Supplementary Material to "Testing the Equality of Spectral Density Operators for Functional Processes" by A. Leucht, E. Paparoditis, D. Rademacher and T. Sapatinas.

This Supplementary Material contains some technical tools, some useful results on Hilbertian linear process, spectral density, periodogram and tensor operators as well as the proofs of Proposition 1, Lemma 2 and Lemma 3 of the main document.

1. Auxiliary results

1.1. Technical tools.

Definition 1.1 (Hilbert-Schmidt-Tensorproduct)**.** Let $\mathcal{H}$ be a (separable) Hilbert space, then for each $u, v \in \mathcal{H}$, the equation

$$u \otimes v(x) := \langle x, v \rangle u, \quad x \in \mathcal{H} \tag{1.1}$$

defines a Hilbert-Schmidt operator $u \otimes v \in HS(\mathcal{H}, \mathcal{H}) =: HS(\mathcal{H})$, the space of Hilbert Schmidt operators from $\mathcal{H}$ to $\mathcal{H}$.

Let $\mathcal{L}(\mathcal{H})$ be the set of all linear bounded operators $A : \mathcal{H} \to \mathcal{H}$. For A and B operators AB denotes the composition $AB(x) = A(B(x))$ for any $x \in \mathcal{H}$.

Remark 1.1. (a) If $\{\varphi_n \mid n \in \mathbb{N}\}$ is an orthonormal basis of $\mathcal{H}$, then $\{\varphi_n \otimes \varphi_m \mid n, m \in \mathbb{N}\}$ is an orthonormal basis of $HS(\mathcal{H})$.
(b) Since the set of Hilbert-Schmidt operators $HS(\mathcal{H})$ is a Hilbert space itself with inner product

$$\langle A, B \rangle_{HS} := \sum_{i \in \mathbb{N}} \langle A(\varphi_i), B(\varphi_i) \rangle = \operatorname{tr}(B^* A), \quad A, B \in HS(\mathcal{H}),$$

where $\{\varphi_i \mid i \in \mathbb{N}\}$ is any orthonormal basis of $\mathcal{H}$, equation (1.1) also applies: For $A, B \in HS(\mathcal{H})$ the equation

$$A \otimes B(C) := \langle C, B \rangle_{HS} A = \operatorname{tr}(B^* C) A, \quad C \in HS(\mathcal{H}) \tag{1.2}$$

defines a Hilbert-Schmidt operator $A \otimes B \in HS(HS(\mathcal{H}), HS(\mathcal{H})) = HS(HS(\mathcal{H}))$.
(c) In case $\mathcal{H} = L^2([0,1], \mathbb{C})$ observe that

$$\langle f \otimes g(u), v \rangle = \left\langle \int_0^1 f(\cdot) \overline{g(\sigma)} u(\sigma)\, d\sigma, v \right\rangle,$$

i.e. $f \otimes g$ is an integral operator with kernel $k(\tau, \sigma) = f(\tau)\overline{g(\sigma)}$.

Lemma 1.1. *For $u, v, f, g \in \mathcal{H}$ and $A, B \in \mathcal{L}(\mathcal{H})$ it holds*

(i) $(\alpha u) \otimes (\beta v) = \overline{\alpha}\ \beta (u \otimes v), \quad \alpha, \beta \in \mathbb{C}$
(ii) $(u + v) \otimes (f + g) = u \otimes f + u \otimes g + v \otimes f + v \otimes g$
(iii) $(u \otimes v)^* = v \otimes u$
(iv) $A(u) \otimes B(v) = A(u \otimes v) B^*$
(v) $\langle f \otimes g\ ,\ u \otimes v \rangle_{HS} = \langle f, u \rangle\ \langle v, g \rangle$.

Proof: Follows immediately from the definition. □

Lemma 1.2. *Let $\mathcal{H}$ a separable Hilbert space and $A, B \in HS(\mathcal{H})$, then*

$$\langle A, B\rangle_{HS} = \langle B^*, A^*\rangle_{HS}.$$

In particular $\langle A, B\rangle \in \mathbb{R}$ if A, B are self-adjoint.

Proof: Since $\langle A, B\rangle_{HS} = \text{trace}(B^*A)$ it suffices to show $\text{trace}(AB) = \text{trace}(BA)$. Let $\{\varphi_n \mid n \in \mathbb{N}\}$ be a orthonormal basis of $\mathcal{H}$, then $B(u) = \sum_n \langle B(u), \varphi_n\rangle\varphi_n$, $u \in \mathcal{H}$, and consequently $AB(u) = \sum_n \langle u, B^*(\varphi_n)\rangle A(\varphi_n)$, i.e.

$$AB = \sum_n \langle \cdot, B^*(\varphi_n)\rangle A(\varphi_n)$$

and similarly $BA = \sum_n \langle \cdot, A^*(\varphi_n)\rangle B(\varphi_n)$. It follows

$$\text{trace}(AB) = \sum_m \langle AB(\varphi_m), \varphi_m\rangle = \sum_{m,n} \langle B(\varphi_m), \varphi_n\rangle\langle \varphi_n, A^*(\varphi_m)\rangle$$
$$= \sum_{m,n} \langle \langle \varphi_n, A^*(\varphi_m)\rangle B(\varphi_m), \varphi_n\rangle = \sum_n \langle BA(\varphi_n), \varphi_n\rangle = \text{trace}(BA).$$

□

Definition 1.2 (Operator tensor product)**.** Let $\mathcal{H}$ be a (separable) Hilbert space and $A, B \in \mathcal{L}(\mathcal{H})$, then the equation

$$A \otimes_{op} B(C) := ACB^*, \quad C \in \mathcal{L}(\mathcal{H}) \tag{1.3}$$

defines a linear operator $A \otimes_{op} B \in \mathcal{L}(\mathcal{L}(\mathcal{H}))$.

Remark 1.2. (a) $A \otimes_{op} B|_{HS(\mathcal{H})} : HS(\mathcal{H}) \to HS(\mathcal{H})$, since $\|ACB^*\|_{HS} \leq \|A\|_{\mathcal{L}}\|C\|_{HS}\|B\|_{\mathcal{L}}$.
(b) In case $A, B \in HS(\mathcal{H})$ it holds

$$\|A \otimes_{op} B\|^2_{HS(HS)} = \sum_{i,j} \|A \otimes_{op} B(e_i \otimes_2 e_j)\|^2_{HS} = \sum_{i,j} \|A(e_i) \otimes B(e_j)\|^2_{HS}$$
$$= \sum_{i,j} \|A(e_i)\|^2\|B(e_j)\|^2 = \|A\|^2_{HS}\|B\|^2_{HS} < \infty,$$

so $A \otimes_{op} B \in HS(HS)$. For $A, B, C, D \in HS(\mathcal{H})$ a similar calculation yields

$$\langle A \otimes_{op} B, C \otimes_{op} D\rangle_{HS(HS)} = \sum_{i,j} \langle A(e_i) \otimes B(e_j), C(e_i) \otimes D(e_j)\rangle_{HS}$$
$$= \sum_{i,j} \langle A(e_i), C(e_i)\rangle\langle D(e_j), B(e_j)\rangle$$
$$= \langle A, C\rangle_{HS}\langle D, B\rangle_{HS}.$$

(c) Suppose $\mathcal{H} = L^2([0,1],\mathbb{C})$ and $A, B, C \in HS(L^2([0,1],\mathbb{C}))$ have corresponding kernerls $a, b, c \in L^2([0,1]^2,\mathbb{C})$. Then

$$\langle A \otimes_{op} B(C)(f), g\rangle = \langle ACB^*(f), g\rangle$$
$$= \Big\langle \int_0^1 \Big(\int_0^1 \int_0^1 a(\cdot,\nu)\overline{b(\sigma,\tau)}c(\nu,\tau)\, d\nu\, d\tau\Big) f(\sigma)\, d\sigma, g\Big\rangle$$

so $A \otimes_{op} B|_{HS}$ is an integral operator with corresponding kernel

$$k(\mu,\sigma,\nu,\tau) = a(\mu,\nu)\overline{b(\sigma,\tau)}.$$

Lemma 1.3. *Let $\mathcal{H}$ be a (separable) Hilbert space, $f, g \in \mathcal{H}$, $\alpha, \beta \in \mathbb{C}$ and $A, B, C, D, E, F \in \mathcal{L}(\mathcal{H})$, then*

(i) $\|A \otimes_{op} B\|_{\mathcal{L}(\mathcal{L})} \le \|A\|_{\mathcal{L}}\|B\|_{\mathcal{L}}$
(ii) $(A + B) \otimes_{op} (C + D) = A \otimes_{op} C + A \otimes_{op} D + B \otimes_{op} C + B \otimes_{op} D$
(iii) $(\alpha A) \otimes_{op} (\beta B) = \alpha\overline{\beta} A \otimes_{op} B$
(iv) $A \otimes_{op} B(f \otimes g) = A(f) \otimes B(g)$
(v) $(A \otimes_{op} B)(C \otimes_{op} D) = AC \otimes_{op} BD$

and when restricted on $HS(\mathcal{H})$ and $C, D \in HS(\mathcal{H})$

(vi) $(A \otimes_{op} B)^* = A^* \otimes_{op} B^*$
(vii) $A \otimes_{op} B\ C \otimes D\ (E \otimes_{op} F)^* = \Big(A \otimes_{op} B(C)\Big) \otimes \Big(E \otimes_{op} F(D)\Big)$

Proof: Follows immediately from the definition. For (vi) notice that by lemma 1.2

$$\langle A \otimes_{op} B(C), D\rangle = \langle ACB^*, D\rangle = \langle CB^*, A^*D\rangle = \langle D^*A, BC^*\rangle$$
$$=\langle B^*D^*A, C^*\rangle = \langle C, A^*DB\rangle = \langle C, A^* \otimes_{op} B^*(D)\rangle$$

for $C, D \in HS(\mathcal{H})$. □

Remark 1.3. From (v) and (vi) we have in particular

$$A \otimes_{op} B\ C \otimes_{op} D\ (E \otimes_{op} F)^* = \Big(A \otimes_{op} E(C)\Big) \otimes_{op} \Big(B \otimes_{op} F(D)\Big)$$

Theorem 1.1 (Complexification of a real Hilbert space, cf. Kardison& Ringrose, Exercise 2.8.3 (p. 161))**.** *Suppose $\mathcal{H}_0$ is a real Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}_0} \colon \mathcal{H}_0 \times \mathcal{H}_0 \to \mathbb{R}$. Define on $\mathcal{H}_0 \times \mathcal{H}_0$ the following algebraic structure*

(a) $+\colon (\mathcal{H}_0\times\mathcal{H}_0)\times(\mathcal{H}_0\times\mathcal{H}_0) \to \mathcal{H}_0\times\mathcal{H}_0, \quad (u_0,u_1)+(v_0,v_1) := (u_0+v_0, u_1+v_1);$
(b) $\cdot\colon \mathbb{C}\times(\mathcal{H}_0\times\mathcal{H}_0) \to (\mathcal{H}_0\times\mathcal{H}_0), \quad (\alpha_0+i\alpha_1)\cdot(u_0,u_1) := (\alpha_0u_0-\alpha_1u_1,\ \alpha_1u_0+\alpha_0u_1);$
(c) $\langle\cdot,\cdot\rangle\colon (\mathcal{H}_0\times\mathcal{H}_0)\times(\mathcal{H}_0\times\mathcal{H}_0) \to \mathbb{C},$
$\Big\langle(u_0,u_1),(v_0,v_1)\Big\rangle := \langle u_0,v_0\rangle_{\mathcal{H}_0} + \langle u_1,v_1\rangle_{\mathcal{H}_0} + i\langle u_1,v_0\rangle_{\mathcal{H}_0} - i\langle u_0,v_1\rangle_{\mathcal{H}_0};$
(d) $\|\cdot\|\colon (\mathcal{H}_0\times\mathcal{H}_0) \to [0,\infty), \quad \|(u_0,u_1)\|^2 = \|u_0\|^2_{\mathcal{H}_0} + \|u_1\|^2_{\mathcal{H}_0},$

then $\mathcal{H}_0 \times \mathcal{H}_0$ becomes a complex Hilbert space, denoted $\mathcal{H}$. There is a natural involution on $\mathcal{H}$ defined by

(e) $^{-}\colon (\mathcal{H}_0) \times (\mathcal{H}_0) \to (\mathcal{H}_0) \times (\mathcal{H}_0), \quad \overline{(u_0, u_1)} := (u_0, -u_1)$.

The set $\mathcal{H}_{\mathbb{R}} := \{(u, 0) \mid u \in \mathcal{H}_0\} \subset \mathcal{H}$ *is a closed real-linear subspace of* $\mathcal{H}$ *and*

$$\mathcal{H} = \mathcal{H}_{\mathbb{R}} + i\mathcal{H}_{\mathbb{R}} = \{h + ik \mid h, k \in \mathcal{H}_{\mathbb{R}}\}.$$

The mapping $\mathcal{H}_0 \ni u \mapsto (u, 0) \in \mathcal{H}_{\mathbb{R}}$ *is an isometric isomorphism, i.e.* $\|u\|_{\mathcal{H}_0} = \|(u, 0)\|$.

