -\infty, \min(0,y)) \\
&\quad + \frac{\nu-\mu}{2}\frac{\exp\{-\nu\max(0,y) + \mu\min(0,y)\}}{\nu e^{-\mu|y|} - \mu e^{-\nu|y|}}TL(\theta \mid \nu+\mu, \max(0,y), +\infty) \\
&\quad + \frac{\mu+\nu}{2}\frac{\exp\{-\mu|y|\} - \exp\{-\nu|y|\}}{\nu e^{-\mu|y|} - \mu e^{-\nu|y|}} \\
&\quad \times TL(\theta \mid \operatorname{sign}(y)(\nu-\mu), \min(0,y), \max(0,y)).
\end{aligned}$$

The posterior distribution is the CTL distribution with $a = \min(0, y)$, $b = \max(0, y)$, $\lambda_1 = \mu + \nu$, $\lambda_2 = \mathrm{sign}(y)(\nu - \mu)$, $\alpha_1(y) = \frac{\nu-\mu}{2}\frac{\exp\{-\mu\max(0,y)+\nu\min(0,y)\}}{\nu e^{-\mu|y|}-\mu e^{-\nu|y|}}$ and $\alpha_2(y) = \frac{\nu-\mu}{2}\frac{\exp\{-\nu\max(0,y)+\mu\min(0,y)\}}{\nu e^{-\mu|y|}-\mu e^{-\nu|y|}}$. The posterior odds ratio for the component at zero in the mixture is easily seen to be $\omega(y) = \frac{1-p}{p}\frac{1-(\mu/\nu)^2}{1-\frac{\mu}{\nu}e^{-(\nu-\mu)|y|}}$, completing the proof of Part (1) of Theorem 3.4.

The function $\gamma(y)$, after some straightforward calculations, is expressed as $\gamma(y) = \mathrm{sign}(y)\frac{p}{1-p}\frac{\mu\nu}{\nu^2-\mu^2}(e^{-(\nu-\mu)|y|} - 1)$. Therefore, the threshold $\lambda > 0$ satisfies the equation $\gamma(\lambda) = -1$ which gives $\lambda = \frac{1}{\mu-\nu}\log(1 + \frac{1-p}{p}\frac{\mu^2-\nu^2}{\nu\mu})$. In case $\nu > \mu$ and $\frac{1-p}{p}\frac{\mu^2-\nu^2}{\nu\mu} + 1 \le 0$, a finite solution of $\gamma(\lambda) = -1$, however, does not exist, as for any $y > 0$ we have $|\gamma(y)| \le 1$. According to the formula for the posterior median given in Proposition 2.2, this implies that the posterior median is zero for any observed value y. Therefore we can rewrite the formula for the threshold $\lambda > 0$ taking this case into account, i.e., $\lambda = \frac{1}{\mu-\nu}\log(\max[0, 1 + \frac{1-p}{p}\frac{\mu^2-\nu^2}{\nu\mu}])$.

We find now the posterior median in the considered case. For $y > \lambda$, the equation for the posterior median $m_\theta(y)$ is given by $\tilde{H}(m_\theta(y) \mid y) = 0.5(1-\omega(y))$. Since we know that $0 \le m_\theta(y) \le y$, the equation for the posterior median can be rewritten as

$$\frac{1-p}{p}\nu e^{-\mu y}(1-(\mu/\nu)^2) = \mu(e^{-\mu y} - e^{-\nu y}) - (\mu+\nu)(e^{-\mu y} - e^{-\nu y})TL(m_\theta(y) \mid \nu-\mu, 0, y)$$

which, after some simple algebra, gives

$$m_\theta(y) = \frac{1}{\mu-\nu}\log\left(\frac{\nu}{\mu+\nu} + \frac{\mu}{\mu+\nu}e^{-(\nu-\mu)y} + \frac{1-p}{\nu p}(\nu-\mu)\right).$$

Since the posterior median is an antisymmetric function, the equation for the posterior median for $|y| > \lambda$ is given by

$$m_\theta(y) = \mathrm{sign}(y)\frac{1}{\mu-\nu}\log\left(\frac{\mu}{\mu+\nu}e^{-(\nu-\mu)|y|} + \frac{\nu}{\mu+\nu} + \frac{1-p}{p}\frac{\nu-\mu}{\nu}\right),$$

completing the proof of Part (2) of Theorem 3.4.

