Supplement to:
"Moving Block and Tapered Block Bootstrap for Functional Time Series with an Application to the K-Sample Mean Problem"

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This supplement contains the proofs of Lemma 5.2 and Theorem 2.2 as well as some additional numerical results.

1 Proofs of Lemma 5.2 and Theorem 2.2

Proof of Lemma 5.2. Consider (i). Note first that, using Lemma 2.1 of Hörmann and Kokoszka (2010), the sequence $\{\langle Y_i, y\rangle,\ i = 1, 2, \ldots\}$ is L^2-m-approximable, since $\sum_{m\geq 1}\left(\mathbb{E}|\langle Y_i - Y_{i,m}, y\rangle|^2\right)^{1/2} \leq \|y\|\sum_{m\geq 1}\left(\mathbb{E}\|Y_i - Y_{i,m}\|^2\right)^{1/2} < \infty$. Therefore, by Lemma 4.1 of Hörmann and Kokoszka (2010), we get that

$$\sum_{i=-\infty}^{\infty} |\mathbb{E}\langle Y_0, y\rangle\langle Y_i, y\rangle| < \infty. \tag{1.1}$$

Also, note that if $w_b(i)$ is of the form (3) of the main paper, then

$$\frac{\mathcal{W}_h}{bw * w(h/b)} \to 1,$$

where $\mathcal{W}_h = \sum_{i=1}^{b-h} w_1(i)w_b(i+h)$, $h = 0, 1, \ldots, b-1$, and $w * w$ denotes is the self-convolution of w. Therefore, since $\|w_b\|_2^2 = \mathcal{W}_0$, we get, for any fixed h, as $n \to \infty$,

$$\frac{\mathcal{W}_h}{\|w_b\|_2^2} = \frac{\mathcal{W}_h}{bw * w(h/b)}\frac{bw * w(0)}{\mathcal{W}_0}\frac{bw * w(h/b)}{bw * w(0)} \to 1. \tag{1.2}$$

Furthermore, by Cauchy-Schwarz's inequality, it is easily seen that $\sum_{i=1}^{b-h} w_b(i)w_b(i+h) \leq \sum_{i=1}^{b} w_b^2(i)$, i.e.,

$$\mathcal{W}_h \leq \|w_b\|_2^2 \quad \text{for } h = 1, 2, \ldots, b-1. \tag{1.3}$$

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To complete the proof of (i), it suffices to prove that $\sum_{h=1}^{b-1}(\mathcal{W}_h/\|w_b\|_2^2)\mathbb{E}\langle Y_0, y\rangle\langle Y_h, y\rangle \to \sum_{h=1}^{\infty}\mathbb{E}\langle Y_0, y\rangle\langle Y_h, y\rangle$. For this, and for b large enough, we use the bound

$$\begin{aligned}
&\left|\sum_{h=1}^{b-1}\frac{\mathcal{W}_h}{\|w_b\|_2^2}\mathbb{E}\langle Y_0, y\rangle\langle Y_h, y\rangle - \sum_{i=1}^{\infty}\mathbb{E}\langle Y_0, y\rangle\langle Y_i, y\rangle\right| \\
&\qquad\leq \left|\sum_{h=1}^{m}\frac{\mathcal{W}_h}{\|w_b\|_2^2}\mathbb{E}\langle Y_0, y\rangle\langle Y_h, y\rangle - \sum_{i=1}^{m}\mathbb{E}\langle Y_0, y\rangle\langle Y_i, y\rangle\right| \\
&\qquad\quad + \left|\sum_{h=m+1}^{b-1}\frac{\mathcal{W}_h}{\|w_b\|_2^2}\mathbb{E}\langle Y_0, y\rangle\langle Y_h, y\rangle - \sum_{i=m+1}^{b-1}\mathbb{E}\langle Y_0, y\rangle\langle Y_i, y\rangle\right| \\
&\qquad\quad + \left|\sum_{i=b}^{\infty}\mathbb{E}\langle Y_0, y\rangle\langle Y_i, y\rangle\right|. \qquad (1.4)
\end{aligned}$$

Because of (1.2) and (1.1), the first and the last term are $o(1)$. Concerning the second term, we show that there exists $m_0 \in \mathbb{N}$ such that

$$\limsup_{n\to\infty}\left|\sum_{h=m+1}^{b-1}\frac{\mathcal{W}_h}{\|w_b\|_2^2}\mathbb{E}\langle Y_0, y\rangle\langle Y_h, y\rangle - \sum_{i=m+1}^{b-1}\mathbb{E}\langle Y_0, y\rangle\langle Y_i, y\rangle\right| = 0$$

for $m = m_0$. By using Assumption (1) of the main paper, expression (1.3), the facts that $\mathcal{W}_h \geq 0$ and that $\langle Y_0, y\rangle$ and $\langle Y_{i,i}, y\rangle$ are independent for $i \geq m+1$, we get that, for every $\epsilon > 0$, $\exists\, m_1 \in \mathbb{N}$ such that, for every $m \geq m_1$,

$$\begin{aligned}
\left|\sum_{h=m+1}^{b-1}\left(\frac{\mathcal{W}_h}{\|w_b\|_2^2} - 1\right)\mathbb{E}(\langle Y_0, y\rangle\langle Y_h, y\rangle)\right| &\leq \sum_{i=m+1}^{\infty}|\mathbb{E}\langle Y_0, y\rangle\langle Y_i, y\rangle| \\
&= \sum_{i=m+1}^{\infty}|\mathbb{E}\langle Y_0, y\rangle\langle Y_i - Y_{i,i}, y\rangle| \\
&\leq \|y\|^2\left(\mathbb{E}\|Y_0\|^2\right)^{1/2}\sum_{i=m+1}^{\infty}\left(\mathbb{E}\|Y_i - Y_{i,i}\|^2\right)^{1/2} < \epsilon, \qquad (1.5)
\end{aligned}$$

because of expression (3) of the main paper.

Consider next assertion (ii). Notice first that,

$$\iint\left\{\frac{1}{n}\sum_{t=1}^{n}Y_t(u)Y_t(v) - \mathbb{E}[Y_0(u)Y_0(v)]\right\}^2 = o_P(1).$$

Hence, and since the summands of $Y_i(u)Y_{i+h}(v)$ and $Y_{i+h}(v)Y_i(u)$ can be handled similarly, it suffices to show that

$$\iint\left\{\sum_{h=1}^{b-1}\frac{\mathcal{W}_h}{\|w_b\|_2^2}\frac{1}{n}\sum_{t=1}^{n-h}Y_t(u)Y_{t+h}(v) - \sum_{t\geq 1}\mathbb{E}[Y_0(u)Y_t(v)]\right\}^2 = o_P(1). \qquad (1.6)$$

By expressions (1.2) and (1.3), the proof of (1.6) is analogous to the proof of (A.2) of Horváth *et al.* (2013). This completes the proof of the lemma.