Remark 1.4. (a) Notice that for $\alpha \in \mathbb{C}$ and $u = (u_0, u_1), v = (v_0, v_1) \in \mathcal{H}$ we have

$$\overline{\alpha u} = \overline{\alpha} \cdot \overline{u}, \quad \overline{u + v} = \overline{u} + \overline{v}, \quad \overline{\overline{u}} = u, \quad \langle \overline{u}, \overline{v} \rangle = \overline{\langle u, v \rangle}.$$

Also $u = \overline{u}$ if and only if $u \in \mathcal{H}_0$.
(b) Any operator $A \in \mathcal{L}(\mathcal{H}_0)$ can be extended to an operator $A_c \in \mathcal{L}(\mathcal{H})$ by defining $A_c(u) := (A(u_0), A(u_1))$ for $u = (u_0, u_1) \in \mathcal{H}$. The correspondence $A \mapsto A_c$ preserves all algebraic properties of the operator. For example

(i) $(\alpha A)_c = \alpha A_c$ for $\alpha \in \mathbb{R}$
(ii) $(A + B)_c = A_c + B_c$
(iii) $(AB)_c = A_c B_c$
(iv) $(A^*)_c = A_c^*$

(c) If $A \in \mathcal{L}(\mathcal{H}_0)$ and $\alpha = \alpha_0 + i\alpha_1 \in \mathbb{C}$ we define $\alpha A \in \mathcal{L}(\mathcal{H}_0, \mathcal{H})$ by $\alpha A(u_0) := (\alpha_0 A(u_0), \alpha_1 A(u_0))$. This operator can then be extended to an operator $\alpha A \in \mathcal{L}(\mathcal{H})$ by defining

$$\alpha A(u) := (\alpha_0 A(u_0) - \alpha_1 A(u_1), \alpha_1 A(u_0) + \alpha_0 A(u_1)), \quad u = (u_0, u_1) \in \mathcal{H}.$$

Notice that the same extension is obtained by first extending $A \in \mathcal{L}(\mathcal{H}_0)$ to $A_c \in \mathcal{L}(\mathcal{H})$ and then multiplying by α.
(d) Suppose $\{\varphi_n \mid n \in \mathbb{N}\}$ is a basis of $\mathcal{H}_0$ and $u = \sum_n u_n \varphi_n,\ v = \sum_n v_n \varphi_n \in \mathcal{H}_0$, $u_n, v_n \in \mathbb{R}$. Then

$$(u, v) = (u, 0) + i(v, 0) = \sum_n (u_n + iv_n)(\varphi_n, 0),$$

so $\{(\varphi_n, 0) \mid n \in \mathbb{N}\}$ is also a basis for the complexified Hilbert space $\mathcal{H}$. If $\{\varphi_n \mid n \in \mathbb{N}\}$ is orthonormal, than so is $\{(\varphi_n, 0) \mid n \in \mathbb{N}\}$.

Definition 1.3 (Conjugate and Transposed Operator)**.** Let $\mathcal{H} = \mathcal{H}_0 + i\mathcal{H}_0$ be a complexified Hilbert space and $A \in \mathcal{L}(\mathcal{H})$. Then define the conjugate operator $\overline{A} \in \mathcal{L}(\mathcal{H})$ by

$$\overline{A}(u) := \overline{A(\overline{u})}, \quad u \in \mathcal{H}$$

and the transposed operator $A^T \in \mathcal{L}(\mathcal{H})$ by $A^T := \overline{A}^*$.

Remark 1.5. (a) The definitions are motivated from the finite dimensional case, i.e., $\mathcal{H} = \mathbb{C}^n$. In that case any operator $A \in \mathcal{L}(\mathbb{C}^n)$ is represented by a matrix $A = (a_{ij}) \in C^{n\times n}$ and the conjugated matrix is simply $\overline{A} = (\overline{a}_{ij})$. We then have

$$\overline{A}x = \left(\sum_{j=1}^{n} \overline{a}_{ij} x_j\right)_i = \left(\overline{\sum_{j=1}^{n} a_{ij}\overline{x}_j}\right)_i = \overline{A\overline{x}}$$

and $\overline{A}^* = A^T = \overline{A^*}$.
(b) If $A = A_c \in \mathcal{L}(\mathcal{H})$ is the extension of an operator $A \in \mathcal{L}(\mathcal{H}_0)$, then $\overline{A} = A$ and therefore $A^T = A^*$.

Lemma 1.4. *Let $\mathcal{H} = \mathcal{H}_0 + i\mathcal{H}_0$ be a complexified Hilbert space, $\alpha \in \mathbb{C}$, $f, g \in \mathcal{H}$ and $A, B \in \mathcal{L}(\mathcal{H})$, then*

(i) $\|A^T\|_{\mathcal{L}} = \|\overline{A}\|_{\mathcal{L}} = \|A\|_{\mathcal{L}}$
(ii) $(A+B)^T = A^T + B^T$ *and* $\overline{A+B} = \overline{A} + \overline{B}$
(iii) $(AB)^T = B^T A^T$ *and* $\overline{AB} = \overline{A}\ \overline{B}$
(iv) $(\alpha A)^T = \alpha A^T$ *and* $\overline{\alpha A} = \overline{\alpha}\overline{A}$
(v) $A^T = \overline{A}^* = \overline{A^*}$
(vi) $(A^T)^T = A$ *and* $\overline{\overline{A}} = A$
(vii) $(f\otimes g)^T = \overline{g} \otimes \overline{f}$ *and* $\overline{f\otimes g} = \overline{f} \otimes \overline{g}$

If $A, B \in HS(\mathcal{H})$, then

(viii) $A^T, \overline{A} \in HS(\mathcal{H})$ *and* $\|A\|_{HS} = \|A^T\|_{HS} = \|\overline{A}\|_{HS}$
(ix) $\langle A^T, B^T\rangle_{HS} = \langle A, B\rangle_{HS}$ *and* $\langle \overline{A}, \overline{B}\rangle_{HS} = \overline{\langle A, B\rangle}_{HS}$

Remark 1.6. From (v) and (vi) we have in particular

$$\overline{A}^T = A^* = \overline{A^T}, \quad (A^T)^* = \overline{A} = (A^*)^T.$$

Proof: (i) By remark 1.4 $\|u\| = \|\overline{u}\|$, so

$$\|A^T\|_{\mathcal{L}} = \sup_{\|u\|=1} \|A^T(u)\| = \sup_{\|u\|=1} \|\overline{A}^*(u)\| = \sup_{\|u\|=1} \|\overline{A}(u)\| = \sup_{\|\overline{u}\|=1} \|A(\overline{u})\| = \|A\|_{\mathcal{L}}.$$

(v) For $u, v \in \mathcal{H}$

$$\langle \overline{A}^*(u), v\rangle = \langle u, \overline{A}(v)\rangle = \overline{\langle \overline{u}, A(\overline{v})\rangle} = \overline{\langle A^*\overline{u}, \overline{v}\rangle} = \langle \overline{A^*}(u), v\rangle.$$

(vii) For $u \in \mathcal{H}$

$$(f\otimes g)^T(u) = \overline{g\otimes f(\overline{u})} = \overline{\langle \overline{u}, f\rangle g} = \langle u, \overline{f}\rangle \overline{g} = \overline{g} \otimes \overline{f}(u)$$

and similarly $\overline{f\otimes g}(u) = \overline{f} \otimes \overline{g}(u)$.
(ii), (iii), (iv) and (vi) follow immediatly from the definitions.
(viii) Note that $\{(\varphi_n, 0) \mid n \in \mathbb{N}\}$ is a orthonormal basis for $\mathcal{H}$ if $\{\varphi_n \mid n \in \mathbb{N}\}$ is one for $\mathcal{H}_0$ (cf. remark 1.4 (d)), so that $\|\overline{A}\|^2_{HS} = \sum_n \|\overline{A(\varphi_n, 0)}\|^2 = \|A\|^2_{HS}$ and likewise

$\|A^T\|_{HS} = \|A\|_{HS}$.
(ix) By remark 1.4 (a)

$$\langle \overline{A}, \overline{B} \rangle = \sum_n \langle \overline{A(\varphi_n, 0)}, \overline{B(\varphi_n, 0)} \rangle = \overline{\sum_n \langle A(\varphi_n, 0), B(\varphi_n, 0) \rangle} = \overline{\langle A, B \rangle}$$

and therefore Lemma 1.2 yields $\langle A^T, B^T \rangle = \overline{\langle A^*, B^* \rangle} = \overline{\langle B, A \rangle} = \langle A, B \rangle$. $\square$

Definition 1.4 (Transposed Operator tensor product)**.** Let $\mathcal{H} = \mathcal{H}_0 + i\mathcal{H}_0$ be a complexified Hilbert space and $A, B \in \mathcal{L}(\mathcal{H})$, then the equation

$$A \otimes_{op}^T B(C) := AC^T B^T$$

defines a linear operator $A \otimes_{op}^T B \in \mathcal{L}(\mathcal{L}(\mathcal{H}))$.

Remark 1.7. (a) $A \otimes_{op}^T B\Big|_{HS(\mathcal{H})} : HS(\mathcal{H}) \to HS(\mathcal{H})$ since $\|AC^T B^T\|_{HS} \leq \|A\|_{\mathcal{L}} \|C\|_{HS} \|B\|_{\mathcal{L}}$.
(b) In case $A, B \in HS(\mathcal{H})$ by lemma 1.1 and lemma 1.5

$$\begin{aligned} \|A \otimes_{op}^T B\|_{HS(HS)}^2 &= \sum_{i,j} \|A(\varphi_j) \otimes \overline{B(\varphi_i)}\|_{HS}^2 = \sum_j \|A(\varphi_j)\|^2 \sum_i \|B(\varphi_i)\|^2 \\ &= \|A\|_{HS}^2 \|B\|_{HS}^2, \end{aligned}$$

so $A \otimes_{op}^T B \in HS(HS)$. For $A, B, C, D \in HS(\mathcal{H})$ a similar calculation yields

$$\begin{aligned} &\langle A \otimes_{op}^T B, C \otimes_{op}^T D \rangle_{HS(HS)} = \sum_{i,j} \langle A(\varphi_j) \otimes \overline{B(\varphi_i)}, C(\varphi_j) \otimes \overline{D(\varphi_i)} \rangle_{HS} \\ =& \sum_{i,j} \langle A(\varphi_j), C(\varphi_j) \rangle \langle \overline{D(\varphi_i)}, \overline{B(\varphi_i)} \rangle = \sum_j \langle A(\varphi_j), C(\varphi_j) \rangle \sum_j \overline{\langle D(\varphi_i), B(\varphi_i) \rangle} \\ =& \langle A, C \rangle \langle B, D \rangle \end{aligned}$$

(c) Suppose $\mathcal{H} = L^2([0,1], \mathbb{C})$ and $A, B, C \in HS(L^2([0,1], \mathbb{C}))$ have corresponding kernels $a, b, c \in L^2([0,1]^2, \mathbb{C})$. Then

$$\begin{aligned} &\langle A \otimes_{op}^T B(C)(f), g \rangle = \langle AC^T B^T(f), g \rangle \\ =& \left\langle \int_0^1 \left(\int_0^1 \int_0^1 a(\cdot, \tau) b(\sigma, \nu) c(\nu, \tau) \, d\nu \, d\tau \right) f(\sigma) \, d\sigma, g \right\rangle \end{aligned}$$

so $A \otimes_{op}^T B\Big|_{HS}$ is an integral operator with corresponding kernel

$$k(\mu, \sigma, \nu, \tau) = a(\mu, \tau) b(\sigma, \nu).$$

Lemma 1.5. *Let $\mathcal{H} = \mathcal{H}_0 + i\mathcal{H}_0$ be a complexified Hilbert space, $f, g \in \mathcal{H}$, $\alpha, \beta \in \mathbb{C}$ and $A, B, C, D \in \mathcal{L}(\mathcal{H})$, then*

(i) $\|A \otimes_{op}^T B\|_{\mathcal{L}(\mathcal{L})} \leq \|A\|_{\mathcal{L}} \|B\|_{\mathcal{L}}$
(ii) $(A + B) \otimes_{op}^T (C + D) = A \otimes_{op}^T C + A \otimes_{op}^T D + B \otimes_{op}^T C + B \otimes_{op}^T D$

(iii) $(\alpha A) \otimes_{op}^T (\beta B) = \alpha\beta A \otimes_{op}^T B$
(iv) $A \otimes_{op}^T B(f \otimes g) = A(\overline{g}) \otimes \overline{B(f)}$
(v) $(A \otimes_{op}^T B)(C \otimes_{op}^T D) = AD \otimes_{op} \overline{BC}$
(vi) $(A \otimes_{op} B)(C \otimes_{op}^T D) = AC \otimes_{op}^T \overline{B}D$
(vii) $(A \otimes_{op}^T B)(C \otimes_{op} D) = A\overline{D} \otimes_{op}^T BC$

and when restricted on $HS(\mathcal{H})$

(viii) $(A \otimes_{op}^T B)^* = B^* \otimes_{op}^T A^*$

Proof: Follows immediatly from the definition, Lemma 1.1 and Lemma 1.4. For (viii) notice that by Lemma 1.2

$$\langle A \otimes_{op}^T B(C), D\rangle = \langle AC^T B^T, D\rangle = \langle \overline{C^* B^*}, A^* D\rangle = \overline{\langle C^* B^*, \overline{A^* D}\rangle}$$

$$= \langle A^T \overline{D}, C^* B^*\rangle = \langle BC, \overline{D}^* (A^T)^*\rangle = \langle C, B^* D^T \overline{A}\rangle = \langle C, B^* \otimes_{op}^T A^*(D)\rangle$$

for $C, D \in HS(\mathcal{H})$. □

1.2. **Spectral density operator.** Let $\mathcal{H}_0$ denote a real separable Hilbert space and $\mathcal{H}$ its complexification, i.e. $\mathcal{H} = \mathcal{H}_0 + i\mathcal{H}_0$ (cf. Theorem 1.1). For example, the complexification of $\mathcal{H}_0 = L^2([0,1],\mathbb{R})$ is the space $\mathcal{H} = L^2([0,1],\mathbb{C})$. Suppose $(X_t)_{t\in\mathbb{Z}}$ is a centred stationary process taking values in $\mathcal{H}_0$ such that the auto-covariance operators $\mathcal{R}_{X,t} = E[X_0 \otimes X_h]$ are summable in nuclear norm, i.e.

$$\sum_{t\in\mathbb{Z}} \|\mathcal{R}_{X,t}\|_{\mathcal{N}} < \infty.$$

Then, as in Panaretos and Tavakoli (2013), one can define for $\mathcal{N}(\mathcal{H})$ the space of nuclear operators from $\mathcal{H}$ to $\mathcal{H}$, a *spectral density operator* $\mathcal{F}_{X,\lambda} \in \mathcal{N}(\mathcal{H})$ through

$$F_{X,\lambda} = \frac{1}{2\pi} \sum_{t\in\mathbb{Z}} \mathcal{R}_{X,t} e^{-ih\lambda}, \qquad \lambda \in [-\pi,\pi].$$

The operator $\mathcal{F}_{X,\lambda}$ is self-adjoint, non-negative ($\langle \mathcal{F}_{X,\lambda}(u), u\rangle \geq 0$) and satisfies the inverse relation

$$\mathcal{R}_{X,t} = \int_{-\pi}^{\pi} \mathcal{F}_{X,\lambda}\, e^{it\lambda}\, \mathrm{d}\lambda, \qquad t \in \mathbb{Z},$$

where integration is taken in Bochner sense.

