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Supplementary material for 'Bootstrap-based testing of equality of mean functions or equality of covariance operators for functional data'

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1. PROOFS OF THEOREMS 1 AND 2

1·1. *Proof of Theorem 1*

Let $\mathcal{S}$ be the Hilbert space of Hilbert–Schmidt operators endowed with the inner product $\langle \Psi_1, \Psi_2 \rangle_{\mathcal{S}} = \sum_{j=1}^{\infty} \langle \Psi_1(e_j), \Psi_2(e_j) \rangle$ for $\Psi_1, \Psi_2 \in \mathcal{S}$, where $\{e_j : j = 1, 2, \ldots\}$ is an orthonormal basis in $\mathcal{H}$. Notice that $\widehat{\mathcal{C}}_i^* \in \mathcal{S}$ $(i = 1, 2)$. Since

$$\overline{X}_{i,n_i}^* = \overline{X}_{i,n_i} + O_P(n_i^{-1/2}), \qquad n_i^{-1/2} \sum_{j=1}^{n_i} (X_{i,j}^* - \overline{X}_{i,n_i}^*) = O_P(1), \quad i = 1, 2,$$

we get

$$\begin{aligned} \widehat{\mathcal{C}}_i^* &= \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{i,j}^* - \overline{X}_{i,n_i}^*) \otimes (X_{i,j}^* - \overline{X}_{i,n_i}^*) \\ &= \frac{1}{n_i} \sum_{j=1}^{n_i} (X_{i,j}^* - \overline{X}_{i,n_i}) \otimes (X_{i,j}^* - \overline{X}_{i,n_i}) + O_P(n^{-1}), \quad i = 1, 2, \end{aligned}$$

where the random variables $(X_{i,j}^* - \overline{X}_{i,n_i}) \otimes (X_{i,j}^* - \overline{X}_{i,n_i})$ are, conditional on X_N, independent and identically distributed. By a central limit theorem for triangular arrays of independent and identically distributed $\mathcal{S}$-valued random variables (see, e.g., Politis & Romano (1992, Theorem 4.2)), we get, conditionally on X_N, that $n_i^{1/2}(\widehat{\mathcal{C}}_i^* - \widehat{\mathcal{C}}_N)$ converges weakly to a Gaussian random element $\mathcal{U}$ in $\mathcal{S}$ with mean zero and covariance operator $\mathcal{B} = \theta\mathcal{B}_1 + (1-\theta)\mathcal{B}_2$ as $n_i \to \infty$. Here, $\mathcal{B}_i$ is the covariance operator of the limiting Gaussian random element U_i to which $n_i^{1/2}(\widehat{\mathcal{C}}_i - \mathcal{C}_i)$ converges weakly as $n_i \to \infty$.

By the independence of the bootstrap random samples between the two populations, we have, conditional on X_N,

$$\begin{aligned} T_N^* &= N \|\widehat{\mathcal{C}}_1^* - \widehat{\mathcal{C}}_2^*\|_{\mathcal{S}}^2 \\ &= N \langle \widehat{\mathcal{C}}_1^* - \widehat{\mathcal{C}}_N, \widehat{\mathcal{C}}_1^* - \widehat{\mathcal{C}}_N \rangle_{\mathcal{S}} + N \langle \widehat{\mathcal{C}}_2^* - \widehat{\mathcal{C}}_N, \widehat{\mathcal{C}}_2^* - \widehat{\mathcal{C}}_N \rangle_{\mathcal{S}} \\ &= \frac{N}{n_1} \|n_1^{1/2}(\widehat{\mathcal{C}}_1^* - \widehat{\mathcal{C}}_N)\|_{\mathcal{S}}^2 + \frac{N}{n_2} \|n_2^{1/2}(\widehat{\mathcal{C}}_2^* - \widehat{\mathcal{C}}_N)\|_{\mathcal{S}}^2. \end{aligned}$$

Hence, taking into account the above results and that $n_1/N \to \theta$, we have that $N\|\widehat{\mathcal{C}}_1^* - \widehat{\mathcal{C}}_2^*\|_{\mathcal{S}}^2$ converges weakly to $\sum_{l=1}^{\infty} \tilde{\lambda}_l Z_l^2$ as $n_1, n_2 \to \infty$, where $\tilde{\lambda}_l$ $(l \geq 1)$ are the eigenvalues of the operator $\widetilde{\mathcal{B}} = \theta^{-1}\mathcal{B} + (1-\theta)^{-1}\mathcal{B}$ and Z_l $(l \geq 1)$ are independent standard (real-valued) Gaussian distributed random variables. Since $\mathcal{B}_1 = \mathcal{B}_2$, the assertion follows.