For the asymptotic behavior of the posterior median as $y \to \infty$ we consider the following two cases. For $\nu < \mu$, we have

$$\begin{aligned} m_\theta(y) &= y - \mathrm{sign}(y)\frac{1}{\mu-\nu}\log\left(1+\frac{\nu}{\mu}\right) \\ &\quad + \mathrm{sign}(y)\frac{1}{\mu-\nu}\frac{\nu}{\mu}e^{-(\mu-\nu)|y|}\left(1+\frac{1-p}{p}\frac{\nu^2-\mu^2}{\nu^2}\right)(1+o(1)), \end{aligned}$$

while, for $\nu > \mu$ and $p > \frac{\nu^2-\mu^2}{\nu\mu+\nu^2-\mu^2}$, we get

$$\begin{aligned} m_\theta(y) &= \mathrm{sign}(y)\frac{1}{\mu-\nu}\log\left(\frac{\nu}{\mu+\nu} + \frac{1-p}{p}\frac{\nu-\mu}{\nu}\right) \\ &\quad + \mathrm{sign}(y)\frac{1}{\mu-\nu}\frac{\mu/\nu}{\frac{1-p}{p}\frac{\nu^2-\mu^2}{\nu^2}+1}e^{-(\nu-\mu)|y|}(1+o(1)), \end{aligned}$$

completing the proof of Part (3) of Theorem 3.4, and hence Theorem 3.4 is proved.

We consider now the case of equal parameters. For the marginal density $\eta(y)$ we get

$$\eta(y) = \frac{\nu}{4}(1 + \nu|y|)\exp\{-\nu|y|\},$$

while for the posterior distribution function, denoting again $p_0 = \min(0, y)$ and $p_1 = \max(0, y)$, and after some simple algebra, we obtain

$$\begin{aligned}\tilde{H}(\theta \mid y) &= \frac{\nu^2}{4\eta(y)} \int_{-\infty}^{\theta} \exp\{-\nu|y-u| - \nu|u|\} du \\ &= \frac{1}{2(1+\nu|y|)}[TL(\theta \mid 2\nu, -\infty, p_0) + 2\nu|y|\mathbf{1}(\theta \geq p_0)\text{Uniform}(\theta \mid p_0, p_1) \\ &\qquad + \mathbf{1}(\theta \geq p_1)TL(\theta \mid 2\nu, p_1, +\infty)].\end{aligned}$$

The posterior distribution is the CTL distribution with $a = \min(0, y)$, $b = \max(0, y)$, $\lambda_1 = 2\nu$, $\lambda_2 = 0$, $\alpha_1(y) = \alpha_2(y) = \frac{1}{2(1+\nu|y|)}$. The posterior odds ratio for the component at zero in the mixture is easily seen to be $\omega(y) = \frac{2(1-p)}{p(1+\nu|y|)}$, completing the proof of Part (1) of Theorem 3.5.

The function $\gamma(y)$, after some straightforward calculations, is expressed as $\gamma(y) = -\frac{p}{1-p}\frac{\nu y}{2}$. Therefore, the threshold $\lambda > 0$ satisfies the equation $\gamma(\lambda) = -1$ which gives $\lambda = \frac{2}{\nu}\frac{1-p}{p}$. We now find the posterior median in the considered case. For $y > \lambda$, the equation for the posterior median $m_\theta(y)$ is given by $\tilde{H}(m_\theta(y) \mid y) = 0.5(1 - \omega(y))$. Since we know that $0 \leq m_\theta(y) \leq y$, the equation for the posterior median can be rewritten as $\frac{1}{1+\nu y}[1 + 2\nu m_\theta(y)] - 1 = -\frac{1-p}{p}\frac{2}{1+\nu y}$ which gives $m_\theta(y) = \frac{y}{2} - \frac{1-p}{\nu p}$. Since the posterior median is an antisymmetric function, the equation for the posterior median for $|y| > \lambda$ is given by