1.3. **Hilbertian linear processes.** Suppose $(X_t)_{t\in\mathbb{Z}}$ is the linear process $X_t = \sum_{j\in\mathbb{Z}} A_j(\varepsilon_{t-j})$ where $(\varepsilon_t)_{t\in\mathbb{Z}}$ are zero mean, i.i.d. random elements in a (real) Hilbert space $\mathcal{H}_0$ with $C_\varepsilon = E[\varepsilon_0 \otimes \varepsilon_0]$, $E\|\varepsilon_0\|^2 < \infty$ and $\sum_{j\in\mathbb{Z}} \|A_j\|_{\mathcal{L}} < \infty$ ($A_j \in \mathcal{L}(\mathcal{H}_0)$ for all j). Then $\mathcal{R}_{X,t} = \sum_{j\in\mathbb{Z}} A_{j+t} \otimes_{op} A_j(C_\varepsilon)$ and

$$\sum_{t\in\mathbb{Z}} \|\mathcal{R}_{X,t}\|_{\mathcal{N}} \leq \sum_{t\in\mathbb{Z}} \sum_{j\in\mathbb{Z}} \|A_{j+t}\|_{\mathcal{L}}\ \|C_\varepsilon\|_{\mathcal{N}}\ \|A_j\|_{\mathcal{L}} = E\|\varepsilon_0\|^2 \left(\sum_{j\in\mathbb{Z}} \|A_j\|_{\mathcal{L}} \right)^2 < \infty. \quad (1.4)$$

Therefore (Cerovecki & Hörmann (2015), Theorem 4),

$$
\begin{aligned}
\mathcal{F}_{X,\lambda} &= \frac{1}{2\pi}\sum_{t\in\mathbb{Z}} \mathcal{R}_{X,t}\ e^{-it\lambda} \\
&= \frac{1}{2\pi}\sum_{t\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}} A_{j+t} \otimes_{op} A_j(C_\varepsilon)\right) e^{-i(j+t-j)\lambda} \\
&= \frac{1}{2\pi}\sum_{t,j\in\mathbb{Z}} A_{j+h}\ e^{-i(j+t)\omega} C_\varepsilon A_j^*\ e^{ij\lambda} \\
&= \frac{1}{2\pi}\sum_{j\in\mathbb{Z}} \left(\sum_{t\in\mathbb{Z}} A_{j+t}\ e^{-i(j+t)\lambda}\right) C_\varepsilon A_j^*\ e^{ij\lambda} \\
&= \frac{1}{2\pi}\left(\sum_{l\in\mathbb{Z}} A_l\ e^{-il\omega}\right) C_\varepsilon \left(\sum_{j\in\mathbb{Z}} A_j\ e^{-ij\omega}\right)^* \\
&= \frac{1}{2\pi} A(e^{-i\omega}) \otimes_{op} A(e^{-i\omega})(C_\varepsilon), \qquad (1.5)
\end{aligned}
$$

where $A(\lambda) := \sum_{j\in\mathbb{Z}} A_j \lambda^j \in \mathcal{L}(\mathcal{H})$ for $\lambda \in \{z \in \mathbb{C} \mid |z| \leq 1\}$ (cf. Remark 1.4 (c)). From (1.5) we can deduce for any basis $\{e_n\}$ of $\mathcal{H}$ that

$$
\begin{aligned}
\operatorname{trace}(\mathcal{F}_{X,\lambda}) &= \sum_{n\in\mathbb{N}} \left\langle \frac{1}{2\pi}[A(e^{-i\lambda}) \otimes_{op} A(e^{-i\lambda})(C_\varepsilon)](e_n), e_n \right\rangle \\
&= \sum_{n\in\mathbb{N}} \left\langle \frac{1}{2\pi}[A(e^{-i\lambda}) \otimes_{op} A(e^{-i\lambda})(E\varepsilon_0 \otimes \varepsilon_0)](e_n), e_n \right\rangle \\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{N}} E\left\langle [A(e^{-i\lambda}) \otimes_{op} A(e^{-i\lambda})(\varepsilon_0 \otimes \varepsilon_0)](e_n), e_n \right\rangle \\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{N}} E\left\langle [A(e^{-i\lambda})(\varepsilon_0) \otimes A(e^{-i\lambda})(\varepsilon_0)](e_n), e_n \right\rangle \\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{N}} E\left\langle \left\langle e_n, A(e^{-i\lambda})(\varepsilon_0)\right\rangle A(e^{-i\lambda})(\varepsilon_0), e_n \right\rangle \\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{N}} E\left\langle A(e^{-i\lambda})(\varepsilon_0), e_n \right\rangle \left\langle e_n, A(e^{-i\lambda})(\varepsilon_0)\right\rangle \\
&= \frac{1}{2\pi}\sum_{n\in\mathbb{N}} E\Big| \left\langle A(e^{-i\lambda})(\varepsilon_0), e_n \right\rangle \Big|^2 \\
&= \frac{1}{2\pi} E\|A(e^{-i\lambda})(\varepsilon_0)\|^2 \qquad (1.6)
\end{aligned}
$$

1.4. **The periodogram operator.**

Definition 1.5 (Periodogram Operator)**.** Suppose $(X_t)_{t\in\mathbb{Z}}$ is a centred stationary time series taking values in $\mathcal{H}_0$ and satisfying $E\|X_0\|^4 < \infty$. Define the *periodogram operator* $I_{X,\lambda}$ by

$$I_{X,\lambda} := \frac{1}{2\pi}\, D_{X,\lambda} \otimes D_{X,\lambda}, \quad \omega \in [-\pi, \pi]$$

where $D_{X,\lambda}$ denotes the *discrete Fourier transform*

$$D_{X,\lambda} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} X_t e^{-it\lambda}, \quad \lambda \in [-\pi, \pi].$$

Remark 1.8. By Lemma 1.1

$$I_{X,\lambda} = \frac{1}{2\pi T} \sum_{t,s=1}^{T} X_t \otimes X_s e^{-i(t-s)\lambda}.$$

and in case $\mathcal{H} = L^2([0,1], \mathbb{C})$

$$I_{X,\lambda}(u)(\tau) = \int_0^1 \frac{1}{2\pi T} \sum_{t,s=1}^{T} X_t(\tau) X_s(\sigma) e^{-i(t-s)\lambda} u(\sigma)\, d\sigma = \int_0^1 p_{X,\lambda}(\tau,\sigma) u(\sigma)\, d\sigma.$$

The next theorem establishes a useful relation for the periodogram operator of a Hilbertian linear process.

Theorem 1.2. *Suppose $(X_t)_{t\in\mathbb{Z}}$ is the linear process $X_t = \sum_{j\in\mathbb{Z}} A_j(\varepsilon_{t-j})$ where $(\varepsilon_t)_{t\in\mathbb{Z}}$ are i.i.d. centred random elements in a (real) Hilbert space $\mathcal{H}_0$ with $E\|\varepsilon_0\|^4 < \infty$ and $\sum_{j\in\mathbb{Z}} |j|^{1/2} \|A_j\|_{\mathcal{L}} < \infty$ ($A_j \in \mathcal{L}(\mathcal{H}_0)$ for all j). Then*

$$I_{X,\lambda} = A\Big(e^{-i\lambda}\Big) \otimes_{op} A\Big(e^{-i\lambda}\Big)\Big(I_{\varepsilon,\lambda}\Big) + R_{T,\lambda}$$

and $\sup_{\lambda_t \in [0,\pi]} E\|R_{T,\lambda_t}\|_{HS}^2 = \mathcal{O}(1/T)$*, where* $\lambda_t = 2\pi t/T$*,* $t \in \{0, 1, 2, \dots, [T/2]\}$.

Proof: To start with

$$\begin{aligned} D_{X,\lambda} &= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} X_t e^{-it\lambda} = \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \left(\sum_{j\in\mathbb{Z}} A_j(\varepsilon_{t-j}) \right) e^{-i(t-j+j)\lambda} \\ &= \frac{1}{\sqrt{T}} \sum_{t=1}^{T} \sum_{j\in\mathbb{Z}} A_j \Big(\varepsilon_{t-j} e^{-i(t-j)\omega} \Big) e^{-ij\lambda} = \frac{1}{\sqrt{T}} \sum_{j\in\mathbb{Z}} A_j \left(\sum_{t=1}^{T} \varepsilon_{t-j} e^{-i(t-j)\lambda} \right) e^{-ij\lambda} \\ &= \frac{1}{\sqrt{T}} \sum_{j\in\mathbb{Z}} A_j \left(\sum_{l=1-j}^{T-j} \varepsilon_l e^{-il\lambda} \right) e^{-ij\lambda} \\ &= \frac{1}{\sqrt{T}} \sum_{j\in\mathbb{Z}} A_j \left(\sum_{l=1}^{T} \varepsilon_l e^{-il\lambda} + \sum_{l=1-j}^{T-j} \varepsilon_l e^{-il\lambda} - \sum_{l=1}^{n} \varepsilon_l e^{-il\lambda} \right) e^{-ij\lambda} \\ &= \sum_{j\in\mathbb{Z}} A_j \Big(D_{\varepsilon,\lambda} \Big) e^{-ij\lambda} + \frac{1}{\sqrt{T}} \sum_{j\in\mathbb{Z}} A_j \Big(U_{T,j} \Big) e^{-ij\omega}, \end{aligned}$$

where

$$U_{T,j} := \sum_{l=1-j}^{T-j} \varepsilon_l e^{-il\lambda} - \sum_{l=1}^{T} \varepsilon_l e^{-il\lambda}.$$

Thus, defining

$$Y_{T,\lambda} := \frac{1}{\sqrt{T}} \sum_{j\in\mathbb{Z}} A_j\Big(U_{T,j}\Big)e^{-ij\lambda},$$

it holds

$$D_{X,\lambda} = A\Big(e^{-i\lambda}\Big)\Big(D_{\varepsilon,\lambda}\Big) + Y_{T,\lambda}.$$

Now Lemma 1.1 yields

$$\begin{aligned}
I_{X,\lambda} =& \frac{1}{2\pi}\left[A\Big(e^{-i\lambda}\Big)\Big(D_{\varepsilon,\lambda}\Big) + Y_{T,\lambda}\right] \otimes \left[A\Big(e^{-i\lambda}\Big)\Big(D_{\varepsilon,\lambda}\Big) + Y_{T,\lambda}\right] \\
=& A\Big(e^{-i\lambda}\Big) \otimes_{op} A\Big(e^{-i\lambda}\Big)\Big(I_{\varepsilon,\lambda}\Big) + \frac{1}{2\pi}\Bigg[A\Big(e^{-i\lambda}\Big)\Big(D_{\varepsilon,\lambda}\Big) \otimes Y_{T,\lambda} \\
&+ \Big(A\Big(e^{-i\lambda}\Big)\Big(D_{\varepsilon,\lambda}\Big) \otimes Y_{T,\lambda}\Big)^* + Y_{T,\lambda} \otimes Y_{T,\lambda}\Bigg] \\
=& A\Big(e^{-i\lambda}\Big) \otimes_{op} A\Big(e^{-i\lambda}\Big)\Big(I_{\varepsilon,\lambda}\Big) + R_{T,\lambda}.
\end{aligned}$$

For the remainder we have

$$\begin{aligned}
2\pi\|R_{T,\lambda}\|_{HS} &\leq 2\|A\Big(e^{-i\lambda}\Big)\Big(D_{\varepsilon,\lambda}\Big) \otimes Y_{T,\lambda}\|_{HS} + \|Y_{T,\lambda} \otimes Y_{T,\lambda}\|_{HS} \\
&= 2\|A\Big(e^{-i\omega}\Big)D_{\varepsilon,\lambda} \otimes Y_{T,\lambda}\|_{HS} + \|Y_{T,\lambda}\|^2 \\
&\leq 2\|D_{\varepsilon,\lambda} \otimes Y_{T,\lambda}\|_{HS}\ \|A\Big(e^{-i\lambda}\Big)\|_{\mathcal{L}} + \|Y_{n,\lambda}\|^2 \\
&\leq 2\underbrace{\left(\sum_{j\in\mathbb{Z}} \|A_j\|_{\mathcal{L}}\right)}_{=:\kappa_0} \|D_{\varepsilon,\lambda}\|\ \|Y_{T,\lambda}\| + \|Y_{T,\lambda}\|^2.
\end{aligned}$$