Proof of Theorem 2.2. Let $S_n^* = \sqrt{n}(\overline{X}_n^* - \mathbb{E}^*(\overline{X}_n^*))$ and, as in Theorem 2.1 of the main paper, we have that $S_n^* = k^{-1/2}\sum_{i=1}^{k}[U_i^* - \mathbb{E}^*(U_i^*)]$, where $U_i^* = b^{-1/2}(X^*_{(i-1)b+1} + X^*_{(i-1)b+2} + \ldots + X^*_{ib})$, $i = 1, 2, \ldots, k$, are i.i.d. random variables, $\langle S_n^*, y\rangle = k^{-1/2}\sum_{i=1}^{k}[W_i^* - \mathbb{E}^*(W_i^*)]$ with $W_i^* = \langle U_i^*, y\rangle$, $i = 1, 2, \ldots, k$, and $\mu^* = \mathbb{E}^*(W_1^*)$. Let C be the covariance operator with kernel

$$c(u, v) = \mathbb{E}[Y_0(u)Y_0(v)] + \sum_{h\geq 1}\mathbb{E}[Y_0(u)Y_h(v)] + \sum_{h\geq 1}\mathbb{E}[Y_0(v)Y_h(u)], \quad u, v \in [0, 1]^2,$$

$N = n - b + 1$, $\|w_b\|_1 = \sum_{i=1}^{b} w_b(t)$ and $\|w_b\|_2^2 = \sum_{t=1}^{b} w_b^2(t)$. Finally, let $X_i = Y_i - \overline{Y}_n$, $i = 1, 2, \ldots, n$, and

$$U_i = \frac{1}{\|w_b\|_2}\left(w_b(1)X_i + w_b(2)X_{i+1}, \ldots + w_b(b)X_{i+b-1}\right), \quad i = 1, 2, \ldots, N.$$

It suffices to prove that

(L1) $\langle S_n^*, y\rangle \xrightarrow{d} N(0, \sigma^2(y))$ for every $y \in L^2$, where $\sigma^2(y) = \langle C(y), y\rangle$, and that

(L2) the sequence $\{S_n^*, n \in \mathbb{N}\}$ is tight.

To prove (L1), we establish that, as $n \to \infty$,

$$\mathrm{Var}^*\left(\langle S_n^*, y\rangle\right) \xrightarrow{P} \sigma^2(y) \tag{1.7}$$

and that

$$\frac{\langle S_n^*, y\rangle}{\sqrt{\mathrm{Var}^*(\langle S_n^*, y\rangle)}} \xrightarrow{d} N(0, 1). \tag{1.8}$$

To see (1.7), note first that $\mathrm{Var}^*\left(\langle S_n^*, y\rangle\right) = k^{-1}\sum_{i=1}^{k}\mathrm{Var}^*(W_i^* - \mathbb{E}^*(W_i^*)) = \mathrm{Var}^*(W_1^*)$ and that

$$\begin{aligned}
\mathrm{Var}^*(W_1^*) &= \frac{1}{N}\sum_{i=1}^{N}\left[\langle U_i, y\rangle - \frac{1}{N}\sum_{j=1}^{N}\langle U_j, y\rangle\right]^2 \\
&= \frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{\|w_b\|_2}\sum_{t=1}^{b} w_b(t)\langle Y_{i+t-1}, y\rangle\right]^2 - \left[\frac{1}{N}\sum_{j=1}^{N}\frac{1}{\|w_b\|_2}\sum_{s=1}^{b} w_b(s)\langle Y_{j+s-1}, y\rangle\right]^2. \qquad (1.9)
\end{aligned}$$

We next show that

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\|w_b\|_2}\sum_{s=1}^{b} w_b(s)\langle Y_{i+s-1}, y\rangle = O_p(\frac{b}{\sqrt{n}}). \tag{1.10}$$

Toward this, note that

$$\frac{1}{N}\sum_{i=1}^{N}\frac{1}{\|w_b\|_2}\sum_{s=1}^{b} w_b(s)\langle Y_{i+s-1}, y\rangle$$

$$= \frac{1}{N}\frac{\|w_b\|_1}{\|w_b\|_2}\left[\sum_{i=1}^{n}\langle Y_i, y\rangle - \sum_{j=1}^{b-1}\left(1-\frac{\sum_{s=1}^{j} w_b(s)}{\|w_b\|_1}\right)\langle Y_j, y\rangle - \sum_{j=1}^{b-1}\left(1-\frac{\sum_{t=b-j+1}^{b} w_b(t)}{\|w_b\|_1}\right)\langle Y_{n-j+1}, y\rangle\right], \tag{1.11}$$

and that

$$\mathbb{E}\left[\frac{1}{N}\sum_{j=1}^{N}\frac{1}{\|w_b\|_2}\sum_{s=1}^{b} w_b(s)\langle Y_{i+s-1}, y\rangle\right] = 0. \tag{1.12}$$

Furthermore, using the decomposition

$$\begin{aligned}
&\left[\sum_{i=1}^{n}\langle Y_i, y\rangle - \sum_{j=1}^{b-1}\left(1-\frac{\sum_{s=1}^{j} w_b(s)}{\|w_b\|_1}\right)\langle Y_j, y\rangle - \sum_{j=1}^{b-1}\left(1-\frac{\sum_{t=b-j+1}^{b} w_b(t)}{\|w_b\|_1}\right)\langle Y_{n-j+1}, y\rangle\right]^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\langle Y_i, y\rangle\langle Y_j, y\rangle + \sum_{i=1}^{b-1}\sum_{j=1}^{b-1}\left(1-\frac{\sum_{t=1}^{i} w_b(t)}{\|w_b\|_1}\right)\left(1-\frac{\sum_{s=1}^{j} w_b(s)}{\|w_b\|_1}\right)\langle Y_i, y\rangle\langle Y_j, y\rangle \\
&+ \sum_{i=1}^{b-1}\sum_{j=1}^{b-1}\left(1-\frac{\sum_{s=b-i+1}^{b} w_b(s)}{\|w_b\|_1}\right)\left(1-\frac{\sum_{t=b-j+1}^{b} w_b(t)}{\|w_b\|_1}\right)\langle Y_{n-i+1}, y\rangle\langle Y_{n-j+1}, y\rangle \\
&- 2\sum_{i=1}^{b-1}\sum_{j=1}^{b-1}\left(1-\frac{\sum_{s=1}^{j} w_b(s)}{\|w_b\|_1}\right)\left(1-\frac{\sum_{t=b-i+1}^{b} w_b(t)}{\|w_b\|_1}\right)\langle Y_{n-i+1}, y\rangle\langle Y_j, y\rangle \\
&- 2\sum_{i=1}^{n}\sum_{j=1}^{b-1}\left(1-\frac{\sum_{s=1}^{j} w_b(s)}{\|w_b\|_1}\right)\langle Y_i, y\rangle\langle Y_j, y\rangle \\
&\qquad - 2\sum_{i=1}^{n}\sum_{j=1}^{b-1}\left(1-\frac{\sum_{t=b-j+1}^{b} w_b(t)}{\|w_b\|_1}\right)\langle Y_i, y\rangle\langle Y_{n-j+1}, y\rangle,
\end{aligned} \tag{1.13}$$

we get, by equation (1.11), the fact that $\|w_b\|_2 = O(b^{1/2})$, $\|w_b\|_1 = O(b)$ and the same arguments as those used to obtain equation (24) of the main paper, that

$$\begin{aligned}
\mathbb{E}\left[\frac{1}{N}\sum_{j=1}^{N}\frac{1}{\|w_b\|_2}\sum_{s=1}^{b} w_b(s)\langle Y_{i+s-1}, y\rangle\right]^2 &= \frac{\|w_b\|_1^2}{N^2\|w_b\|_2^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}[\langle Y_i, y\rangle\langle Y_j, y\rangle] + O(b^2/n) \\
&= O(b/n) + O(b^2/n) = O(b^2/n).
\end{aligned} \tag{1.14}$$