$$m_\theta(y) = \frac{y}{2} - \text{sign}(y)\frac{1-p}{\nu p},$$

completing the proof of Part (2) of Theorem 3.5, and hence Theorem 3.5 is proved.

5.8 *Proof of Theorem* 3.6. (*The* t_1-t_3 *case*)

First of all, we need to calculate the integral of the following function (with parameter y)

$$\varphi(u)h(y-u) = \frac{2}{\pi^2\sqrt{3}} \frac{1}{(1+(u-y)^2)(1+u^2/3)^2}.$$

We present the ratio above as the sum of ratios

$$\frac{1}{(1+(u-y)^2)(1+u^2/3)^2} = \frac{A(u-y)+B}{1+(u-y)^2} + \frac{Cu+D}{1+u^2/3} + \frac{Fu+G}{(1+u^2/3)^2},$$

where coefficients A, B, C, D, F and G may depend on parameter y. Multiplying by the common denominator and equating coefficients for each power of u, after some simple algebra, we have the following equations for the coefficients

$$\begin{aligned}&A + 3C = 0 \\ &B - yA + 3D - 6yC = 0\end{aligned}$$

$$\begin{aligned}
&3C + 2yD + 2yG - (y^2+1)(C+F) = 0\\
&6C + 2yD - (y^2+4)C - 3F = 0\\
&2B - 2yA + (y^2+4)D - 6yC + 3G - 6yF = 0\\
&B - yA + (y^2+1)(D+G) = 1.
\end{aligned}$$

The solution of this system finds the expressions $B = B(y)$, $C = C(y)$, $D = D(y)$, $F = F(y)$ and $G = G(y)$ given in Subsection 3.5. Therefore the marginal density $\eta(y)$ is given by

$$\eta(y) = \int_{\mathbb{R}} \varphi(u)h(y-u)du = \frac{2}{\pi\sqrt{3}}(B + \sqrt{3}(D + G/2)),$$

while the posterior distribution function is given by

$$\begin{aligned}
\tilde{H}(\theta \mid y) &= \frac{1}{\eta(y)}\int_{y-\theta}^{+\infty} \varphi(u)h(y-u)du\\
&= \frac{1}{B+\sqrt{3}(D+G/2)}\frac{1}{2\pi}\left(3C\log\frac{1+\theta^2}{3+(y-\theta)^2} + B(\pi + 2\arctan\theta)\right.\\
&\qquad + \sqrt{3}(D+G/2)\left(\pi + 2\arctan\frac{\theta-y}{\sqrt{3}}\right)\\
&\qquad \left. + \frac{G(\theta-y)+3F}{1+(y-\theta)^2/3}\right).
\end{aligned}$$

Substituting the values of the coefficients $B = B(y)$, $C = C(y)$, $D = D(y)$, $F = F(y)$ and $G = G(y)$ into the equations for $\eta(y)$ and $\tilde{H}(\theta \mid y)$, we get the following

$$\eta(y) = \frac{1}{\pi}\frac{y^6 + 6(4+\sqrt{3})y^4 + 36y^2 + 8(3\sqrt{3}-4)}{9y^8 + 48y^6 + 38y^4 - 52y^2 + 48},$$

and

$$\begin{aligned}
\tilde{H}(\theta \mid y) &= \frac{1}{y^6 + 6(4+\sqrt{3})y^4 + 36y^2 + 8(3\sqrt{3}-4)}\frac{1}{\pi}\\
&\times\left(12\sqrt{3}y(y^2+2)\log\frac{1+\theta^2}{3+(y-\theta)^2}\right.\\
&\quad + 6\sqrt{3}(y^4+4)\left(\frac{\pi}{2} + \arctan\theta\right)\\
&\quad + (y^6 + 24y^4 + 36y^2 - 32)\left(\frac{\pi}{2} + \arctan\left(\frac{\theta-y}{\sqrt{3}}\right)\right)\\
&\quad \left. + \frac{1}{\sqrt{3}}\frac{(y^6+6y^4-12y^2-8)\theta - y(y^6-60y^2-32)}{1+(y-\theta)^2/3}\right).
\end{aligned}$$