Since $(a+b)^2 \leq 2a^2 + 2b^2$ for $a, b \in \mathbb{R}$ it follows

$$4\pi^2\|R_{T,\omega}\|_{HS}^2 \leq 8\kappa_0^2\ \|D_{\varepsilon,\lambda}\|^2\ \|Y_{T,\lambda}\|^2 + 2\|Y_{T,\lambda}\|^4$$

and therefore by Cauchy Schwarz

$$4\pi^2 E\|R_{T,\lambda}\|_{HS}^2 \leq 8\kappa_0^2\ \Big(E\|D_{\varepsilon,\lambda}\|^4\Big)^{1/2}\Big(E\|Y_{T,\lambda}\|^4\Big)^{1/2} + 2E\|Y_{T,\lambda}\|^4.$$

Applying Cauchy Schwarz again gives

$$E\|Y_{T,\lambda}\|^4 = E\left\|\frac{1}{\sqrt{T}}\sum_{j\in\mathbb{Z}} A_j\Big(U_{T,j}\Big)e^{-ij\lambda}\right\|^4 \le \frac{1}{T^2}E\left(\sum_{j\in\mathbb{Z}}\|A_j\|_{\mathcal{L}}\|U_{T,j}\|\right)^4$$
$$\le \frac{1}{T^2}\left(\sum_{j\in\mathbb{Z}}\Big(E\|A_j\|_{\mathcal{L}}^4\|U_{T,j}\|^4\Big)^{1/4}\right)^4 = \frac{1}{T^2}\left(\sum_{j\in\mathbb{Z}}\|A_j\|_{\mathcal{L}}\Big(E\|U_{T,j}\|^4\Big)^{1/4}\right)^4.$$

Depending on $j \in \mathbb{Z}$ we have for $U_{T,j}$

$$U_{T,j} = \sum_{l=1-j}^{T-j}\varepsilon_l e^{-il\lambda} - \sum_{l=1}^{T}\varepsilon_l e^{-il\lambda} = \begin{cases} \sum_{l=1+|j|}^{T+|j|}\varepsilon_l e^{-il\lambda} - \sum_{l=1}^{T}\varepsilon_l e^{-il\lambda}, & j \le -T \\ \sum_{l=T+1}^{T+|j|}\varepsilon_l e^{-il\lambda} - \sum_{l=1}^{|j|}\varepsilon_l e^{-il\lambda}, & -T < j < 0 \\ 0, & j = 0 \\ \sum_{l=1-j}^{0}\varepsilon_l e^{-il\lambda} - \sum_{l=T-j+1}^{T}\varepsilon_l e^{-il\lambda}, & 0 < j < T \\ \sum_{l=1-j}^{T-j}\varepsilon_l e^{-il\lambda} - \sum_{l=1}^{T}\varepsilon_l e^{-il\lambda}, & T \le j, \end{cases}$$

that is $U_{T,j}$ is the sum of $2\min\{|j|, T\}$ independent random variables. Thus (cf. Brockwell & Davis (1991), Problem 10.14)

$$E\|U_{T,j}\|^4 \le E\left(\sum_l \|\varepsilon_l\|\right)^4 \le 2\min\{|j|,T\}E\|\varepsilon_0\|^4 + 3\left((2\min\{|j|,T\})^2(E\|\varepsilon_0\|^2)^2\right)$$
$$\le 2|j|E\|\varepsilon_0\|^4 + 12|j|^2\left(E\|\varepsilon_0\|^2\right)^2$$

and therefore

$$E\|Y_{T,\lambda}\|^4 \le \frac{1}{T^2}\left(\sum_{j\in\mathbb{Z}}\|A_j\|_{\mathcal{L}}\Big(2|j|E\|\varepsilon_0\|^4 + 12|j|^2\left(E\|\varepsilon_0\|^2\right)^2\Big)^{1/4}\right)^4$$
$$\le \frac{1}{T^2}\left(\sum_{j\in\mathbb{Z}}\|A_j\|_{\mathcal{L}}\Big(\underbrace{\left(2|j| + 12|j|^2\right)}_{\le 14|j|^2}E\|\varepsilon_0\|^4\Big)^{1/4}\right)^4$$
$$\le \frac{1}{T^2}\left(\sqrt[4]{14E\|\varepsilon_0\|^4}\sum_{j\in\mathbb{Z}}\sqrt{|j|}\|A_j\|_{\mathcal{L}}\right)^4 \in \mathcal{O}(1/T^2)$$

Since

$$E\|D_{\varepsilon,\lambda}\|^4 \le \frac{1}{T}E\|\varepsilon_0\|^4 + 3E\|\varepsilon_0\|^2$$

it follows that

$$4\pi^2 \|R_{T,\lambda}\|_{HS}^2 \leq 8\kappa_0 \underbrace{\left(E\|D_{\varepsilon,\lambda}\|^4\right)^{1/2}}_{<\infty} \underbrace{\left(E\|Y_{T,\lambda}\|^4\right)^{1/2}}_{\in\mathcal{O}(1/T)} + \underbrace{2E\|Y_{T,\lambda}\|^4}_{\mathcal{O}(1/T^2)} \in \mathcal{O}(1/T).$$

□

Lemma 1.6. *Suppose X, Y are independent random elements talking values in a separable Hilbert space $\mathcal{H}$ and $E\|X\| < \infty$, $E\|Y\| < \infty$. Then*

$$E\langle X, Y\rangle = \langle EX, EY\rangle.$$

Proof: Let $\{\varphi_n \mid n \in \mathbb{N}\}$ a ONB of $\mathcal{H}$, then $E\langle X, Y\rangle = E\sum_n \langle X, \varphi_n\rangle\langle\varphi_n, Y\rangle$. Since by Cauchy-Schwarz and Parsevals identity

$$E\sum_n |\langle X, \varphi_n\rangle\langle\varphi_n, Y\rangle| \leq E\sqrt{\sum_n |\langle X, \varphi_n\rangle|^2}\sqrt{\sum_n |\langle Y, \varphi_n\rangle|^2} = E\|X\|E\|Y\| < \infty,$$

Fubini's theorem yields

$$E\langle X, Y\rangle = \sum_n \langle EX, \varphi_n\rangle\langle\varphi_n, EY\rangle = \langle EX, EY\rangle.$$

□

Lemma 1.7 (4th Moment of a strong White Noise)**.** *Suppose $(\varepsilon_t)_{t\in\mathbb{Z}}$ are i.i.d. centred random elements in a real Hilbert space $\mathcal{H}_0$ with covariance operator $C_\varepsilon = E[\varepsilon_0 \otimes \varepsilon_0]$ and $E\|\varepsilon_0\|^4 < \infty$, then*

$$E\Big[(\varepsilon_t \otimes \varepsilon_s) \otimes (\varepsilon_k \otimes \varepsilon_l)\Big] = \begin{cases} E\Big[(\varepsilon_0 \otimes \varepsilon_0) \otimes (\varepsilon_0 \otimes \varepsilon_0)\Big], & t = s = k = l \\ C_\varepsilon \otimes_{op} C_\varepsilon, & t = s \neq k = l \\ C_\varepsilon \otimes_{op} C_\varepsilon, & t = k \neq s = l \\ C_\varepsilon \otimes_{op}^T C_\varepsilon, & t = l \neq k = s \\ 0, & \textit{otherwise.} \end{cases}$$

Proof: To start with

$$E\|(\varepsilon_t \otimes \varepsilon_s) \otimes (\varepsilon_k \otimes \varepsilon_l)\|_{HS(HS)} = E\|\varepsilon_t\|\|\varepsilon_s\|\|\varepsilon_k\|\|\varepsilon_l\| \leq E\|\varepsilon_0\|^4 < \infty,$$

so $E(\varepsilon_t \otimes \varepsilon_s) \otimes (\varepsilon_k \otimes \varepsilon_l) \in HS(HS(\mathcal{H}_0))$ exists and is uniquely determined by

$$\begin{aligned} &\langle E(\varepsilon_t \otimes \varepsilon_s) \otimes (\varepsilon_k \otimes \varepsilon_l), (f \otimes g) \otimes (u \otimes v)\rangle \\ &= E\,\langle(\varepsilon_t \otimes \varepsilon_s) \otimes (\varepsilon_k \otimes \varepsilon_l), (f \otimes g) \otimes (u \otimes v)\rangle \\ &= E\langle\varepsilon_t, f\rangle\langle\varepsilon_s, g\rangle\langle\varepsilon_k, u\rangle\langle\varepsilon_l, v\rangle \end{aligned}$$

for $f, g, u, v \in \mathcal{H}_0$ since $\overline{\text{span}}\{(f \otimes g) \otimes (u \otimes v) \mid f, g, u, v \in \mathcal{H}_0\} = HS(HS(\mathcal{H}_0))$ (cf. Werner, Satz VI.6.2 (g)).
1. In case $t = s \neq k = l$ by Lemma 1.1

$$\begin{aligned} E\langle \varepsilon_t, f\rangle\langle \varepsilon_t, g\rangle\langle \varepsilon_k, u\rangle\langle \varepsilon_k, v\rangle =&\langle E\varepsilon_t \otimes \varepsilon_t, f \otimes g\rangle\langle E\varepsilon_k \otimes \varepsilon_k, u \otimes v\rangle \\ =&\langle C_\varepsilon \otimes_{op} C_\varepsilon, (f \otimes g) \otimes (u \otimes v)\rangle, \end{aligned}$$

so $E(\varepsilon_t \otimes \varepsilon_t) \otimes (\varepsilon_k \otimes \varepsilon_k) = C_\varepsilon \otimes_{op} C_\varepsilon$.
2. In case $t = k \neq s = l$ we have

$$\begin{aligned} &E\langle \varepsilon_t, f\rangle\langle \varepsilon_s, g\rangle\langle \varepsilon_t, u\rangle\langle \varepsilon_s, v\rangle = \langle C_\varepsilon, f \otimes u\rangle\langle C_\varepsilon, g \otimes v\rangle \\ =&\sum_i \langle C_\varepsilon(\varphi_i), f \otimes u(\varphi_i)\rangle \sum_j \langle C_\varepsilon(\varphi_j), g \otimes v(\varphi_j)\rangle \\ =&\langle u, C_\varepsilon(f)\rangle\langle v, C_\varepsilon(g)\rangle = \langle C_\varepsilon(f) \otimes C_\varepsilon(g), u \otimes v\rangle \\ =&\left\langle C_\varepsilon \otimes_{op} C_\varepsilon(\sum_{i,j} \langle f, \varphi_i\rangle\langle g, \varphi_j\rangle \varphi_i \otimes \varphi_j), \sum_{k,l} \langle u, \varphi_k\rangle\langle v, \varphi_l\rangle \varphi_k \otimes_2 \varphi_l \right\rangle \\ =&\sum_{i,j,k,l} \langle f, \varphi_i\rangle\langle g, \varphi_j\rangle\langle u, \varphi_k\rangle\langle v, \varphi_l\rangle \sum_{n,m} \langle\langle \varphi_n \otimes \varphi_m, \varphi_k \otimes \varphi_l\rangle \varphi_i \otimes \varphi_j, C_\varepsilon \otimes_{op} C_\varepsilon(\varphi_n \otimes \varphi_m)\rangle \\ =&\sum_{i,j,k,l} \langle f, \varphi_i\rangle\langle g, \varphi_j\rangle\langle u, \varphi_k\rangle\langle v, \varphi_l\rangle \langle (\varphi_i \otimes \varphi_j) \otimes (\varphi_k \otimes \varphi_l), C_\varepsilon \otimes_{op} C_\varepsilon\rangle \\ =&\langle C_\varepsilon \otimes_{op} C_\varepsilon, (f \otimes g) \otimes (u \otimes v)\rangle, \end{aligned}$$

so $E(\varepsilon_t \otimes \varepsilon_s) \otimes (\varepsilon_t \otimes \varepsilon_s) = C_\varepsilon \otimes_{op} C_\varepsilon$.
3. In case $t = l \neq k = s$ we have

$$\begin{aligned} &E\langle \varepsilon_t, f\rangle\langle \varepsilon_s, g\rangle\langle \varepsilon_s, u\rangle\langle \varepsilon_t, v\rangle = \langle C_\varepsilon, f \otimes v\rangle\langle C_\varepsilon, g \otimes u\rangle \\ =&\langle C_\varepsilon(f) \otimes C_\varepsilon(g), v \otimes u\rangle = \langle u \otimes v, C_\varepsilon \otimes_{op}^T C_\varepsilon(f \otimes g)\rangle \\ =&\sum_{i,j,k,l} \langle f, \varphi_i\rangle\langle g, \varphi_j\rangle\langle u, \varphi_k\rangle\langle v, \varphi_l\rangle \sum_{n,m} \left\langle \langle \varphi_n \otimes \varphi_m, \varphi_i \otimes \varphi_j\rangle \varphi_k \otimes \varphi_l, C_\varepsilon \otimes_{op}^T C_\varepsilon(\varphi_n \otimes \varphi_m)\right\rangle \\ =&\sum_{i,j,k,l} \langle f, \varphi_i\rangle\langle g, \varphi_j\rangle\langle u, \varphi_k\rangle\langle v, \varphi_l\rangle \left\langle (\varphi_k \otimes \varphi_l) \otimes (\varphi_i \otimes \varphi_j), C_\varepsilon \otimes_{op}^T C_\varepsilon\right\rangle \\ =&\langle C_\varepsilon \otimes_{op}^T C_\varepsilon, (f \otimes g) \otimes (u \otimes v)\rangle, \end{aligned}$$

so $E(\varepsilon_t \otimes \varepsilon_s) \otimes (\varepsilon_t \otimes \varepsilon_s) = C_\varepsilon \otimes_{op}^T C_\varepsilon$.
4. In case $t = s = k = l$ we have

$$E\langle \varepsilon_t, f\rangle\langle \varepsilon_t, g\rangle\langle \varepsilon_t, u\rangle\langle \varepsilon_t, v\rangle = \langle E(\varepsilon_0 \otimes \varepsilon_0) \otimes (\varepsilon_0 \otimes \varepsilon_0), (f \otimes g) \otimes_2 (u \otimes v)\rangle$$