From (1.12) and (1.14), assertion (1.10) follows. Consider next the first term of the right hand side of equation (1.9). For this, we have

$$\begin{aligned}
&\frac{1}{N}\sum_{i=1}^{N}\left[\frac{1}{\|w_b\|_2}\sum_{t=1}^{b} w_b(t)\langle Y_{i+t-1}, y\rangle\right]^2 \\
&\quad = \frac{1}{N}\frac{1}{\|w_b\|_2^2}\left\{\sum_{i=1}^{n}\|w_b\|_2^2\langle Y_i, y\rangle\langle Y_i, y\rangle + \sum_{h=1}^{b-1}\mathcal{W}_h\sum_{i=1}^{n-h}[\langle Y_i, y\rangle\langle Y_{i+h}, y\rangle + \langle Y_{i+h}, y\rangle\langle Y_i, y\rangle]\right. \\
&\quad - \sum_{s=1}^{b-1}\left(\|w_b\|_2^2 - \sum_{t=1}^{s} w_b^2(t)\right)\langle Y_s, y\rangle\langle Y_s, y\rangle - \sum_{s=1}^{b-1}\left(\|w_b\|_2^2 - \sum_{t=b-s+1}^{b} w_b^2(t)\right)\langle Y_{n-s+1}, y\rangle\langle Y_{n-s+1}, y\rangle
\end{aligned}$$

$$
- \sum_{h=1}^{b-1} \sum_{i=1}^{b-h} \left(\mathcal{W}_h - \sum_{t=1}^{i} w_b(t) w_b(t+h) \right) [\langle Y_i, y \rangle \langle Y_{i+h}, y \rangle + \langle Y_{i+h}, y \rangle \langle Y_i, y \rangle]
$$

$$
- \sum_{h=1}^{b-1} \sum_{i=1}^{b-h} \left(\mathcal{W}_h - \sum_{t=b-i-h+1}^{b-h} w_b(t) w_b(t+h) \right) [\langle Y_{n-i+1}, y \rangle \langle Y_{n-i+1-h}, y \rangle
$$

$$
\left. + \langle Y_{n-i+1-h}, y \rangle \langle Y_{n-i+1}, y \rangle] \right\},
$$

from which it follows that

$$
\frac{1}{N} \sum_{i=1}^{N} \left[\frac{1}{\|w_b\|_2} \sum_{t=1}^{b} w_b(t) \langle Y_{i+t-1}, y \rangle \right]^2
$$

$$
= \frac{1}{N} \sum_{i=1}^{n} \langle Y_i, y \rangle \langle Y_i, y \rangle + \sum_{h=1}^{b-1} \frac{\mathcal{W}_h}{\|w_b\|_2^2} \frac{1}{N} \sum_{i=1}^{n-h} [\langle Y_i, y \rangle \langle Y_{i+h}, y \rangle + \langle Y_{i+h}, y \rangle \langle Y_i, y \rangle]
$$

$$
+ O_p(b/n) + O_p(b^2/n).
$$

Hence, using expressions (1.9) and (1.10), we get,

$$
\mathrm{Var}^*(W_1^*) = \iint \tilde{c}_N(u, v) y(u) y(v) \mathrm{d}u \mathrm{d}v + O_p(b^2/n), \tag{1.15}
$$

where

$$
\tilde{c}_N(u, v) = \frac{1}{N} \sum_{i=1}^{n} Y_i(u) Y_i(v) + \sum_{h=1}^{b-1} \frac{\mathcal{W}_h}{\|w_b\|_2^2} \frac{1}{N} \sum_{i=1}^{n-h} [Y_i(u) Y_{i+h}(v) + Y_{i+h}(u) Y_i(v)]. \tag{1.16}
$$

Using Lemma 5.2 (ii) of the main paper and Cauchy-Schwarz's inequality, we conclude that, as $n \to \infty$,

$$
\left| \iint (\tilde{c}_n(u, v) - c(u, v)) y(u) y(v) \mathrm{d}u \mathrm{d}v \right| \leq \left(\iint \{\tilde{c}_n(u, v) - c(u, v)\}^2 \mathrm{d}u \mathrm{d}v \right)^{1/2} \|y\|^2 = o_P(1). \tag{1.17}
$$

where $\tilde{c}_n(u, v) = (N/n) \tilde{c}_N(u, v)$. Thus,

$$
\iint \tilde{c}_n(u, v) y(u) y(v) \mathrm{d}u \mathrm{d}v \xrightarrow{P} \iint c(u, v) y(u) y(v) \mathrm{d}u \mathrm{d}v
$$

and, using equation (1.15),

$$
\mathrm{Var}^* \langle S_n^*, y \rangle = \frac{n}{N} \iint c_n(u, v) y(u) y(v) \mathrm{d}u \mathrm{d}v + O_p(b^2/n)
$$

$$
\xrightarrow{P} \iint c(u, v) y(u) y(v) \mathrm{d}u \mathrm{d}v = \sigma^2(y), \tag{1.18}
$$

as $n \to \infty$. To prove (1.8), as stated in the proof of Theorem 2.1 of the main paper, we must establish Lindeberg's condition.

For this, let $W_i = \langle U_i, y \rangle$, $i = 1, 2, \ldots, n$, and note that, by (1.9), we have

$$
W_i - \mu^* = \langle U_i, y \rangle - \frac{1}{N} \sum_{j=1}^{N} \langle U_j, y \rangle
$$

$$
\begin{aligned}
&= \frac{1}{\|w_b\|_2} \sum_{t=1}^{b} w_b(t)\langle X_{i+t-1}, y\rangle - \frac{1}{N}\sum_{j=1}^{N} \frac{1}{\|w_b\|_2}\sum_{s=1}^{b} w_b(s)\langle X_{j+s-1}, y\rangle \\
&= \frac{1}{\|w_b\|_2} \sum_{t=1}^{b} w_b(t)\langle Y_{i+t-1}, y\rangle - \frac{1}{N}\sum_{j=1}^{N} \frac{1}{\|w_b\|_2}\sum_{s=1}^{b} w_b(s)\langle Y_{j+s-1}, y\rangle \\
&= W_i^Y - \frac{1}{N}\sum_{j=1}^{N} W_j^Y = W_i^Y - \mu_Y^*, \qquad (1.19)
\end{aligned}
$$

with an obvious notation for W_i^Y and μ_Y^*. Hence, using (33) of the main paper and Markov's inequality, we have, for any $\delta > 0$ and for any $\varepsilon > 0$, that

$$
\begin{aligned}
P&\left(\frac{1}{k}\sum_{t=1}^{k} \mathbb{E}^*\left[(W_t^* - \mu^*)^2 \mathbb{1}(|W_t^* - \mu^*| > \varepsilon\tau_k^*)\right] > \delta\right) \\
&\leq \delta^{-1}\mathbb{E}\left\{\mathbb{E}^*\left[(W_1^* - \mu^*)^2 \mathbb{1}(|W_1^* - \mu^*| > \varepsilon\tau_k^*)\right]\right\} \\
&= \delta^{-1}\mathbb{E}\left[(W_1^Y - \mu_Y^*)^2 \mathbb{1}(|W_1^Y - \mu_Y^*| > \varepsilon\tau_k^*)\right] \\
&\leq 4\delta^{-1}\left[\mathbb{E}(W_1^Y)^2 \mathbb{1}(|W_1^Y| > \varepsilon\tau_k^*/2) + \mathbb{E}(\mu_Y^*)^2 \mathbb{1}(|\mu_Y^*| > \varepsilon\tau_k^*/2)\right] \\
&\leq 4\delta^{-1}\left[\mathbb{E}(W_1^Y)^2 \mathbb{1}(|W_1^Y| > \varepsilon\tau_k^*/2) + \mathbb{E}(\mu_Y^*)^2\right]. \qquad (1.20)
\end{aligned}
$$

Since $\mathbb{E}(W_1^Y)^2 = \sum_{|h|<b}\left(\frac{\mathcal{W}_{|h|}}{\|w_b\|_2^2}\right) E[\langle Y_0, y\rangle\langle Y_h, y\rangle]$, we get, by Lemma 5.2 (i) of the main paper that,

$$
\mathbb{E}(W_1^Y)^2 \xrightarrow{P} \iint c(u,v)y(u)y(v)\mathrm{d}u\mathrm{d}v,
$$

and, by the dominated convergence theorem, that $\lim_{n\to\infty} \mathbb{E}(W_1^Y)^2 \mathbb{1}(|W_1^Y| > \varepsilon\tau_k^*/2) = 0$. Using this result and expression (1.14), it follows that the bound in (1.20) converges to 0 as $n \to \infty$, which establishes Lindeberg's condition.