The posterior odds ratio for the component at zero in the mixture is easily seen to be

$$\omega(y) = \frac{(1-p)}{p(B(y) + \sqrt{3}(D(y) + G(y)/2))(1+y^2/3)^2},$$

completing the proof of Part (1) of Theorem 3.6.

Expressions (3.23) and (3.24) follow easily now by substituting the found expressions for $\tilde{H}(\theta \mid y)$, $\eta(y)$ and $\omega(y)$ into the equations $\tilde{H}(m_\theta(y) \mid y) = 0.5(1-\omega(y))$ for $y > \lambda$ and

$2\tilde{H}(0 \mid \lambda) - 1 = -\omega(\lambda)$ for the posterior median $m_\theta(y)$ and threshold $\lambda > 0$ respectively, proving Part (2) of Theorem 3.6

To work out the asymptotic behavior of the posterior median, we consider the equation $\tilde{H}(m_\theta(y) \mid y) = 0.5(1 - \omega(y))$ for $y \to +\infty$. It is not difficult to see that, in this case, we have

$$\begin{aligned}
&\frac{1}{y^6 + 6(4+\sqrt{3})y^4 + 36y^2 + 8(3\sqrt{3}-4)} \frac{1}{\pi} \\
&\quad \times \Bigg(12\sqrt{3}y(y^2+2) \log \frac{1 + m_\theta(y)^2}{3 + (y - m_\theta(y))^2} \\
&\qquad + 6\sqrt{3}(y^4+4)\arctan\theta + (y^6 + 24y^4 + 36y^2 - 32)\arctan \frac{m_\theta(y) - y}{\sqrt{3}} \\
&\qquad + \frac{1}{\sqrt{3}}((y^6 + 6y^4 - 12y^2 - 8)\theta - y(y^6 - 60y^2 - 32))\frac{1}{1 + (y - m_\theta(y))^2/3} \Bigg) \\
&= -\frac{1-p}{p} \frac{(9y^8 + 48y^6 + 38y^4 - 52y^2 + 48)}{\sqrt{3}(1 + y^2/3)^2(y^6 + 6(4+\sqrt{3})y^4 + 36y^2 + 8(3\sqrt{3}-4))}.
\end{aligned}$$

Take now the leading order of each component, i.e.,

$$\begin{aligned}
&\frac{1}{\pi}\Bigg(12\sqrt{3}y^{-3} \log \frac{1 + m_\theta(y)^2}{3 + (y - m_\theta(y))^2} + 6\sqrt{3}y^{-2} \arctan m_\theta(y) \\
&\qquad + \arctan\left(\frac{m_\theta(y) - y}{\sqrt{3}}\right) + \frac{(m_\theta(y) - y)/\sqrt{3}}{1 + (y - m_\theta(y))^2/3}\Bigg) \\
&= -\frac{1-p}{p} 27\sqrt{3}y^{-2}(1 + o(1)).
\end{aligned}$$