5. In any other case $E\langle \varepsilon_t, f\rangle\langle \varepsilon_s, g\rangle\langle \varepsilon_k, u\rangle\langle \varepsilon_l, v\rangle = 0$ by independence. $\square$

Theorem 1.3. *Suppose $(\varepsilon_t)_{t\in\mathbb{Z}}$ are i.i.d. centred random elements in a real Hilbert space $\mathcal{H}_0$ with covariance operator $C_\varepsilon = E[\varepsilon_0 \otimes \varepsilon_0]$ and $E\|\varepsilon_0\|^4 < \infty$. The covariance of the corresponding periodogram operator is then given for $\lambda_1, \lambda_2 \in [-\pi, \pi]$ by*

$$\begin{aligned}\mathrm{Cov}(I_{\varepsilon,\lambda_1}, I_{\varepsilon,\lambda_2}) =& \frac{1}{4\pi^2 T}\mathrm{cum}(\varepsilon_0, \varepsilon_0, \varepsilon_0, \varepsilon_0) + \frac{1}{4\pi^2 T} F_T(\lambda_1 - \lambda_2)\ C_\varepsilon \otimes_{op} C_\varepsilon \\ &+ \frac{1}{4\pi^2 T} F_T(\lambda_1 + \lambda_2)\ C_\varepsilon \otimes_{op}^T C_\varepsilon,\end{aligned}$$

where $F_T(\lambda)$ denotes the T*th Fejer kernel,*

$$F_T(\lambda) = \frac{1}{T}\left|\sum_{t=1}^{T} e^{-it\lambda}\right|^2 = \begin{cases} \frac{1}{T}\frac{\sin^2(T\lambda/2)}{\sin^2(\lambda/2)}, & \lambda \notin 2\pi\mathbb{Z} \\ T, & \lambda \in 2\pi\mathbb{Z}. \end{cases}$$

and $\mathrm{cum}(\varepsilon_0, \varepsilon_0, \varepsilon_0, \varepsilon_0) := \Lambda - C_\varepsilon \otimes C_\varepsilon - C_\varepsilon \otimes_{op} C_\varepsilon - C_\varepsilon \otimes_{op}^T C_\varepsilon$ *with* $\Lambda := E[(\varepsilon_0 \otimes \varepsilon_0) \otimes (\varepsilon_0 \otimes \varepsilon_0)]$.

Proof: Since $EI_{\varepsilon,\lambda} = C_\varepsilon/2\pi$ we have

$$\begin{aligned}\mathrm{Cov}(I_{\varepsilon,\lambda_1}, I_{\varepsilon,\lambda_2}) :=& E\Big[(I_{\varepsilon,\lambda_1} - EI_{\varepsilon,\lambda_1}) \otimes (I_{\varepsilon,\lambda_2} - EI_{\varepsilon,\lambda_2})\Big] \\ =& E\Big[I_{\varepsilon,\lambda_1} \otimes I_{\varepsilon,\lambda_2}\Big] - \frac{1}{4\pi^2} C_\varepsilon \otimes C_\varepsilon.\end{aligned}$$

Lemma 1.7 yields then

$$
\begin{aligned}
&E\Big[I_{\varepsilon,\lambda_1}\otimes I_{\varepsilon,\lambda_2}\Big]\\
=&E\left[\left(\frac{1}{2\pi T}\sum_{t,s=1}^{T}\varepsilon_t\otimes\varepsilon_s\ e^{-i(t-s)\lambda_1}\right)\otimes\left(\frac{1}{2\pi T}\sum_{k,l=1}^{T}\varepsilon_k\otimes\varepsilon_l\ e^{-i(k-l)\lambda_2}\right)\right]\\
=&\frac{1}{(2\pi T)^2}\sum_{t,s,k,l=1}^{T}e^{i(t-s)\lambda_1}e^{-i(k-l)\lambda_2}\ E\Big[(\varepsilon_t\otimes\varepsilon_s)\otimes(\varepsilon_k\otimes\varepsilon_l)\Big]\\
=&\frac{1}{(2\pi T)^2}\Bigg(T\Lambda+T(T-1)C_\varepsilon\otimes_{op}C_\varepsilon+\sum_{\substack{t,s=1\\t\neq s}}^{T}e^{-i(t-s)\lambda_1}e^{-i(s-t)\lambda_2}C_\varepsilon\otimes_{op}C_\varepsilon\\
&+\sum_{\substack{t,s=1\\t\neq s}}^{T}e^{-i(t-s)\lambda_1}e^{-i(t-s)\lambda_2}C_\varepsilon\otimes_{op}^{T}C_\varepsilon\Bigg)\\
=&\frac{1}{(2\pi T)^2}\Bigg\{T\Lambda+(T^2-T)C_\varepsilon\otimes C_\varepsilon+\left(\left|\sum_{t=1}^{T}e^{-it(\lambda_1-\lambda_2)}\right|^2-T\right)C_\varepsilon\otimes_{op}C_\varepsilon\\
&+\left(\left|\sum_{t=1}^{T}e^{-it(\lambda_1+\lambda_2)}\right|^2-T\right)C_\varepsilon\otimes_{op}^{T}C_\varepsilon\Bigg\}\\
=&\frac{1}{4\pi^2T}\Big(\Lambda-C_\varepsilon\otimes C_\varepsilon-C_\varepsilon\otimes_{op}C_\varepsilon-C_\varepsilon\otimes_{op}^{T}C_\varepsilon\Big)+\frac{1}{4\pi^2}C_\varepsilon\otimes C_\varepsilon\\
&+\frac{1}{4\pi^2T^2}\left|\sum_{t=1}^{T}e^{-it(\lambda_1-\lambda_2)}\right|^2C_\varepsilon\otimes_{op}C_\varepsilon+\frac{1}{4\pi^2T^2}\left|\sum_{t=1}^{T}e^{-it(\lambda_1+\lambda_2)}\right|^2C_\varepsilon\otimes_{op}^{T}C_\varepsilon\\
=&\frac{1}{4\pi^2T}\mathrm{cum}(\varepsilon_0,\varepsilon_0,\varepsilon_0,\varepsilon_0)+\frac{1}{4\pi^2T}F_T(\lambda_1-\lambda_2)\ C_\varepsilon\otimes_{op}C_\varepsilon\\
&+\frac{1}{4\pi^2T}F_T(\lambda_1+\lambda_2)\ C_\varepsilon\otimes_{op}^{T}C_\varepsilon+\frac{1}{4\pi^2}C_\varepsilon\otimes C_\varepsilon
\end{aligned}
$$

□

Remark 1.9. Notice that if $\varepsilon_0 \sim \mathcal{N}(0, C_\varepsilon)$ then for $f, g, u, v \in \mathcal{H}_0$ the vector $(\langle \varepsilon_0, f\rangle, \langle \varepsilon_0, g\rangle, \langle \varepsilon_0, u\rangle, \langle \varepsilon_0, v\rangle)$ is multivariate normal distributed, so by Isserlis theorem

$$\begin{aligned}
&E\langle(\varepsilon_0 \otimes \varepsilon_0) \otimes (\varepsilon_0 \otimes \varepsilon_0), (f \otimes g) \otimes (u \otimes v)\rangle \\
=&E\langle \varepsilon_0, f\rangle\langle \varepsilon_0, g\rangle E\langle \varepsilon_0, u\rangle\langle \varepsilon_0, v\rangle + E\langle \varepsilon_0, f\rangle\langle \varepsilon_0, u\rangle E\langle \varepsilon_0, g\rangle\langle \varepsilon_0, v\rangle \\
&+ E\langle \varepsilon_0, f\rangle\langle \varepsilon_0, v\rangle E\langle \varepsilon_0, g\rangle\langle \varepsilon_0, u\rangle \\
=&\langle C_\varepsilon, f \otimes g\rangle\langle C_\varepsilon, u \otimes v\rangle + \langle C_\varepsilon, f \otimes u\rangle\langle C_\varepsilon, g \otimes v\rangle + \langle C_\varepsilon, f \otimes v\rangle\langle C_\varepsilon, g \otimes u\rangle \\
=&\langle C_\varepsilon \otimes C_\varepsilon + C_\varepsilon \otimes_{op} C_\varepsilon + C_\varepsilon \otimes_{op}^T C_\varepsilon, (f \otimes g) \otimes (u \otimes v)\rangle.
\end{aligned}$$

Therefore $E[(\varepsilon_0 \otimes \varepsilon_0) \otimes (\varepsilon_0 \otimes \varepsilon_0)] = C_\varepsilon \otimes_{op} C_\varepsilon + C_\varepsilon \otimes_{op} C_\varepsilon + C_\varepsilon \otimes_{op}^T C_\varepsilon$ and $\mathrm{cum}(\varepsilon_0, \varepsilon_0, \varepsilon_0, \varepsilon_0) = 0$.

2. Proofs of Proposition 1, Lemma 2 and Lemma 3

2.1. **Proof of Proposition 1.** By Assumption 2, we have $b^{-1/2}\Big(\widehat{\mu}_0 - \widetilde{\mu}_0\Big) \xrightarrow{P} 0$ and $\widehat{\theta}_0 \xrightarrow{P} \widetilde{\theta}_0$. Furthermore,

$$\begin{aligned}
\sqrt{b}T\,\mathcal{U}_T - b^{-1/2}\widetilde{\mu}_0 =& \sqrt{b}T\int_{-\pi}^{\pi} \|E\widehat{\mathcal{F}}_{X,\lambda} - E\widehat{\mathcal{F}}_{Y,\lambda}\|_{HS}^2 d\lambda \\
&+ \sqrt{b}T\int_{-\pi}^{\pi} \|\Big(\widehat{\mathcal{F}}_{X,\lambda} - E\widehat{\mathcal{F}}_{X,\lambda}\Big) - \Big(\widehat{\mathcal{F}}_{Y,\lambda} - E\widehat{\mathcal{F}}_{Y,\lambda}\Big)\|_{HS}^2 d\lambda - b^{-1/2}\widetilde{\mu}_0 \\
&+ 2\sqrt{b}T\int_{-\pi}^{\pi} \langle\Big(\widehat{\mathcal{F}}_{X,\lambda} - E\widehat{\mathcal{F}}_{X,\lambda}\Big) - \Big(\widehat{\mathcal{F}}_{Y,\lambda} - E\widehat{\mathcal{F}}_{Y,\lambda}\Big), E\widehat{\mathcal{F}}_{X,\lambda} - E\widehat{\mathcal{F}}_{Y,\lambda}\rangle_{HS} d\lambda.
\end{aligned} \tag{2.1}$$

The assertion of the proposition follows since $\sqrt{b}T\int_{-\pi}^{\pi} \|E\widehat{\mathcal{F}}_{X,\lambda} - E\widehat{\mathcal{F}}_{Y,\lambda}\|_{HS}^2 d\lambda = \sqrt{b}T\int_{-\pi}^{\pi} \|\mathcal{F}_{X,\lambda} - \mathcal{F}_{Y,\lambda}\|_{HS}^2 d\lambda + o(1)$ since $\|E\widehat{\mathcal{F}}_{X,\lambda} - \mathcal{F}_{X,\lambda}\|_{HS} = \|E\widehat{\mathcal{F}}_{Y,\lambda} - \mathcal{F}_{Y,\lambda}\|_{HS} = O(b^2)$ unifomly in λ. Furthermore, $\sqrt{b}T\int_{-\pi}^{\pi} \|\Big(\widehat{\mathcal{F}}_{X,\lambda} - E\widehat{\mathcal{F}}_{X,\lambda}\Big) - \Big(\widehat{\mathcal{F}}_{Y,\lambda} - E\widehat{\mathcal{F}}_{Y,\lambda}\Big)\|_{HS}^2 d\lambda - b^{-1/2}\widetilde{\mu}_0 = O_P(1)$, while the last term in (2.1) is $O_P(\sqrt{T})$ since $\sqrt{b}T$ times the expression of the inner product is bounded by

$$\begin{aligned}
&\sqrt{T}\Big\{\|\sqrt{Tb}(\widehat{\mathcal{F}}_{X,\lambda} - E\widehat{\mathcal{F}}_{X,\lambda})\|_{HS} + \|\sqrt{Tb}(\widehat{\mathcal{F}}_{Y,\lambda} - E\widehat{\mathcal{F}}_{Y,\lambda})\|_{HS}\Big\}\Big\{\|E\widehat{\mathcal{F}}_{X,\lambda} - E\widehat{\mathcal{F}}_{Y,\lambda}\|_{HS}\Big\} \\
&= O_P(\sqrt{T}).
\end{aligned}$$

Notice that the last equality follows because the sequences $\sqrt{Tb}(\widehat{\mathcal{F}}_{X,\lambda} - E\widehat{\mathcal{F}}_{X,\lambda})$ and $\sqrt{Tb}(\widehat{\mathcal{F}}_{Y,\lambda} - E\widehat{\mathcal{F}}_{Y,\lambda})$ are bounded in probability, uniformly in λ, since as in Theorem 3.7 of Panaretos and Tavakoli (2013), both sequences converge weakly to Gaussian elements in $L^2([0,1]^2, \mathbb{C})$ with covariance kernels that can be bounded uniformly in λ.