Consider now $(L2)$. For this, it suffices to verify that conditions (a)-(e) of Theorem 2.1 of the main paper are satisfied. Note that, by letting $y = e_j$ in expression (1.18), property (b) follows with $\Sigma_j = \iint c(u,v)e_j(u)e_j(v)\mathrm{d}u\mathrm{d}v$. To prove (c), note that, by Proposition 6 of Hörmanm *et al.* (2015), since the stochastic process $\{Y_t, t \in \mathbb{Z}\}$ is L^2-m-approximable, the covariance operator C with kernel $c(\cdot,\cdot)$ is trace class. Therefore, $\sum_{j\geq 1}\Sigma_j = \sum_{j\geq 1}\iint c(u,v)e_j(u)e_j(v)\mathrm{d}u\mathrm{d}v = \sum_{j\geq 1}\lambda_j < \infty$, where λ_j, $j \geq 1$ are the eigenvalues of the covariance operator C. To establish (d), let first

$$
U_i^Y = \frac{1}{\|w_b\|_2}\left(w_b(1)Y_i + w_b(2)Y_{i+1}, \ldots + w_b(b)Y_{i+b-1}\right), \quad i = 1, 2, \ldots, N.
$$

Then, using equation (1.9), we have

$$
\mathrm{Var}^*(\langle U_1^*, e_j\rangle) = \frac{1}{N}\sum_{i=1}^{N}\langle U_i^Y, e_j\rangle^2 - \left[\frac{1}{N}\sum_{i=1}^{N}\langle U_i^Y, e_j\rangle\right]^2. \qquad (1.21)
$$

From expressions (1.11) and (1.13), we get,

$$
\sum_{j\geq 1}\left[\frac{1}{N}\sum_{i=1}^{N}\langle U_i^Y, e_j\rangle\right]^2
$$

$$
\begin{aligned}
&= \frac{1}{N^2}\frac{\|w_b\|_1^2}{\|w_b\|_2^2}\Bigg[\sum_{j\geq 1}\sum_{i=1}^{n}\sum_{t=1}^{n}\langle Y_i, e_j\rangle\langle Y_t, e_j\rangle \\
&\quad + \sum_{j\geq 1}\sum_{i=1}^{b-1}\sum_{t=1}^{b-1}\left(1-\frac{\sum_{s=1}^{i} w_b(s)}{\|w_b\|_1}\right)\left(1-\frac{\sum_{s=1}^{j} w_b(s)}{\|w_b\|_1}\right)\langle Y_i, e_j\rangle\langle Y_s, e_j\rangle \\
&\quad + \sum_{j\geq 1}\sum_{i=1}^{b-1}\sum_{s=1}^{b-1}\left(1-\frac{\sum_{t=b-i+1}^{b} w_b(t)}{\|w_b\|_1}\right)\left(1-\frac{\sum_{t=b-j+1}^{b} w_b(t)}{\|w_b\|_1}\right)\langle Y_{n-i+1}, e_j\rangle\langle Y_{n-s+1}, e_j\rangle \\
&\quad - 2\sum_{j\geq 1}\sum_{i=1}^{b-1}\sum_{s=1}^{b-1}\left(1-\frac{\sum_{t=1}^{j} w_b(t)}{\|w_b\|_1}\right)\left(1-\frac{\sum_{t=b-i+1}^{b} w_b(t)}{\|w_b\|_1}\right)\langle Y_{n-i+1}, e_j\rangle\langle Y_s, e_j\rangle \\
&\quad - 2\sum_{j\geq 1}\sum_{i=1}^{n}\sum_{s=1}^{b-1}\left(1-\frac{\sum_{t=1}^{j} w_b(t)}{\|w_b\|_1}\right)\langle Y_i, e_j\rangle\langle Y_s, e_j\rangle \\
&\quad - 2\sum_{j\geq 1}\sum_{i=1}^{n}\sum_{s=1}^{b-1}\left(1-\frac{\sum_{t=b-s+1}^{b} w_b(t)}{\|w_b\|_1}\right)\langle Y_i, e_j\rangle\langle Y_{n-s+1}, e_j\rangle\Bigg]. \qquad (1.22)
\end{aligned}
$$

Hence, and because $\langle x, y\rangle = \sum_{j\geq 1}\langle x, e_j\rangle\langle y, e_j\rangle$,

$$
\begin{aligned}
&\sum_{j\geq 1}\left[\frac{1}{N}\sum_{i=1}^{N}\langle U_i^Y, e_j\rangle\right]^2 \\
&\quad = \frac{1}{N^2}\frac{\|w_b\|_1^2}{\|w_b\|_2^2}\Bigg\{\sum_{i=1}^{n}\sum_{t=1}^{n}\langle Y_i, Y_t\rangle + \sum_{i=1}^{b-1}\sum_{t=1}^{b-1}\left(1-\frac{\sum_{s=1}^{i} w_b(s)}{\|w_b\|_1}\right)\left(1-\frac{\sum_{s=1}^{j} w_b(s)}{\|w_b\|_1}\right)\langle Y_i, Y_s\rangle \\
&\qquad + \sum_{i=1}^{b-1}\sum_{s=1}^{b-1}\left(1-\frac{\sum_{t=b-i+1}^{b} w_b(t)}{\|w_b\|_1}\right)\left(1-\frac{\sum_{t=b-j+1}^{b} w_b(t)}{\|w_b\|_1}\right)\langle Y_{n-i+1}, Y_{n-s+1}\rangle \\
&\qquad - 2\sum_{i=1}^{b-1}\sum_{s=1}^{b-1}\left(1-\frac{\sum_{t=1}^{j} w_b(t)}{\|w_b\|_1}\right)\left(1-\frac{\sum_{t=b-i+1}^{b} w_b(t)}{\|w_b\|_1}\right)\langle Y_{n-i+1}, Y_s\rangle \\
&\qquad - 2\sum_{i=1}^{n}\sum_{s=1}^{b-1}\left(1-\frac{\sum_{t=1}^{j} w_b(t)}{\|w_b\|_1}\right)\langle Y_i, Y_s\rangle \\
&\qquad - 2\sum_{i=1}^{n}\sum_{s=1}^{b-1}\left(1-\frac{\sum_{t=b-s+1}^{b} w_b(t)}{\|w_b\|_1}\right)\langle Y_i, Y_{n-s+1}\rangle\Bigg\}.
\end{aligned}
$$

Therefore, by using (40) of the main paper, we get

$$
\sum_{j\geq 1}\left[\frac{1}{N}\sum_{i=1}^{N}\langle U_i^Y, e_j\rangle\right]^2 = \frac{1}{N^2}\frac{\|w_b\|_1^2}{\|w_b\|_2^2}\sum_{i=1}^{n}\sum_{t=1}^{n}\langle Y_i, Y_t\rangle + O_P(b^2/n) = O_p(b^2/n) = o_p(1). \qquad (1.23)
$$