There are two possibilities: $m_\theta(y) \to +\infty$ or $m_\theta(y) \to$ constant, as $y \to +\infty$. Since in the latter case we get a contradiction ($\arctan(\frac{c-y}{\sqrt{3}}) \to 0$), the former case must be true, i.e.,

$$\arctan\left(\frac{m_\theta(y) - y}{\sqrt{3}}\right) + \frac{(m_\theta(y) - y)/\sqrt{3}}{1 + (y - m_\theta(y))^2/3} \to 0,$$

which implies that $m_\theta(y) - y \to 0$. Denote now $f(y) = y - \theta(y)$. Then

$$\begin{aligned}
&6y^{-2}\sqrt{3}\arctan(y - f(y)) - \arctan \frac{f(y)}{\sqrt{3}} - \frac{1}{\sqrt{3}} f(y) \frac{1}{1 + f(y)^2/3} \\
&= -\pi \frac{1-p}{p} 27\sqrt{3}y^{-2}(1 + o(1)),
\end{aligned}$$

implying that $f(y) = \frac{3}{y}(1 + o(1))$. Therefore the asymptotic behavior of the posterior median is given by

$$m_\theta(y) = y - \frac{3}{y}(1 + o(1)), \qquad \text{as} \quad y \to \infty,$$

completing the proof of Part (3) of Theorem 3.6, and hence Theorem 3.6 is proved.

5.9 *Proof of Theorem* 3.7. (*The (scaled)* t_1-t_1 *case*)
The marginal density $\eta(y)$ in this case is given by

$$\eta(y) = \int_{\mathbb{R}} \varphi(y-u)h(u)du = \frac{\tau\sigma}{\pi^2}\int_{\mathbb{R}} \frac{du}{(\sigma^2+(u-y)^2)(\tau^2+u^2)}.$$

To calculate the above integral, we find the coefficients (which may depend on y) in the following expansion

$$\frac{1}{(\sigma^2+(u-y)^2)(\tau^2+u^2)} = \frac{A(u-y)+B}{\sigma^2+(u-y)^2} + \frac{Cu+D}{\tau^2+u^2}.$$

Multiplying by the common denominator and equating coefficients for each power of u, after some simple algebra, we have the following equations for the coefficients

$$\begin{aligned}
&A+C=0\\
&B-Ay-2Cy+D=0\\
&A\tau^2+C(y^2+\sigma^2)-2Dy=0\\
&(B-Ay)\tau^2+D(y^2+\sigma^2)=1.
\end{aligned}$$

The solution of this system finds the expressions $B=B(y)$, $C=C(y)$ and $D=D(y)$ given in Subsection 3.6. Therefore, after some simple algebra, the marginal density $\eta(y)$ is expressed as

$$\eta(y) = \frac{1}{\pi}\frac{(\tau+\sigma)(y^2+(\tau-\sigma)^2)}{4y^2\tau^2+(y^2+\sigma^2-\tau^2)^2},$$

and the posterior distribution function is given by

$$\begin{aligned}
\tilde{H}(\theta \mid y) = \frac{1}{\pi(B(y)\tau+D(y)\sigma)}\Bigg(&\frac{\tau\sigma}{2}C(y)\log\left(\frac{\theta^2+\tau^2}{(\theta-y)^2+\sigma^2}\right)\\
&+B(y)\tau\left(\frac{\pi}{2}+\arctan\left(\frac{\theta-y}{\sigma}\right)\right)\\
&+D(y)\sigma\left(\frac{\pi}{2}+\arctan\left(\frac{\theta}{\tau}\right)\right)\Bigg).
\end{aligned}$$

The posterior odds ratio for the component at zero in the mixture is easily seen to be

$$\omega(y) = \frac{(1-p)\sigma}{p(\sigma^2+y^2)(B(y)\tau+D(y)\sigma)},$$

thus completing the proof of Part (1) of Theorem 3.7.

The threshold $\lambda>0$ satisfies the equation $\gamma(\lambda)=-1$ which gives

$$\tau\sigma C(\lambda)\log\left(\frac{\lambda^2+\sigma^2}{\tau^2}\right) + 2B(\lambda)\tau\arctan\left(\frac{\lambda}{\sigma}\right) = \frac{1-p}{p}\frac{\pi\sigma}{\sigma^2+\lambda^2}.$$

If we take the limit $y\to+\infty$ in $|\gamma(y)|\le 1$ (which implies that the posterior median is identically zero) we get $\frac{1-p}{p}\ge\frac{\tau}{\sigma}$. Thus a finite solution exists if and only if $p>\frac{\sigma}{\tau+\sigma}$, otherwise the posterior median is zero for all $y\in\mathbb{R}$.