2.2. **Proof of Lemma 2.** We show that mean and variance of $\sqrt{b}T\, M_{T,0} - b^{-1/2}\,\mu_0$ tend to zero asymptotically. For the mean, we have

$$
\begin{aligned}
&E M_{T,0}\\
=&E \int_{-\pi}^{\pi} \frac{1}{b^2T^2} \sum_{t=-N}^{N} W^2\left(\frac{\lambda-\lambda_t}{b}\right) \left\{ \left\|Q^c_{X,\lambda_t}\right\|_{HS}^2 + \left\|Q^c_{Y,\lambda_t}\right\|_{HS}^2 - 2\left\langle Q^c_{X,\lambda_t}, Q^c_{Y,\lambda_t}\right\rangle_{HS} \right\} d\lambda
\end{aligned}
$$

For the first part, we get

$$
\begin{aligned}
&\sqrt{b}T \int_{-\pi}^{\pi} \frac{1}{b^2T^2} \sum_{t=-N}^{N} W^2\left(\frac{\lambda-\lambda_t}{b}\right) E\left\|Q^c_{X,\lambda_t}\right\|_{HS}^2 d\lambda\\
&= \frac{1}{\sqrt{b}T} \sum_{t=-N}^{N} \int_{-\pi/b}^{\pi/b} W^2\left(\lambda - \frac{\lambda_t}{b}\right) d\lambda\, E\left\|Q^c_{X,\lambda_t}\right\|_{HS}^2\\
&= \int_{-\pi}^{\pi} W^2(\lambda) d\lambda\, \frac{1}{\sqrt{b}T} \sum_{t=-N}^{N} E\left\|Q^c_{X,\lambda_t}\right\|_{HS}^2 + o(1). \qquad (2.2)
\end{aligned}
$$

Using that $\sum_{s=1}^{T} e^{is\lambda_t} = 0$, Remark 1.8 and Lemma 1.1 give

$$
\begin{aligned}
&\frac{1}{\sqrt{b}T} \sum_{t=-N}^{N} E\left\|Q^c_{X,\lambda_t}\right\|_{HS}^2\\
&= \frac{1}{\sqrt{b}T} \sum_{t=-N}^{N} E\left\|A(e^{-i\lambda_t}) I^c_{\varepsilon,\lambda_t} A(e^{-i\lambda_t})^*\right\|_{HS}^2\\
&\le \frac{1}{4\pi^2}\frac{1}{\sqrt{b}T} \sum_{t=-N}^{N} \frac{1}{T^2} \sum_{s_1\neq s_2=1}^{T} E\left\|A(e^{-i\lambda_t})(\varepsilon_{s_1}\otimes\varepsilon_{s_2})A(e^{-i\lambda_t})^*\right\|_{HS}^2\\
&+ \frac{1}{4\pi^2}\frac{1}{\sqrt{b}T} \sum_{t=-N}^{N} \frac{1}{T^2} \sum_{s_1\neq s_2=1}^{T} e^{-2i\lambda_t(s_1-s_2)}\\
&\quad \times E\left\langle A(e^{-i\lambda_t})(\varepsilon_{s_1}\otimes\varepsilon_{s_2})A(e^{-i\lambda_t})^*, A(e^{-i\lambda_t})(\varepsilon_{s_2}\otimes\varepsilon_{s_1})A(e^{-i\lambda_t})^*\right\rangle_{HS} + o(1)\\
&= \frac{1}{4\pi^2}\frac{1}{\sqrt{b}T} \sum_{t=-N}^{N} E\left\|[A(e^{-i\lambda_t})\varepsilon_1]\otimes[A(e^{-i\lambda_t})\varepsilon_2]\right\|_{HS}^2 + o(1)\\
&= \frac{1}{4\pi^2}\frac{1}{\sqrt{b}T} \sum_{t=-N}^{N} \left(E\left\|A(e^{-i\lambda_t})\varepsilon_0\right\|_{L_2}^2\right)^2 + o(1). \qquad (2.3)
\end{aligned}
$$

Using a Riemann approximation for the latter term as well as (1.6) and recalling that $\sum_{j\in\mathbb{Z}} |j|^{1/2}\|A_j\|_{\mathcal{L}} < \infty$ we obtain

$$
\begin{aligned}
&\frac{1}{\sqrt{b}T}\sum_{t=-N}^{N} E\left\|A(e^{-i\lambda_t})I^c_{\varepsilon,\lambda_t}A(e^{-i\lambda_t})^*\right\|^2_{HS}\\
&=\frac{1}{2\pi\sqrt{b}}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}E\left\|A(e^{-i\lambda})\varepsilon_0\right\|^2_{L_2}\right)^2 d\lambda\\
&\quad+O\left(\frac{1}{\sqrt{b}}\right)\sum_{t=-N}^{N}\int_{\pi(t-1)/T}^{\pi t/T}\sum_{j,k\in\mathbb{Z}}[e^{i\lambda_t(j-k)}-e^{i\lambda(j-k)}]E\langle A_j\varepsilon_0, A_k\varepsilon_0\rangle_{L_2}d\lambda\,+\,o(1)\\
&=\frac{1}{2\pi\sqrt{b}}\int_{-\pi}^{\pi}\left(\operatorname{trace}(\mathcal{F}_{X,\lambda})\right)^2 d\lambda\\
&\quad+O\left(\frac{1}{\sqrt{bT}}\right)\sum_{j,k\in\mathbb{Z}}|j-k|^{1/2}E\|[A_j\varepsilon_0]\otimes[A_k\varepsilon_0]\|_{HS}d\lambda\,+\,o(1)\\
&=\frac{1}{2\pi\sqrt{b}}\int_{-\pi}^{\pi}\left(\operatorname{trace}(\mathcal{F}_{X,\lambda})\right)^2 d\lambda\\
&\quad+O\left(\frac{1}{\sqrt{bT}}\right)\sum_{j,k\in\mathbb{Z}}|j-k|^{1/2}\|A_j\|_{\mathcal{L}}\|A_k\|_{\mathcal{L}}\|C_\varepsilon\|_{\mathcal{N}}\,+\,o(1)\\
&=\frac{1}{2\pi\sqrt{b}}\int_{-\pi}^{\pi}\left(\operatorname{trace}(\mathcal{F}_{X,\lambda})\right)^2 d\lambda+o(1).
\end{aligned}
\tag{2.4}
$$

Hence, (2.3) and (2.4) give

$$
\sqrt{b}T\int_{-\pi}^{\pi}\frac{1}{b^2T^2}\sum_{t=-N}^{N}W^2\left(\frac{\lambda-\lambda_t}{b}\right)d\lambda\,E\left\|Q^c_{X,\lambda_t}\right\|^2_{HS}=\frac{1}{2\sqrt{b}}\mu_0+o(1).
$$

With the same arguments,

$$
\sqrt{b}T\int_{-\pi}^{\pi}\frac{1}{b^2T^2}\sum_{t=-N}^{N}W^2\left(\frac{\lambda-\lambda_t}{b}\right)d\lambda\,E\left\|Q^c_{Y,\lambda_t}\right\|^2_{HS}=\frac{1}{2\sqrt{b}}\mu_0+o(1).
$$

By independence of $(e_t)_t$ und $(\varepsilon_t)_t$, we can deduce that the corresponding periodogramm operators are independent. Hence, we obtain that the mixed terms vanish, that is,

$$
E\left\langle A(e^{-i\lambda_t})I^c_{\varepsilon,\lambda_t}A(e^{-i\lambda_t})^*, B(e^{-i\lambda_t})I^c_{e,\lambda_t}B(e^{-i\lambda_t})^*\right\rangle_{HS}=0.
$$

Now, we turn to $var(\sqrt{b}T\, M_{T,0})$:

$$\begin{aligned}
&var(\sqrt{b}T\, M_{T,0})\\
=&\frac{1}{bT^2}\sum_{t_1,t_2=-N}^{N}\int_{-\pi/b}^{\pi/b} W^2\left(\lambda-\frac{\lambda_{t_1}}{b}\right)d\lambda\int_{-\pi/b}^{\pi/b} W^2\left(\lambda-\frac{\lambda_{t_2}}{b}\right)d\lambda\\
&\times cov\Big(\left\|A(e^{-i\lambda_{t_1}})I^c_{\varepsilon,\lambda_{t_1}}A(e^{-i\lambda_{t_1}})^*-B(e^{-i\lambda_{t_1}})I^c_{e,\lambda_{t_1}}B(e^{-i\lambda_{t_1}})^*\right\|^2_{HS},\\
&\left\|A(e^{-i\lambda_{t_2}})I^c_{\varepsilon,\lambda_{t_2}}A(e^{-i\lambda_{t_2}})^*-B(e^{-i\lambda_{t_2}})I^c_{e,\lambda_{t_2}}B(e^{-i\lambda_{t_2}})^*\right\|^2_{HS}\Big)
\end{aligned}$$

Under Gaussianity, all summands with indizes t_1 and t_2 with $|t_1|\neq|t_2|$ vanish due to the independence of the periodograms at the corresponding Fourier frequencies. By standard arguments, we get

$$E\left\|A(e^{-i\lambda_t})I^c_{\varepsilon,\lambda_t}A(e^{-i\lambda_t})^*\right\|^4_{HS}\leq CE\left\|I^c_{\varepsilon,\lambda_t}\right\|^4_{HS}.$$

The latter term can be bounded uniformly in t with similar arguments as in Theorem 1.2. Finally this implies that $var(\sqrt{b}T\, M_{T,0})=O(1/(bT))$. □

2.3. **Proof of Lemma 3.** Recalling $Q^c_{X,\lambda}:=A(e^{-i\lambda})I^c_{\varepsilon,\lambda}A(e^{-i\lambda})^*=A(e^{-i\lambda})\otimes_{op}A(e^{-i\lambda})(I^c_{\varepsilon,\lambda})$ and $Q^c_{Y,\lambda}:=B(e^{-i\lambda})I^c_{e,\lambda}B(e^{-i\lambda})^*=B(e^{-i\lambda})\otimes_{op}B(e^{-i\lambda})(I^c_{e,\lambda})$ gives

$$\begin{aligned}
&var(\sqrt{b}TL_{T,0})\\
=&\frac{1}{b^3T^2}var\left(\int_{-\pi}^{\pi}\sum_{\substack{t_1,t_2=-N\\t_1\neq t_2}}^{N}W\left(\frac{\lambda-\lambda_{t_1}}{b}\right)W\left(\frac{\lambda-\lambda_{t_2}}{b}\right)d\lambda\left\langle Q^c_{X,\lambda_{t_1}}-Q^c_{Y,\lambda_{t_1}},Q^c_{X,\lambda_{t_2}}-Q^c_{Y,\lambda_{t_2}}\right\rangle\right)\\
=&\frac{1}{b^3T^2}\sum_{\substack{t_1,t_2=-N\\t_1\neq t_2}}^{N}\sum_{\substack{s_1,s_2=-N\\s_1\neq s_2}}^{N}\int_{-\pi}^{\pi}W\left(\frac{\lambda-\lambda_{t_1}}{b}\right)W\left(\frac{\lambda-\lambda_{t_2}}{b}\right)d\lambda\int_{-\pi}^{\pi}W\left(\frac{\lambda-\lambda_{s_1}}{b}\right)W\left(\frac{\lambda-\lambda_{s_2}}{b}\right)d\lambda\\
&\times cov\left(\left\langle Q^c_{X,\lambda_{t_1}}-Q^c_{Y,\lambda_{t_1}},Q^c_{X,\lambda_{t_2}}-Q^c_{Y,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{X,\lambda_{s_1}}-Q^c_{Y,\lambda_{s_1}},Q^c_{X,\lambda_{s_2}}-Q^c_{Y,\lambda_{s_2}}\right\rangle\right).
\end{aligned}$$