Consider now, the first term of the right hand side of expression (1.21). By Parseval's identity,

$$
\sum_{j\geq 1}\frac{1}{N}\sum_{i=1}^{N}\langle U_i^Y, e_j\rangle^2 = \frac{1}{N}\sum_{i=1}^{N}\|U_i^Y\|^2
$$

$$
\begin{aligned}
&= \frac{1}{N}\frac{1}{\|w_b\|_2^2}\Bigg\{ \sum_{i=1}^{n} \|w_b\|_2^2 \langle Y_i, Y_i\rangle + \sum_{h=1}^{b-1} \mathcal{W}_h \sum_{i=1}^{n-h} [\langle Y_i, Y_{i+h}\rangle + \langle Y_{i+h}, Y_i\rangle] \\
&\quad - \sum_{s=1}^{b-1} \left(\|w_b\|_2^2 - \sum_{t=1}^{s} w_b^2(t) \right) \langle Y_s, Y_s\rangle \\
&\quad - \sum_{s=1}^{b-1} \left(\|w_b\|_2^2 - \sum_{t=b-s+1}^{b} w_b^2(t) \right) \langle Y_{n-s+1}, Y_{n-s+1}\rangle \\
&\quad - \sum_{h=1}^{b-1} \sum_{i=1}^{b-h} \left(\mathcal{W}_h - \sum_{t=1}^{i} w_b(t) w_b(t+h) \right) [\langle Y_i, Y_{i+h}\rangle + \langle Y_{i+h}, Y_i\rangle] \\
&\quad - \sum_{h=1}^{b-1} \sum_{i=1}^{b-h} \left(\mathcal{W}_h - \sum_{t=b-i-h+1}^{b-h} w_b(t) w_b(t+h) \right) [\langle Y_{n-i+1}, Y_{n-i+1-h}\rangle \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \langle Y_{n-i+1-h}, Y_{n-i+1}\rangle] \Bigg\}.
\end{aligned}
$$

Hence,

$$
\sum_{j\geq 1} \frac{1}{N} \sum_{i=1}^{N} \langle U_i^Y, e_j\rangle^2 = \frac{1}{N}\sum_{i=1}^{n} \langle Y_i, Y_i\rangle + \sum_{h=1}^{b-1} \frac{\mathcal{W}_h}{\|w_b\|_2^2} \frac{1}{N} \sum_{i=1}^{n-h} \langle Y_i, Y_{i+h}\rangle + \langle Y_{i+h}, Y_i\rangle + O_P(b^2/n), \tag{1.24}
$$

and because $N/n \to 1$ as $n \to \infty$ and taking $g_b(h) = \dfrac{\mathcal{W}_{|h|}}{\mathcal{W}_0}$ in Lemma 5.1 of the main paper, in conjunction with expressions (1.2) and (1.3), we get, as $n \to \infty$, that $\sum_{j\geq 1} \dfrac{1}{N} \sum_{i=1}^{N} \langle U_i^Y, e_j\rangle^2 \stackrel{P}{\to} \sum_{i=-\infty}^{\infty} \mathbb{E}(\langle Y_0, Y_i\rangle)$. Thus, using (1.21) and (1.23), we conclude that

$$
\sum_{j\geq 1} \mathrm{Var}^*(\langle U_1^*, e_j\rangle) \stackrel{P}{\to} \sum_{i=-\infty}^{\infty} \mathbb{E}(\langle Y_0, Y_i\rangle) \tag{1.25}
$$

and, using $\sum_{i=-\infty}^{\infty} \mathbb{E}(\langle Y_0, Y_i\rangle) = \sum_{j\geq 1} \lambda_j$, property (d) is established. Finally, (e) is proved using the same arguments as in the corresponding case in Theorem 2.1 of the main paper, and taking into account expressions (1.21), (1.24) and (1.25).

Consider next assertion (ii) of the theorem. It suffices to prove that, as $n \to \infty$, $\|n\mathbb{E}^*(\overline{X}_n^* - \mathbb{E}^*(\overline{X}_n^*)) \otimes (\overline{X}_n^* - \mathbb{E}^*(\overline{X}_n^*)) - 2\pi\mathcal{F}_0\|_{HS} = o_P(1)$. Notice that $n\mathbb{E}^*(\overline{X}_n^* - \mathbb{E}^*(\overline{X}_n^*)) \otimes (\overline{X}_n^* - \mathbb{E}^*(\overline{X}_n^*))$ is an integral operator with kernel

$$
\begin{aligned}
\tilde{d}(u,v) &= \mathbb{E}^*[U_1^*(u) - \mathbb{E}^*(U_1^*(u))][U_1^*(v) - \mathbb{E}^*(U_1^*(v))] \\
&= \frac{1}{N}\sum_{i=1}^{N} U_i^Y(u) U_i^Y(v) - \left(\frac{1}{N}\sum_{j=1}^{N} U_j^Y(u)\right)\left(\frac{1}{N}\sum_{j=1}^{N} U_j^Y(v)\right).
\end{aligned} \tag{1.26}
$$

Now,

$$
\frac{1}{N}\sum_{i=1}^{N} U_i^Y(u) U_i^Y(v)
$$

$$= \frac{1}{N}\frac{1}{\|w_b\|_2^2}\Bigg\{\sum_{i=1}^{n}\|w_b\|_2^2 Y_i(u)Y_i(v) + \sum_{h=1}^{b-1}\mathcal{W}_h\sum_{i=1}^{n-h}[Y_i(u)Y_{i+h}(v)+Y_{i+h}(u)Y_i(v)]$$

$$-\sum_{s=1}^{b-1}\left(\|w_b\|_2^2-\sum_{t=1}^{s}w_b^2(t)\right)Y_s(u)Y_s(v)-\sum_{s=1}^{b-1}\left(\|w_b\|_2^2-\sum_{t=b-s+1}^{b}w_b^2(t)\right)Y_{n-s+1}(u)Y_{n-s+1}(v)$$

$$-\sum_{h=1}^{b-1}\sum_{i=1}^{b-h}\left(\mathcal{W}_h-\sum_{t=1}^{i}w_b(t)w_b(t+h)\right)[Y_i(u)Y_{i+h}(v)+Y_{i+h}(u)Y_i(v)]$$

$$-\sum_{h=1}^{b-1}\sum_{i=1}^{b-h}\left(\mathcal{W}_h-\sum_{t=b-i-h+1}^{b-h}w_b(t)w_b(t+h)\right)[Y_{n-i+1}(u)Y_{n-i+1-h}(v)+Y_{n-i+1-h}(v)Y_{n-i+1}(u)]\Bigg\}$$

and

$$\frac{1}{N}\sum_{i=1}^{N}U_i^Y(u)\frac{1}{N}\sum_{j=1}^{N}U_j^Y(v)$$

$$= \frac{1}{N^2}\frac{\|w_b\|_1^2}{\|w_b\|_2^2}\Bigg\{\sum_{i=1}^{n}\sum_{j=1}^{n}Y_i(u)Y_j(v)+\sum_{i=1}^{b-1}\sum_{j=1}^{b-1}\left(1-\frac{\sum_{t=1}^{i}w_b(t)}{\|w_b\|_1}\right)\left(1-\frac{\sum_{s=1}^{j}w_b(s)}{\|w_b\|_1}\right)Y_i(u)Y_j(v)$$

$$+\sum_{i=1}^{b-1}\sum_{j=1}^{b-1}\left(1-\frac{\sum_{s=b-i+1}^{b}w_b(s)}{\|w_b\|_1}\right)\left(1-\frac{\sum_{t=b-j+1}^{b}w_b(t)}{\|w_b\|_1}\right)Y_{n-i+1}(u)Y_{n-j+1}(v)$$

$$-\sum_{i=1}^{b-1}\sum_{j=1}^{b-1}\left(1-\frac{\sum_{s=1}^{j}w_b(s)}{\|w_b\|_1}\right)\left(1-\frac{\sum_{t=b-i+1}^{b}w_b(t)}{\|w_b\|_1}\right)[Y_{n-i+1}(u)Y_j(v)+Y_{n-i+1}(v)Y_j(u)$$

$$-\sum_{i=1}^{n}\sum_{j=1}^{b-1}\left(1-\frac{\sum_{s=1}^{j}w_b(s)}{\|w_b\|_1}\right)[Y_i(u)Y_j(v)+Y_j(u)Y_i(v)]$$

$$-\sum_{i=1}^{n}\sum_{j=1}^{b-1}\left(1-\frac{\sum_{t=b-j+1}^{b}w_b(t)}{\|w_b\|_1}\right)[Y_i(u)Y_{n-j+1}(v)+Y_i(v)Y_{n-j+1}(u)]\Bigg\}.$$