We now find the posterior median in the case $p > \frac{\sigma}{\tau+\sigma}$. For $y > \lambda$, the equation for the posterior median $m_\theta(y)$ is given by $\tilde{H}(m_\theta(y) \mid y) = 0.5(1-\omega(y))$. The equation for the posterior median θ is now expressed as

$$2\tau B(y)\arctan\left(\frac{m_\theta(y)-y}{\sigma}\right) + 2\sigma D(y)\arctan\left(\frac{m_\theta(y)}{\tau}\right) + \sigma\tau C(y)\log\left(\frac{m_\theta(y)^2+\tau^2}{(y-m_\theta(y))^2+\sigma^2}\right) = -\frac{(1-p)\pi\sigma}{p(y^2+\sigma^2)}. \tag{5.1}$$

We now study the asymptotic behavior of the posterior median as the observed value y goes to $+\infty$. Taking the leading terms in coefficients $B(y)$, $C(y)$ and $D(y)$ in the equation for the posterior median, we have

$$\frac{2\tau}{y^2}\arctan\left(\frac{m_\theta(y)-y}{\sigma}\right) + \frac{2\sigma}{y^2}\arctan\left(\frac{m_\theta(y)}{\tau}\right) + \frac{2\sigma\tau}{y^3}\log\left(\frac{m_\theta(y)^2+\tau^2}{(y-m_\theta(y))^2+\sigma^2}\right) \approx -\frac{(1-p)\pi\sigma}{py^2}.$$

The absolute value of the term with the logarithm is bounded by $O(2\log y/y^3) = o(1/y^2)$, transforming the above equation into

$$\tau\arctan\left(\frac{m_\theta(y)-y}{\sigma}\right) + \sigma\arctan\left(\frac{m_\theta(y)}{\tau}\right) \approx -\frac{(1-p)\pi\sigma}{2p}. \tag{5.2}$$

If there exists a limit of $m_\theta(y)$ as $y \to +\infty$ then there are three possibilities

1. $m_\theta(y) \to y - c$, $c = \text{constant} \geq 0$;
2. $m_\theta(y) \to c$, $c = \text{constant} \geq 0$;
3. $m_\theta(y) \to \infty$ and $y - m_\theta(y) \to \infty$.

We now take the limit as $y \to +\infty$ in the equation (5.2) in each of these cases.

1. For $m_\theta(y) \to y - c$, $c = \text{constant} \geq 0$, we have

$$-\tau\arctan\left(\frac{c}{\sigma}\right) + \sigma\frac{\pi}{2} = -\frac{\pi\sigma(1-p)}{2p} \Leftrightarrow \arctan\left(\frac{c}{\sigma}\right) = \frac{\pi\sigma}{2\tau p},$$

and which has a solution if and only if $\sigma < \tau$ and $p > \frac{\sigma}{\tau}$. Then $c = \sigma\tan(\frac{\pi\sigma}{2\tau p})$.

2. For $m_\theta(y) \to c$, $c = \text{constant} \geq 0$, we have

$$-\tau\frac{\pi}{2} + \sigma\arctan\left(\frac{c}{\sigma}\right) = -\frac{\pi\sigma(1-p)}{2p} \Leftrightarrow \arctan\left(\frac{c}{\sigma}\right) = \frac{\pi}{2}\left(\frac{\tau}{\sigma} - \frac{1-p}{p}\right),$$

and which has a solution if and only if $0 < \frac{\tau}{\sigma} - \frac{1-p}{p} < 1$ which is equivalent to $\frac{\sigma}{\sigma+\tau} < p < \frac{\tau}{\sigma}$. Then $c = \tau\tan(\frac{\pi\tau}{2\sigma} - \frac{\pi(1-p)}{2p})$. We saw above that if $p \leq \frac{\sigma}{\sigma+\tau}$ the posterior median is identically zero.