Notice that by Lemma 1.6 we have for example

$$\begin{aligned}
&cov\left(\left\langle Q^c_{X,\lambda_{t_1}},Q^c_{X,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{X,\lambda_{s_1}},Q^c_{Y,\lambda_{s_2}}\right\rangle\right)\\
=&E\Big(\left\langle Q^c_{X,\lambda_{t_1}},Q^c_{X,\lambda_{t_2}}\right\rangle-\underbrace{E\left\langle Q^c_{X,\lambda_{t_1}},Q^c_{X,\lambda_{t_2}}\right\rangle}_{=0,\text{ if }t_1\neq\pm t_2}\Big)\left\langle Q^c_{Y,\lambda_{s_2}},Q^c_{X,\lambda_{s_1}}\right\rangle\\
=&\left\langle EQ^c_{Y,\lambda_{s_2}},E\Big(\left\langle Q^c_{X,\lambda_{t_2}},Q^c_{X,\lambda_{t_1}}\right\rangle-E\left\langle Q^c_{X,\lambda_{t_1}},Q^c_{X,\lambda_{t_2}}\right\rangle\Big)Q^c_{X,\lambda_{s_1}}\right\rangle=0,
\end{aligned}$$

and likewise for similar terms. Thus

$$
\begin{aligned}
&var(\sqrt{b}TL_{T,0}) \\
=&\frac{1}{b^3T^2}\sum_{\substack{t_1,t_2=-N\\t_1\neq t_2}}^{N}\sum_{\substack{s_1,s_2=-N\\s_1\neq s_2}}^{N}\int_{-\pi}^{\pi}W\left(\frac{\lambda-\lambda_{t_1}}{b}\right)W\left(\frac{\lambda-\lambda_{t_2}}{b}\right)d\lambda\int_{-\pi}^{\pi}W\left(\frac{\lambda-\lambda_{s_1}}{b}\right)W\left(\frac{\lambda-\lambda_{s_2}}{b}\right)d\lambda \\
&\times\Big\{cov\left(\left\langle Q^c_{X,\lambda_{t_1}},Q^c_{X,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{X,\lambda_{s_1}},Q^c_{X,\lambda_{s_2}}\right\rangle\right)+cov\left(\left\langle Q^c_{Y,\lambda_{t_1}},Q^c_{Y,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{Y,\lambda_{s_1}},Q^c_{Y,\lambda_{s_2}}\right\rangle\right) \\
&+cov\left(\left\langle Q^c_{X,\lambda_{t_1}},Q^c_{Y,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{X,\lambda_{s_1}},Q^c_{Y,\lambda_{s_2}}\right\rangle\right)+cov\left(\left\langle Q^c_{X,\lambda_{t_1}},Q^c_{Y,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{Y,\lambda_{s_1}},Q^c_{X,\lambda_{s_2}}\right\rangle\right) \\
&+cov\left(\left\langle Q^c_{Y,\lambda_{t_1}},Q^c_{X,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{X,\lambda_{s_1}},Q^c_{Y,\lambda_{s_2}}\right\rangle\right)+cov\left(\left\langle Q^c_{Y,\lambda_{t_1}},Q^c_{X,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{Y,\lambda_{s_1}},Q^c_{X,\lambda_{s_2}}\right\rangle\right)\Big\}.
\end{aligned}
$$

Since $(Q^c_{X,\lambda})^*=Q^c_{X,\lambda}$ (because $(I^c_{\varepsilon,\lambda})^*=I^c_{\varepsilon,\lambda}$) it follows from Lemma 1.2 that

$$
\begin{aligned}
&\sum_{\substack{t_1,t_2=-N\\t_1\neq t_2}}^{N}\sum_{\substack{s_1,s_2=-N\\s_1\neq s_2}}^{N}cov\left(\left\langle Q^c_{Y,\lambda_{t_1}},Q^c_{X,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{Y,\lambda_{s_1}},Q^c_{X,\lambda_{s_2}}\right\rangle\right) \\
=&\sum_{\substack{t_1,t_2=-N\\t_1\neq t_2}}^{N}\sum_{\substack{s_1,s_2=-N\\s_1\neq s_2}}^{N}cov\left(\left\langle Q^c_{X,\lambda_{t_1}},Q^c_{Y,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{X,\lambda_{s_1}},Q^c_{Y,\lambda_{s_2}}\right\rangle\right)
\end{aligned}
$$

and the same holds for the other two mixed terms since $W(\lambda-\lambda_{s1})W(\lambda-\lambda_{s2})=W(\lambda-\lambda_{s2})W(\lambda-\lambda_{s1})$. Therefore

$$
\begin{aligned}
&var(\sqrt{b}TL_{T,0}) \\
=&\frac{1}{b^3T^2}\sum_{\substack{t_1,t_2=-N\\t_1\neq t_2}}^{N}\sum_{\substack{s_1,s_2=-N\\s_1\neq s_2}}^{N}\int_{-\pi}^{\pi}W\left(\frac{\lambda-\lambda_{t_1}}{b}\right)W\left(\frac{\lambda-\lambda_{t_2}}{b}\right)d\lambda\int_{-\pi}^{\pi}W\left(\frac{\lambda-\lambda_{s_1}}{b}\right)W\left(\frac{\lambda-\lambda_{s_2}}{b}\right)d\lambda \\
&\times\Big\{cov\left(\left\langle Q^c_{X,\lambda_{t_1}},Q^c_{X,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{X,\lambda_{s_1}},Q^c_{X,\lambda_{s_2}}\right\rangle\right)+cov\left(\left\langle Q^c_{Y,\lambda_{t_1}},Q^c_{Y,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{Y,\lambda_{s_1}},Q^c_{Y,\lambda_{s_2}}\right\rangle\right) \\
&+4cov\left(\left\langle Q^c_{X,\lambda_{t_1}},Q^c_{Y,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{X,\lambda_{s_1}},Q^c_{Y,\lambda_{s_2}}\right\rangle\right)\Big\}.
\end{aligned}
$$

Now consider the first summand in more detail: Suppose $t_1\neq -t_2$ and $s_1\neq -s_2$, then in the cases

(a) $t_1\neq\pm s_1$, $t_2\neq\pm s_2$
(b) $t_1\neq\pm s_2$, $t_2\neq\pm s_1$

the covariance vanishes due to independence.
(i) Consider the case $t_1 = s_1$, $t_2 = s_2$ (resp. $t_1 = s_2$, $t_2 = s_1$ due to symmetry) then

$$\begin{aligned} cov\left(\left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{t_2}}\right\rangle, \left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{t_2}}\right\rangle\right) &= E\left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{t_2}}\right\rangle \left\langle Q^c_{X,\lambda_{t_2}}, Q^c_{X,\lambda_{t_1}}\right\rangle \\ &= E\left\langle Q^c_{X,\lambda_{t_1}} \otimes Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{t_2}} \otimes Q^c_{X,\lambda_{t_2}}\right\rangle \\ &= \left\langle EQ^c_{X,\lambda_{t_1}} \otimes Q^c_{X,\lambda_{t_1}}, EQ^c_{X,\lambda_{t_2}} \otimes Q^c_{X,\lambda_{t_2}}\right\rangle \end{aligned}$$

and by Theorem 1.3 and Lemma 1.3

$$\begin{aligned} & EQ^c_{X,\lambda_{t_1}} \otimes Q^c_{X,\lambda_{t_1}} \\ =& E\left[A(e^{-i\lambda_{t_1}}) \otimes_{op} A(e^{-i\lambda_{t_1}})(I^c_{\varepsilon,\lambda_{t_1}})\right] \otimes \left[A(e^{-i\lambda_{t_1}}) \otimes_{op} A(e^{-i\lambda_{t_1}})(I^c_{\varepsilon,\lambda_{t_1}})\right] \\ =& EA(e^{-i\lambda_{t_1}}) \otimes_{op} A(e^{-i\lambda_{t_1}}) I^c_{\varepsilon,\lambda_{t_1}} \otimes I^c_{\varepsilon,\lambda_{t_1}} \left(A(e^{-i\lambda_{t_1}}) \otimes_{op} A(e^{-i\lambda_{t_1}})\right)^* \\ =& A(e^{-i\lambda_{t_1}}) \otimes_{op} A(e^{-i\lambda_{t_1}}) E\left[I^c_{\varepsilon,\lambda_{t_1}} \otimes I^c_{\varepsilon,\lambda_{t_1}}\right] \left(A(e^{-i\lambda_{t_1}}) \otimes_{op} A(e^{-i\lambda_{t_1}})\right)^* \\ =& A(e^{-i\lambda_{t_1}}) \otimes_{op} A(e^{-i\lambda_{t_1}}) \frac{1}{4\pi^2} C_\varepsilon \otimes_{op} C_\varepsilon \left(A(e^{-i\lambda_{t_1}}) \otimes_{op} A(e^{-i\lambda_{t_1}})\right)^* \\ =& \frac{1}{2\pi} A(e^{-i\lambda_{t_1}}) \otimes_{op} A(e^{-i\lambda_{t_1}})(C_\varepsilon) \otimes_{op} \frac{1}{2\pi} A(e^{-i\lambda_{t_1}}) \otimes_{op} A(e^{-i\lambda_{t_1}})(C_\varepsilon) \\ =& \mathcal{F}_{X,\lambda_{t_1}} \otimes_{op} \mathcal{F}_{X,\lambda_{t_1}}, \end{aligned}$$

since $\mathrm{cum}_{op}(\varepsilon_0, \varepsilon_0, \varepsilon_0, \varepsilon_0) = 0$ (cf. Remark 1.9) and $F_n(2\lambda_{t_1}) = 0$. Thus (cf. Remark 1.2)

$$\begin{aligned} cov\left(\left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{t_2}}\right\rangle, \left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{t_2}}\right\rangle\right) &= \left\langle \mathcal{F}_{X,\lambda_{t_1}} \otimes_{op} \mathcal{F}_{X,\lambda_{t_1}}, \mathcal{F}_{X,\lambda_{t_2}} \otimes_{op} \mathcal{F}_{X,\lambda_{t_2}}\right\rangle \\ &= \left\langle \mathcal{F}_{X,\lambda_{t_1}}, \mathcal{F}_{X,\lambda_{t_2}}\right\rangle \left\langle \mathcal{F}_{X,\lambda_{t_1}}, \mathcal{F}_{X,\lambda_{t_2}}\right\rangle \\ &= \left\langle \mathcal{F}_{X,\lambda_{t_1}}, \mathcal{F}_{X,\lambda_{t_2}}\right\rangle^2. \end{aligned}$$

Combining both cases ($t_1 = s_1, t_2 = s_2$ and $t_1 = s_2, t_2 = s_1$) one obtains

$$\frac{2}{b^3T^2}\sum_{\substack{t_1,t_2=-N\\t_1\neq\pm t_2}}^{N}\left(\int_{\mathbb{R}} W\left(\frac{\lambda-\lambda_{t_1}}{b}\right) W\left(\frac{\lambda-\lambda_{t_2}}{b}\right) d\lambda\right)^2 E\left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{t_2}}\right\rangle^2$$

$$\overset{s(\lambda)=\lambda/b}{=}\frac{2}{b^3T^2}\sum_{\substack{t_1,t_2=-N\\t_1\neq\pm t_2}}^{N}\left(b\int_{\mathbb{R}} W\left(s-\frac{\lambda_{t_1}}{b}\right) W\underbrace{\left(s-\frac{\lambda_{t_2}}{b}\right)}_{=u(s)} ds\right)^2\left\langle \mathcal{F}_{X,\lambda_{t_1}}, \mathcal{F}_{X,\lambda_{t_2}}\right\rangle^2$$

$$=\frac{2}{bT^2}\sum_{\substack{t_1,t_2=-N\\t_1\neq\pm t_2}}^{N}\left(\int_{\mathbb{R}} W\left(u-\frac{2\pi(t_1-t_2)}{bT}\right) W(u)\,du\right)^2\left\langle \mathcal{F}_{X,\lambda_{t_1}}, \mathcal{F}_{X,\lambda_{t_2}}\right\rangle^2$$

$$=\frac{1}{2\pi^2}\frac{2\pi}{bT}\sum_{t_1=-2N}^{2N}\left(\int_{\mathbb{R}} W\left(u-\frac{2\pi t_1}{bT}\right) W(u)\,du\right)^2\frac{2\pi}{T}\sum_{k=0}^{2N-|t_1|}\left\langle \mathcal{F}_{X,\lambda_{t_1-N+k}}, \mathcal{F}_{X,\lambda_{-N+k}}\right\rangle^2+\mathcal{O}(1/Tb)$$

$$\xrightarrow{T\to\infty}\frac{1}{2\pi^2}\int_{-2\pi}^{2\pi}\left\{\int_{-\pi}^{\pi} W(u-x)W(u)\,du\right\}^2 dx\int_{-\pi}^{\pi}\|\mathcal{F}_{X,\lambda}\|^4_{HS}\,d\lambda.$$

(ii) Consider the case $t_1 = -s_1$, $t_2 = -s_2$ (resp. $t_1 = -s_2$, $t_2 = -s_1$ due to symmetry). With the notation of Definition 1.5 by Lemma 1.4

$$\overline{Q^c_{X,\lambda}} = \overline{A(e^{-i\lambda})(I_{\varepsilon,\lambda}-C_\varepsilon/2\pi)A(e^{-i\lambda})^*} = A(e^{i\lambda})\frac{1}{2\pi}(\overline{D}_{\varepsilon,\lambda}\otimes\overline{D}_{\varepsilon,\lambda}-C_\varepsilon)A(e^{i\lambda})^*$$
$$= A(e^{i\lambda})(I_{\varepsilon,-\lambda}-C_\varepsilon/2\pi)A(e^{i\lambda})^* = Q^c_{X,-\lambda}$$

and therefore

$$\begin{aligned}cov\left(\left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{t_2}}\right\rangle, \left\langle Q^c_{X,\lambda_{-t_1}}, Q^c_{X,\lambda_{-t_2}}\right\rangle\right) &= E\left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{t_2}}\right\rangle\left\langle\overline{Q^c_{X,\lambda_{t_2}}}, \overline{Q^c_{X,\lambda_{t_1}}}\right\rangle\\ &= E\left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{t_2}}\right\rangle\overline{\left\langle Q^c_{X,\lambda_{t_2}}, Q^c_{X,\lambda_{t_1}}\right\rangle}\\ &= E\left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{t_2}}\right\rangle^2.\end{aligned}$$