Therefore, $\tilde{d}(u,v)=\tilde{c}_N(u,v)+\tilde{R}(u,v)$ where $\tilde{c}_N(u,v)$ is defined in (1.16) and $\tilde{R}(u,v)$ is the remainder term, and

$$\|n\mathbb{E}^*(\overline{X}_n^*-\mathbb{E}^*(\overline{X}_n^*))\otimes(\overline{X}_n^*-\mathbb{E}^*(\overline{X}_n^*))-2\pi\mathcal{F}_0\|_{HS}$$
$$\leq 2\iint[\tilde{c}_N(u,v)-c(u,v)]^2\mathrm{d}u\mathrm{d}v+2\iint[\tilde{R}_N(u,v))]^2\mathrm{d}u\mathrm{d}v.$$

Using similar arguments as those used in the proof of assertion (ii) of Theorem 2.1 of the main paper, it follows that $\iint[\tilde{R}(u,v))]^2\mathrm{d}u\mathrm{d}v=o_p(1)$, from which assertion (ii) follows because of (1.17). □

2 Estimating the standard deviation of the mean function estimator

Recall that realizations of length $n=100$ and $n=500$ from the functional time series models (4) with errors following either the FAR(1) model (7) or the FMA(1) model (8) of the main paper have been generated and the standard deviation, $\sigma(\tau) = \sqrt{c(\tau,\tau)}$ of the normalized sample mean $\sqrt{n}\overline{X}_n(\tau) = (1/\sqrt{n})\sum_{i=1}^n X_i(\tau)$ has been estimated, over a set of $\tau \in \mathcal{I}$, using the MBB, the TBB and the SB procedures. The exact standard deviation has been estimated using $100,000$ replications of the models considered. $R = 1000$ replications of each data generating process have been used where, for each replication, $B = 1000$ bootstrap pseudo-time series have been generated in order to evaluate the bootstrap estimators.

Since the results of both block bootstrap methods are, for small sample sizes, sensitive with respect to the choice of the block size b, we first present some simulations results which demonstrate the capabilities of these block bootstrap methods for functional time series. For this, we present, in some sense, the less biased results that can been obtained using the three different block bootstrap methods. That is, we present the results obtained when the block size b used has been selected as the one which minimizes the absolute averaged relative bias $T^{-1}\sum_{i=1}^T \left|\sigma^*_{j,b}(\tau_i)/\sigma(\tau_i) - 1\right|$ for $j = 1, 2$. Here, $\sigma^*_{1,b}(\tau)$ and $\sigma^*_{2,b}(\tau)$ denote the MBB and TBB estimators of $\sigma(\tau)$, respectively, using the block size b. The same criterion has been used to choose the "best" probability p of the geometric distribution involved in the SB procedure i.e., the one which leads to the smallest overall in the sense described above. For the FAR(1) model and for $n = 100$, the block sizes selected using the described procedure were $b = 5$, $b = 8$ and $p = 0.25$ for the MBB, the TBB and the SB procedure, respectively. For $n = 500$, the corresponding values were $b = 10$, $b = 18$ and $p = 0.1$. For the FMA(1) model, for $n = 100$ and $n = 500$, we obtained the parameters: $b = 4$ and $b = 14$ for the MBB, $b = 6$ and $b = 10$ for the TBB, and $p = 0.5$ and $p = 0.125$ for the SB, respectively. The block bootstrap estimates of $\sigma(\tau)$ obtained using these block sizes for the FAR(1) model are presented in Figure 1 and for the FMA(1) model in Figure 2.

As it is seen from these figures, the TBB estimates perform best with the MBB estimates being better than the SB estimates. For both sample sizes considered, the block bootstrap estimators perform better in the case of the FMA(1) model than in the case of the FAR(1) model while for the FMA(1) model, the TBB estimates are quite good even for $n = 100$ observations. The results using all three bootstrap methods are better for the larger sample size of $n = 500$ curves.

To demonstrate the performance of the suggested simpler rule $b^* = \lceil n^{1/3} \rceil$ to choose the block size b, the TBB estimates using this block size are compared with the estimates obtained using the block size leading to the less biased estimates, as described above. Comparisons for the FAR(1) and for the FMA(1) model are shown in Figure 3 and Figure 4 respectively.

As these figures demonstrate, for both sample sizes and for both models considered, the TBB estimates using the block size b^* perform well, being quite close to the TBB estimates using the "best" block size in the sense described above.

3 TBB-based Test versus Projection-based Tests

We compare the performance of the TBB-based test with the projection-based tests $U^{(1)}_{n_1,n_2}$ and $U^{(2)}_{n_1,n_2}$ proposed in Horváth *et al.* (2013) (see (3.11) and (3.12) in their paper). We adopted their simulation