3. For $m_\theta(y) \to \infty$ and $y - m_\theta(y) \to \infty$, we have

$$\frac{\pi(\sigma-\tau)}{2} = -\frac{\pi\sigma(1-p)}{2p},$$

which implies that $\sigma < \tau$ and $p = \frac{\sigma}{\tau}$. Working out the approximation for the posterior median in this case, using the approximation $\arctan(x) = \frac{\pi}{2} - \frac{1}{x} + o(\frac{1}{x^2})$ as $x \to +\infty$, we have

$$-\frac{2\tau}{y^2}\left(\frac{\pi}{2} - \frac{\sigma}{y - m_\theta(y)}\right) + \frac{2\sigma}{y^2}\left(\frac{\pi}{2} - \frac{\tau}{m_\theta(y)}\right) + \frac{2\sigma\tau}{y^3}\log\left(\frac{m_\theta(y)^2 + \tau^2}{(y - m_\theta(y))^2 + \sigma^2}\right)$$
$$\approx -\frac{(\tau - \sigma)\pi}{y^2}.$$

We now consider two cases: $m_\theta(y) = cy + d$ and $m_\theta(y) = o(y)$, where $c \in (0,1)$. In the latter case, we have

$$\frac{2\tau\sigma}{y^3} - \frac{2\sigma\tau}{y^2 m_\theta(y)} + \frac{4\sigma\tau}{y^3}\log\left(\frac{m_\theta(y)}{y}\right) \approx 0, \quad \text{that is} \quad \frac{y}{m_\theta(y)} - 2\log\left(\frac{y}{m_\theta(y)}\right) \approx 1,$$

i.e., that function $\frac{y}{m_\theta(y)}$ and its logarithm have the same asymptotic behavior which is possible if and only if the function is a constant which does not satisfy the assumed condition $m_\theta(y) = o(y)$. In the first case, the equation for c is

$$\frac{1}{1-c} - \frac{1}{c} + 2\log\left(\frac{c}{1-c}\right) = 0,$$

which is satisfied with $c = \frac{1}{2}$. Expanding all terms in equation (5.1) up to order of $\frac{1}{y^4}$, we get

$$-\frac{2\tau}{y^2}\left(-\frac{\pi(\tau^2 + 3\sigma^2)}{2y^2} + \frac{4d\sigma}{y^2}\right) + \frac{2\sigma}{y^2}\left(-\frac{\pi(3\tau^2 + \sigma^2)}{2y^2} + \frac{4d\tau}{y^2}\right)$$
$$+ \frac{16d\sigma\tau}{y^4} + o\left(\frac{1}{y^4}\right) = -\frac{\pi\sigma^2(\sigma - \tau)}{y^4}.$$

Rearranging the terms, we have $d = -\frac{\pi}{32}\frac{(\tau-\sigma)(\tau-2\sigma)}{\sigma}$, and therefore the asymptote for the posterior median in case $p = \frac{\sigma}{\tau}$ is given by

$$m_\theta(y) = \frac{y}{2} - \frac{\pi}{32}\frac{(\tau - \sigma)(\tau - 2\sigma)}{\sigma} + o(1).$$

Hence, the proof of Part (3) of Theorem 3.7 is completed.