Combining both cases ($t_1 = -s_1, t_2 = -s_2$ and $t_1 = -s_2, t_2 = -s_1$) and proceeding as before yields

$$\begin{aligned}
&\frac{2}{b^3T^2}\sum_{\substack{t_1,t_2=-N\\ t_1\neq\pm t_2}}^{N}\int_{\mathbb{R}} W\left(\frac{\lambda-\lambda_{t_1}}{b}\right)W\left(\frac{\lambda-\lambda_{t_2}}{b}\right)d\lambda\int_{\mathbb{R}} W\left(\frac{\lambda+\lambda_{t_1}}{b}\right)W\left(\frac{\lambda+\lambda_{t_2}}{b}\right)d\lambda\\
&\times E\left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{t_2}}\right\rangle^2\\
=&\frac{2}{bT^2}\sum_{\substack{t_1,t_2=-N\\ t_1\neq\pm t_2}}^{N}\int_{\mathbb{R}} W\left(u-\frac{2\pi(t_1-t_2)}{bT}\right)W(u)\,du\int_{\mathbb{R}} W\left(u+\frac{2\pi(t_1-t_2)}{bT}\right)W(u)\,du\\
&\times\left\langle \mathcal{F}_{X,\lambda_{t_1}}, \mathcal{F}_{X,\lambda_{t_2}}\right\rangle^2\\
\xrightarrow{T\to\infty}&\frac{1}{2\pi^2}\int_{-2\pi}^{2\pi}\int_{-\pi}^{\pi}W(u-x)W(u)\,du\int_{-\pi}^{\pi}W(u+x)W(u)\,du\,dx\int_{-\pi}^{\pi}\|\mathcal{F}_{X,\lambda}\|_{HS}^4\,d\lambda\\
=&\frac{1}{2\pi^2}\int_{-2\pi}^{2\pi}\left\{\int_{-\pi}^{\pi}W(u-x)W(u)\,du\right\}^2dx\int_{-\pi}^{\pi}\|\mathcal{F}_{X,\lambda}\|_{HS}^4\,d\lambda
\end{aligned}$$

(iii) In the remaining cases ($t_1 = s_1, t_2 = -s_2;\ t_1 = -s_1, t_2 = s_2;\ t_1 = s_2, t_2 = -s_1;\ t_1 = -s_2, t_2 = s_1$) the covariance is bounded by

$$\left(\frac{1}{2\pi}\right)^4\left(\sum_{l\in\mathbb{Z}}\|A_l\|_{\mathcal{L}}\right)^8\|C_\varepsilon\|_{\mathcal{N}}^4 =: \kappa.$$

For example in case $t_1 = s_1, t_2 = -s_2$ (resp. $t_1 = s_2, t_2 = -s_1$ due to symmetry) we have

$$\begin{aligned}
&cov\left(\left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{t_2}}\right\rangle, \left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{-t_2}}\right\rangle\right)\\
=&\left\langle E(Q^c_{X,\lambda_{t_1}}\otimes Q^c_{X,\lambda_{t_1}}), E(Q^c_{X,\lambda_{t_2}}\otimes Q^c_{X,-\lambda_{t_2}})\right\rangle\\
=&\left\langle \mathcal{F}_{X,\lambda_{t_1}}\otimes_{op}\mathcal{F}_{X,\lambda_{t_1}}, \mathcal{F}_{X,\lambda_{t_2}}\otimes^T_{op}\mathcal{F}_{X,-\lambda_{t_2}}\right\rangle
\end{aligned}$$

since by Theorem 1.3, Lemma 1.3, Lemma 1.4 and Lemma 1.5

$$\begin{aligned}
&E(Q^c_{X,\lambda_{t_2}} \otimes Q^c_{X,-\lambda_{t_2}}) \\
=&E\left(\left[A(e^{-i\lambda_{t_2}}) \otimes_{op} A(e^{-i\lambda_{t_2}})(I^c_{\varepsilon,\lambda_{t_2}})\right]\right) \otimes \left[A(e^{i\lambda_{t_2}}) \otimes_{op} A(e^{i\lambda_{t_2}})(I^c_{\varepsilon,-\lambda_{t_2}})\right] \\
=&E\left(A(e^{-i\lambda_{t_2}}) \otimes_{op} A(e^{-i\lambda_{t_2}}) I^c_{\varepsilon,\lambda_{t_2}} \otimes I^c_{\varepsilon,-\lambda_{t_2}}\right)\left(A(e^{i\lambda_{t_2}}) \otimes_{op} A(e^{i\lambda_{t_2}})\right)^* \\
=&A(e^{-i\lambda_{t_2}}) \otimes_{op} A(e^{-i\lambda_{t_2}}) E\left[I^c_{\varepsilon,\lambda_{t_2}} \otimes I^c_{\varepsilon,-\lambda_{t_2}}\right] A(e^{i\lambda_{t_2}})^* \otimes_{op} A(e^{i\lambda_{t_2}})^* \\
=&A(e^{-i\lambda_{t_2}}) \otimes_{op} A(e^{-i\lambda_{t_2}}) \frac{1}{4\pi^2} C_\varepsilon \otimes^T_{op} C_\varepsilon A(e^{i\lambda_{t_2}})^* \otimes_{op} A(e^{i\lambda_{t_2}})^* \\
=&\frac{1}{2\pi} A(e^{-i\lambda_{t_2}}) C_\varepsilon \overline{A(e^{i\lambda_{t_2}})^*} \otimes^T_{op} \frac{1}{2\pi} \overline{A(e^{-i\lambda_{t_2}})} C_\varepsilon A(e^{i\lambda_{t_2}})^* \\
=&\frac{1}{2\pi} A(e^{-i\lambda_{t_2}}) C_\varepsilon A(e^{-i\lambda_{t_2}})^* \otimes^T_{op} \frac{1}{2\pi} A(e^{i\lambda_{t_2}}) C_\varepsilon A(e^{i\lambda_{t_2}})^* \\
=&\mathcal{F}_{X,\lambda_{t_2}} \otimes^T_{op} \mathcal{F}_{X,-\lambda_{t_2}}.
\end{aligned}$$

Therefore by Cauchy Schwarz

$$\begin{aligned}
\left|cov\left(\left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{t_2}}\right\rangle, \left\langle Q^c_{X,\lambda_{t_1}}, Q^c_{X,\lambda_{-t_2}}\right\rangle\right)\right| \leq& \|\mathcal{F}_{X,\lambda_{t_1}} \otimes_{op} \mathcal{F}_{X,\lambda_{t_1}}\| \|\mathcal{F}_{X,\lambda_{t_2}} \otimes^T_{op} \mathcal{F}_{X,-\lambda_{t_2}}\| \\
=& \|\mathcal{F}_{X,\lambda_{t_1}}\|^2 \|\mathcal{F}_{X,\lambda_{t_2}}\| \|F_{X,-\lambda_{t_2}}\|
\end{aligned}$$

and

$$\|F_{X,\lambda}\| \leq \frac{1}{2\pi} \|A(e^{-i\lambda})\|^2_{\mathcal{L}} \|C_\varepsilon\|_{\mathcal{N}} \leq \frac{1}{2\pi} \|C_\varepsilon\|_{\mathcal{N}} \left(\sum_{l\in\mathbb{Z}} \|A_l\|_{\mathcal{L}}\right)^2$$

yields the bound. Furthermore, for example in case $t_1 = s_1, t_2 = -s_2$

$$\begin{aligned}
&\frac{1}{b^3T^2} \sum_{\substack{t_1,t_2=-N \\ t_1\neq\pm t_2}}^{N} \int_{\mathbb{R}} W\left(\frac{\lambda-\lambda_{t_1}}{b}\right) W\left(\frac{\lambda-\lambda_{t_2}}{b}\right) d\lambda \int_{\mathbb{R}} W\left(\frac{\lambda-\lambda_{t_1}}{b}\right) W\left(\frac{\lambda+\lambda_{t_2}}{b}\right) d\lambda \\
&= \frac{1}{bT^2} \underbrace{\sum_{\substack{t_1,t_2=-N \\ t_1\neq\pm t_2}}^{N} \int_{\mathbb{R}} W\left(u - \frac{2\pi(t_1-t_2)}{bT}\right) W(u)\, du \int_{\mathbb{R}} W\left(u - \frac{2\pi(t_1+t_2)}{bT}\right) W(u)\, du}_{\in\mathcal{O}(b^2T^2)} \\
&\in \mathcal{O}(b),
\end{aligned}$$

hence the terms are asymptotically negletable. Proceeding similarly for the other two summands we obtain

$$\frac{1}{b^3T^2}\sum_{\substack{t_1,t_2=-N\\t_1\neq t_2}}^{N}\sum_{\substack{s_1,s_2=-N\\s_1\neq s_2}}^{N}\int_{-\pi}^{\pi}W\left(\frac{\lambda-\lambda_{t_1}}{b}\right)W\left(\frac{\lambda-\lambda_{t_2}}{b}\right)d\lambda\int_{-\pi}^{\pi}W\left(\frac{\lambda-\lambda_{s_1}}{b}\right)W\left(\frac{\lambda-\lambda_{s_2}}{b}\right)d\lambda$$
$$\times\, cov\left(\left\langle Q^c_{Y,\lambda_{t_1}},Q^c_{Y,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{Y,\lambda_{s_1}},Q^c_{Y,\lambda_{s_2}}\right\rangle\right)$$
$$\xrightarrow{T\to\infty}\frac{1}{\pi^2}\int_{-2\pi}^{2\pi}\left\{\int_{-\pi}^{\pi}W(u-x)W(u)\,du\right\}^2dx\int_{-\pi}^{\pi}\|\mathcal{F}_{Y,\lambda}\|^4_{HS}\,d\lambda$$

and

$$\frac{4}{b^3T^2}\sum_{\substack{t_1,t_2=-N\\t_1\neq t_2}}^{N}\sum_{\substack{s_1,s_2=-N\\s_1\neq s_2}}^{N}\int_{-\pi}^{\pi}W\left(\frac{\lambda-\lambda_{t_1}}{b}\right)W\left(\frac{\lambda-\lambda_{t_2}}{b}\right)d\lambda\int_{-\pi}^{\pi}W\left(\frac{\lambda-\lambda_{s_1}}{b}\right)W\left(\frac{\lambda-\lambda_{s_2}}{b}\right)d\lambda$$
$$\times\, cov\left(\left\langle Q^c_{X,\lambda_{t_1}},Q^c_{Y,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{X,\lambda_{s_1}},Q^c_{Y,\lambda_{s_2}}\right\rangle\right)$$
$$\xrightarrow{T\to\infty}\frac{2}{\pi^2}\int_{-2\pi}^{2\pi}\left\{\int_{-\pi}^{\pi}W(u-x)W(u)\,du\right\}^2dx\int_{-\pi}^{\pi}\langle\mathcal{F}_{X,\lambda},\mathcal{F}_{Y,\lambda}\rangle^2\,d\lambda.$$

Notice that for the last summand the cases $t_1=s_1,t_2=s_2$ and $t_1=s_2,t_2=s_1$ resp. $t_1=-s_1,t_2=-s_2$ and $t_1=-s_2,t_2=-s_1$ are not symmetric and due to independence

$$cov\left(\left\langle Q^c_{X,\lambda_{t_1}},Q^c_{Y,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{X,\lambda_{t_2}},Q^c_{Y,\lambda_{t_1}}\right\rangle\right)=\left\langle EQ^c_{X,\lambda_{t_1}},EQ^c_{Y,\lambda_{t_2}}\right\rangle\left\langle EQ^c_{X,\lambda_{t_2}},EQ^c_{Y,\lambda_{t_1}}\right\rangle$$
$$=0.$$

and similarly $cov\left(\left\langle Q^c_{X,\lambda_{t_1}},Q^c_{Y,\lambda_{t_2}}\right\rangle,\left\langle Q^c_{X,\lambda_{-t_2}},Q^c_{Y,\lambda_{-t_1}}\right\rangle\right)=0$.
In the case $t_1=s_1$, $t_2=s_2$ we have

$$cov(\langle Q^c_{X,\lambda_{t_1}},Q^c_{Y,\lambda_{t_2}}\rangle,\langle Q^c_{X,\lambda_{t_1}},Q^c_{Y,\lambda_{t_2}}\rangle)=E\langle Q^c_{X,\lambda_{t_1}},Q^c_{Y,\lambda_{t_2}}\rangle\langle Q^c_{Y,\lambda_{t_2}},Q^c_{X,\lambda_{t_1}}\rangle$$
$$=\langle EQ^c_{X,\lambda_{t_1}}\otimes Q^c_{X,\lambda_{t_1}},EQ^c_{Y,\lambda_{t_2}}\otimes Q^c_{Y,\lambda_{t_2}}\rangle=\ldots=\langle\mathcal{F}_{X,\lambda_{t_1}},\mathcal{F}_{Y,\lambda_{t_2}}\rangle^2$$

and similarly in the case $t_1=-s_1$, $t_2=-s_2$,

$$cov(\langle Q^c_{X,\lambda_{t_1}},Q^c_{Y,\lambda_{t_2}}\rangle,\langle Q^c_{X,\lambda_{-t_1}},Q^c_{Y,\lambda_{-t_2}}\rangle)=\langle EQ^c_{X,\lambda_{t_1}}\otimes Q^c_{X,-\lambda_{t_1}},EQ^c_{Y,\lambda_{t_2}}\otimes Q^c_{Y,-\lambda_{t_2}}\rangle$$
$$=\langle\mathcal{F}_{X,\lambda_{t_1}},\mathcal{F}_{Y,\lambda_{t_2}}\rangle\langle\mathcal{F}_{X,-\lambda_{t_1}},\mathcal{F}_{Y,-\lambda_{t_2}}\rangle=\langle\mathcal{F}_{X,\lambda_{t_1}},\mathcal{F}_{Y,\lambda_{t_2}}\rangle^2$$

Finally under $\mathcal{H}_0$ we have $\mathcal{F}_{X,\lambda}=\mathcal{F}_{Y,\lambda}$ and the assertion follows. □

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