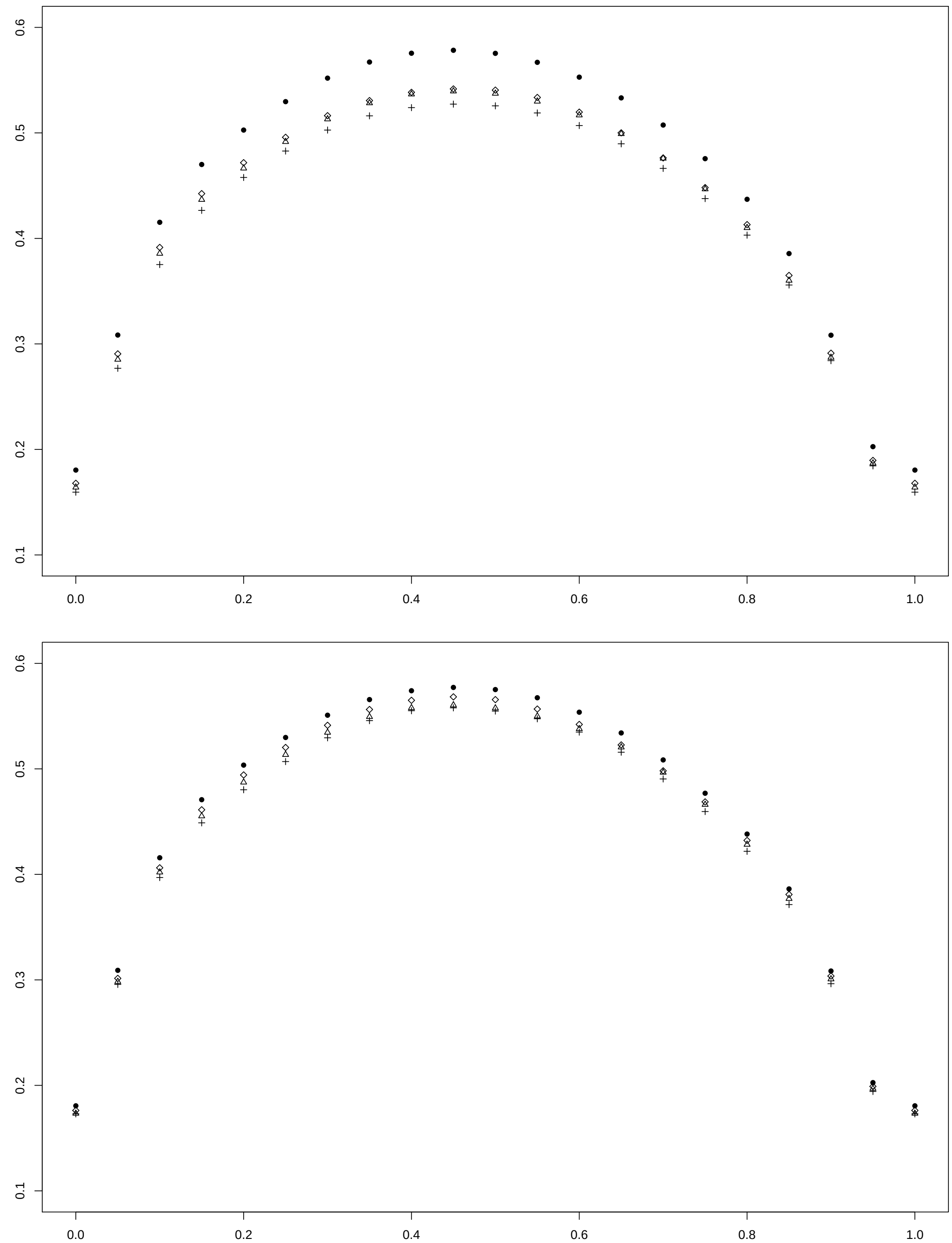

Figure 1: Comparison of different bootstrap estimates of the standard deviation $\sigma(\tau_i)$ of the normalized sample mean $\sqrt{n}\overline{X}_n(\tau_j)$ for FAR(1) time series and for a set of values $\tau_j \in [0,1]$. The first figure refers to $n = 100$ and the second to $n = 500$. The estimated exact standard deviation is denoted by $\bullet$ while the mean estimates of the standard deviation of the TBB are denoted by "$\diamond$", of the MBB by "$\triangle$", and of the SB by "+".

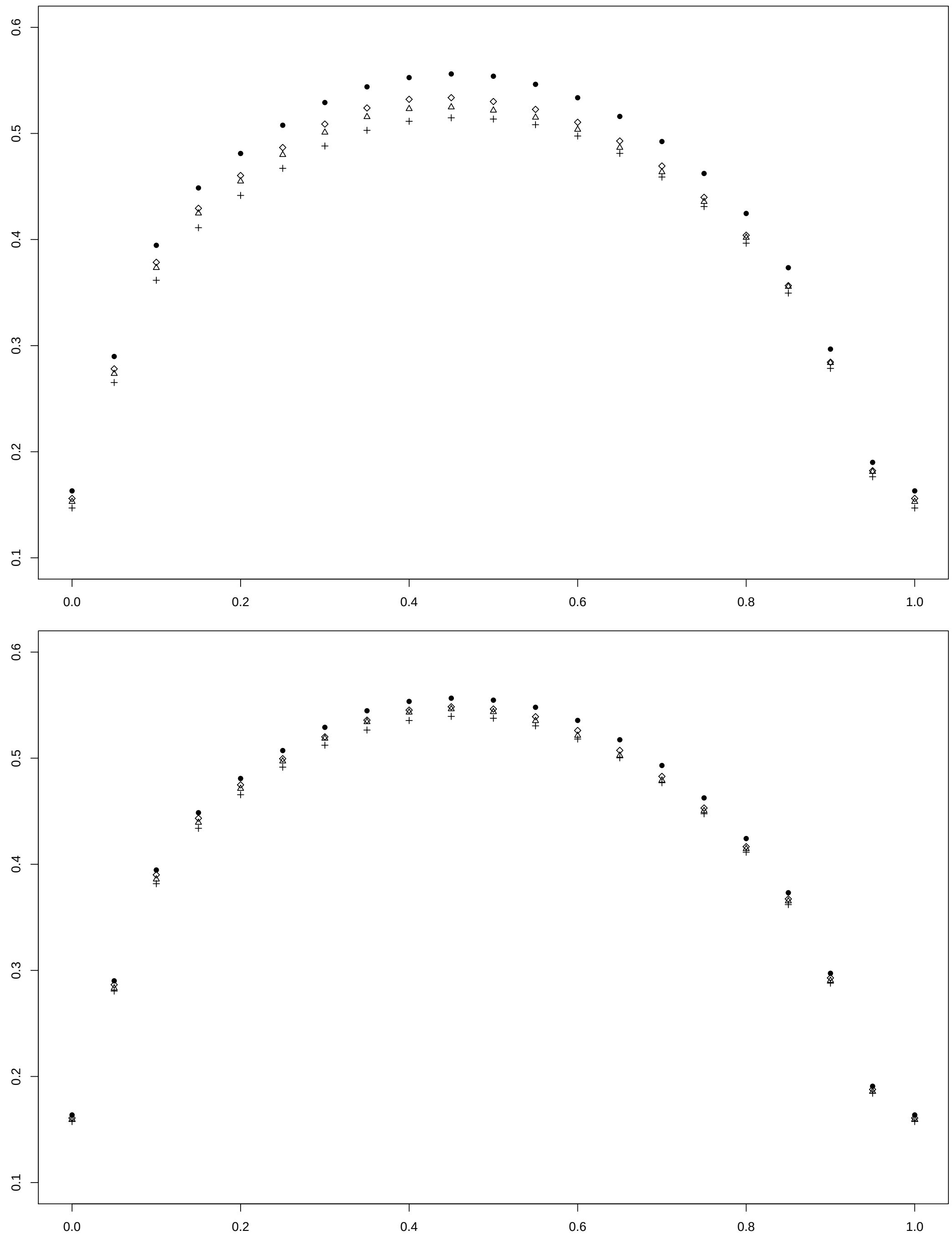

Figure 2: Comparison of different bootstrap estimates of the standard deviation $\sigma(\tau_i)$ of the normalized sample mean $\sqrt{n}\overline{X}_n(\tau_j)$ for FMA(1) time series and for a set of values $\tau_j \in [0,1]$. The first figure refers to $n = 100$ and the second to $n = 500$. The estimated exact standard deviation is denoted by $\bullet$ while the mean estimates of the standard deviation of the TBB are denoted by "$\diamond$", of the MBB by "$\triangle$", and of the SB by "$+$".

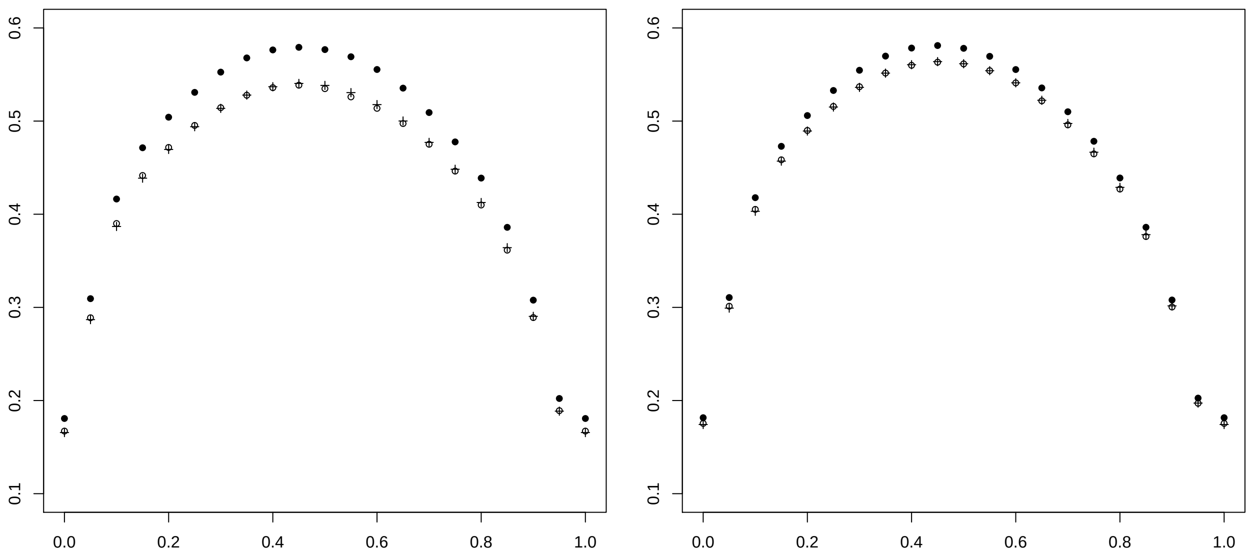

Figure 3: TBB estimates of the standard deviation $\sigma(\tau_i)$ of the normalized sample mean $\sqrt{n}\overline{X}_n(\tau_j)$ for the FAR(1) time series and for a set of values $\tau_j \in [0,1]$ using the "best" block size and the block size $b^* = \lceil n^{1/3} \rceil$. The first figure refers to $n = 100$ and the second to $n = 500$. The estimated exact standard deviation is denoted by $\bullet$ while the TBB estimates using the "best" block size are denoted by "$\circ$" and using the block size b^* by "+".