We now study whether the posterior median is an increasing function. In this case, assumption (A2), which assures the monotonicity of the posterior median, does not hold so we need another way of proving or disproving that the posterior median is an increasing function. We take the equation for the posterior median (3.28) and differentiate it with respect to y, rearranging the terms:

$$\tau\sigma C'(y)\log\frac{m_\theta^2 + \tau^2}{(m_\theta - y)^2 + \sigma^2} - 2\tau B'(y)\arctan\left(\frac{y - m_\theta(y)}{\sigma}\right)$$
$$+ 2\sigma D'(y)\arctan\left(\frac{m_\theta(y)}{\tau}\right) - \frac{2\pi\sigma y(1-p)}{p(y^2 + \sigma^2)^2}$$
$$- \frac{2(y - m_\theta(y))}{(y - m_\theta(y))^2 + \sigma^2}\tau\sigma C(y) - 2\tau B(y)\frac{\sigma}{\sigma^2 + (m_\theta(y) - y)^2}$$
$$= -\frac{2\tau\sigma m_\theta'(y)}{(m_\theta^2(y) + \tau^2)((m_\theta(y) - y)^2 + \sigma^2)}.$$

We now consider the limit of this equation as $y \to +\infty$ in each of the three cases.

1. For $m_\theta(y) \to y - c$, $c =$ constant ≥ 0, and taking the leading behavior of each term, we have

$$-\frac{6\tau\sigma}{y^4}\log y^2 + \frac{4\tau}{y^3}\arctan\left(\frac{c}{\sigma}\right) - \frac{2\sigma\pi}{y^3} - \frac{2\pi\sigma(1-p)}{py^3} - \frac{4c\tau\sigma}{y^3(c^2+\sigma^2)} - \frac{2\tau\sigma}{y^2(\sigma^2+c^2)}$$
$$\approx -\frac{2\tau\sigma m_\theta'(y)}{y^2(c^2+\sigma^2)}.$$

The leading term of the left hand side is $-\frac{2\tau\sigma}{y^2(\sigma^2+c^2)}$ which makes the derivative $m_\theta'(y)$ positive for large y, and thus the posterior median increases for large y.

2. For $m_\theta(y) \to c$, $c =$ constant ≥ 0, which holds if and only if $p < \frac{\sigma}{\tau}$, we have

$$\frac{6\tau\sigma}{y^4}\log y^2 - \frac{2\tau\sigma}{y^4} - \frac{4\sigma}{y^3}\arctan\left(\frac{c}{\tau}\right) - \frac{2\pi\sigma(1-p)}{py^3} - \frac{2\tau\sigma}{y^4} + \frac{2\tau\pi}{y^3} \approx -\frac{2\tau\sigma m_\theta'(y)}{y^2(c^2+\tau^2)},$$

where the constant c is such $\arctan(\frac{c}{\tau}) = \frac{\pi}{2}[\frac{\tau}{\sigma} - \frac{1-p}{p}]$. The leading term on the left hand side is $\frac{\tau\sigma}{y^4}(6\log y^2 - 4)$, which is positive for sufficiently large y making the derivative of the posterior median negative.

3. For $m_\theta(y) \to y/2 + d$ ($\sigma < \tau$ and $p = \frac{\sigma}{\tau}$), the limit as $y \to +\infty$ in this case is given by

$$\frac{6\tau\sigma}{y^4}\log\frac{y^2/4 - yd + d^2 + \sigma^2}{y^2/4 + yd + d^2 + \tau^2} + \frac{4\tau}{y^3}\left(\frac{\pi}{2} - \frac{2\sigma}{y}\right) - \frac{4\sigma}{y^3}\left(\frac{\pi}{2} - \frac{2\tau}{y}\right)$$
$$- \frac{2\pi(\tau-\sigma)}{y^3} - \frac{8\tau\sigma}{y^4} - \frac{8\tau\sigma}{y^4} \approx -\frac{32\tau\sigma m_\theta'(y)}{y^4},$$

which is equivalent to $-\frac{16\tau\sigma}{y^4} \approx -\frac{32\tau\sigma m_\theta'(y)}{y^4}$. Thus the posterior median increases for large values of y.

Thus, in the case $p < \frac{\sigma}{\tau}$ the posterior median decreases and it increases if and only if $\sigma < \tau$ and $p \geq \frac{\sigma}{\tau}$, completing the proof of Part (2) of Theorem 3.7, and hence Theorem 3.7 is proved.

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