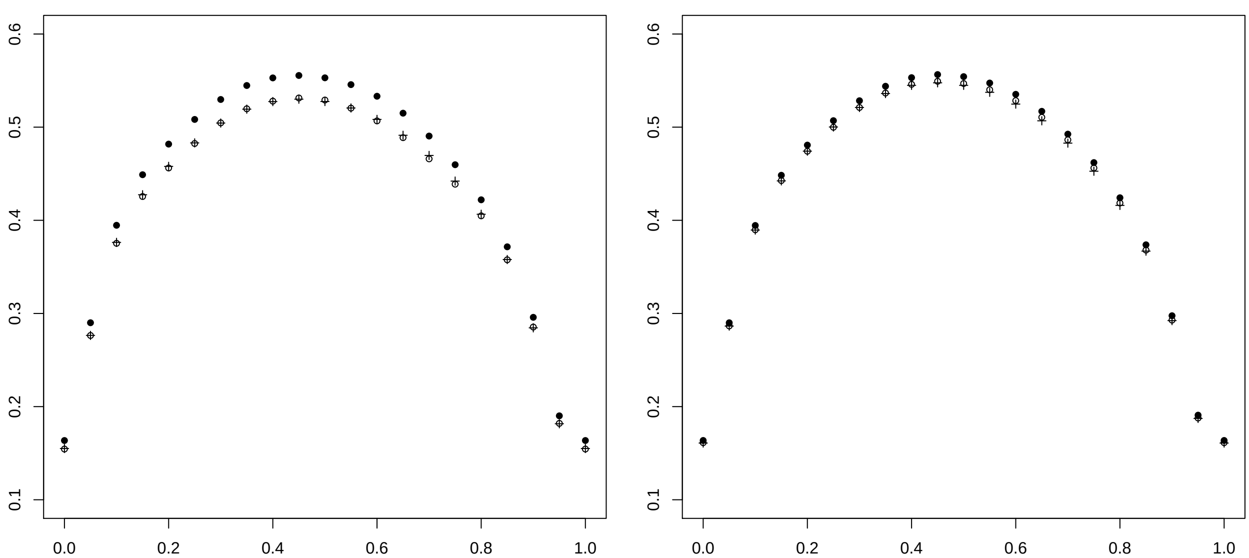

Figure 4: TBB estimates of the standard deviation $\sigma(\tau_i)$ of the normalized sample mean $\sqrt{n}\overline{X}_n(\tau_j)$ for the FMA(1) time series and for a set of values $\tau_j \in [0,1]$ using the "best" block size and the block size $b^* = \lceil n^{1/3} \rceil$. The first figure refers to $n = 100$ and the second to $n = 500$. The estimated exact standard deviation is denoted by $\bullet$ while the TBB estimates using the "best" block size are denoted by "$\circ$" and using the block size b^* by "+".

γ	$\alpha = 0.01$ $U^{(1)}_{n_1,n_2}$	$U^{(2)}_{n_1,n_2}$	TBB	$\alpha = 0.05$ $U^{(1)}_{n_1,n_2}$	$U^{(2)}_{n_1,n_2}$	TBB	$\alpha = 0.10$ $U^{(1)}_{n_1,n_2}$	$U^{(2)}_{n_1,n_2}$	TBB
0.0	0.018	0.019	0.017	0.066	0.072	0.070	0.122	0.135	0.128
			0.016			0.070			0.122
0.2	0.051	0.033	0.058	0.136	0.116	0.149	0.216	0.187	0.235
			0.046			0.142			0.236
0.4	0.194	0.123	0.150	0.359	0.265	0.322	0.467	0.363	0.431
			0.178			0.364			0.476
0.6	0.421	0.296	0.405	0.622	0.518	0.633	0.731	0.625	0.737
			0.425			0.649			0.738
0.8	0.686	0.538	0.684	0.857	0.746	0.847	0.915	0.831	0.920
			0.674			0.849			0.910
1.0	0.874	0.787	0.870	0.959	0.908	0.952	0.981	0.945	0.977
			0.881			0.959			0.987
1.2	0.976	0.937	0.964	0.995	0.981	0.990	0.998	0.992	0.995
			0.973			0.994			0.997

Table 1: Empirical rejection frequencies of the projection-based tests $U^{(1)}_{n_1,n_2}$ and $U^{(2)}_{n_1,n_2}$ are the results reported in Table 2 of Horváth *et al.* (2013). For the TBB-base test, the first line corresponds to the choices $b = 6$ and $b = 8$ and the second line to the choices $b = 6$ and $b = 10$ of the block size for sample sizes $n_1 = 100$ and $n_2 = 200$, respectively.

set up and generated two samples according to the functional time series model (4) with the errors $\varepsilon_{i,t}$ following the FAR(1) model (7) with kernel (9) of the main paper, for $i \in \{1, 2\}$, with mean functions given by $\mu_1(t) = 0$ and $\mu_2(t) = \gamma t(1-t)$ for the first and for the second population, respectively. All curves were approximated using $T = 49$ equidistant points $\tau_1, \tau_2, \ldots, \tau_{49}$ in the unit interval $\mathcal{I}$ and transformed into functional objects using the Fourier basis with 49 basis functions.

We considered sample sizes $n_1 = 100$ and $n_2 = 200$ and block sizes $b = b_1 = 6$ and 8 (for $n_1 = 100$) and $b = b_2 = 6$ and 10 (for $n_2 = 200$). The tests have been applied using three nominal levels, i.e., $\alpha = 0.01$, $\alpha = 0.05$ and $\alpha = 0.1$. All bootstrap calculations were based on $B = 1000$ bootstrap replicates and $R = 1000$ model repetitions. The results obtained are shown in Table 1 for a range of values of γ. Notice that $\gamma = 0$ corresponds to the null hypothesis. The empirical rejection frequencies of the projection-based tests $U^{(1)}_{n_1,n_2}$ and $U^{(2)}_{n_1,n_2}$ are those reported in Table 2 of Horváth *et al.* (2013).

As can be seen from Table 1, the TBB-based test performs well retaining the nominal sizes and having a power which increases as the deviations from H_0 increases, as described by the parameter γ. Compared to the projection-based test $U^{(2)}_{n_1,n_2}$, the TBB-based test performs better while its empirical size and power is similar to that of the projection-based test $U^{(1)}_{n_1,n_2}$. Notice, however, that the TBB-based test is consistent against any alternative for which $\|\mu_1 - \mu_2\| > 0$ which is not the case with the $U^{(1)}_{n_1,n_2}$ (and $U^{(2)}_{n_1,n_2}$) test if such alternatives are orthogonal to the projection space.

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