1·2. *Proof of Theorem 2*

Define

$$Z^+_{n_1,n_2}(t) = \Big[n_1^{-1/2}\sum_{j=1}^{n_1}\{X^+_{1,j}(t) - \overline{X}_N(t)\}, n_2^{-1/2}\sum_{j=1}^{n_2}\{X^+_{2,j}(t) - \overline{X}_N(t)\}\Big], \quad t \in \mathcal{I},$$

and

$$Z^+_{i,n_i}(t) = n_i^{-1/2}\sum_{j=1}^{n_i}\{X^+_{i,j}(t) - \overline{X}_N(t)\}, \quad t \in \mathcal{I}, \;\; i = 1, 2.$$

Notice that, conditionally on X_N, $Z^+_{1,n_1}(t)$ and $Z^+_{2,n_2}(t)$ are independent, have covariance operators $\widehat{\mathcal{C}}_1$ and $\widehat{\mathcal{C}}_2$, respectively, and $X^+_{1,j}(t)$ and $X^+_{2,j}(t)$ have the same mean function $\overline{X}_N(t)$. By a central limit theorem for triangular arrays of independent and identically distributed $\mathcal{H}$-valued random variables (see, e.g., Politis & Romano (1992, Theorem 4.2)), it follows that, conditionally on X_N, Z^+_{i,n_i} converges weakly to a Gaussian random element $\mathcal{U}_i$ with mean zero and covariance operator $\mathcal{C}_i$ as $n_i \to \infty$.

By the independence of Z^+_{1,n_1} and Z^+_{2,n_2}, we have, conditionally on X_N,

$$\begin{aligned} S^+_N &= \frac{n_1 n_2}{N}\int_{\mathcal{I}}\{\overline{X}^+_{1,n_1}(t) - \overline{X}^+_{2,n_2}(t)\}^2 dt \\ &= \frac{n_1 n_2}{N}\int_{\mathcal{I}}\Big[\frac{1}{n_1}\sum_{t=1}^{n_1}\{X^+_{1,j}(t) - \overline{X}_N(t)\} - \frac{1}{n_2}\sum_{t=1}^{n_2}\{X^+_{2,j}(t) - \overline{X}_N(t)\}\Big]^2 dt \\ &= \int_{\mathcal{I}}\Big\{n_2^{1/2}N^{-1/2}Z^+_{1,n_1}(t) - n_1^{1/2}N^{-1/2}Z^+_{2,n_2}(t)\Big\}^2 dt, \end{aligned}$$

from which, and taking into account that $n_1/N \to \theta$, we have that S^+_N converges weakly to $\int_{\mathcal{I}}\Gamma^2(t)dt$ as $n_1, n_2 \to \infty$. Thus, the assertion follows.

2. ADDITIONAL SIMULATION RESULTS

2·1. THE EFFECT OF INCREASING THE NUMBER OF PROJECTIONS

We present in Tables I and II additional empirical size and power for $T_{p,N}$, for testing the null hypothesis of equality of two covariance operators, and for$S^{(1)}_{p,N}$ and $S^{(2)}_{p,N}$, for testing the null hypothesis of equality of two mean functions, using $p = 3$, 6 and 8 functional principal components. We also illustrate in Figure I the quality of the asymptotic $\chi^2_{p(p+1)/2}$ approximation of the distribution of the test statistic $T_{p,N}$ for $p = 2$ and $p = 6$. For this, we estimate the exact distribution of $T_{p,N}$ under the null by generating 2000 replications of the functional data X_N using the simulation set-up of Section 3.1 of the paper, for sample sizes $n_1 = n_2 = 25$. We then compare the kernel density estimate of this exact distribution, obtained using a Gaussian kernel with bandwidth equal to 0.45 ($p = 2$) and 1.5 ($p = 6$), with the corresponding asymptotic χ^2_3 and χ^2_{21} distributions. The chi-square approximation of the distribution of $T_{p,N}$ is getting worse as the truncation parameter p increases. In particular, it overestimates the exact density in the right tail. This overestimation explains why $T_{p,N}$ using chi-square critical values leads to rejection rates that are below the desired nominal size.

γ	Test	$n_1 = n_2 = 25$			$n_1 = n_2 = 50$		
		$\alpha = 1\%$	5%	10%	$\alpha = 1\%$	5%	10%
1.0	$T_{3,N}$	0	1.4	4.3	0	1.1	3.5
	$T_{6,N}$	0	0.1	1.4	0	1.6	5.4
	$T_{8,N}$	0	0	0	0	0.8	3.4
	T_N^*	0.3	2.5	8.2	0.6	3.2	7.6
1.2	$T_{3,N}$	0.1	1.8	4.6	0.3	1.8	5.5
	$T_{6,N}$	0	0.1	1.6	0.1	0.8	4.3
	$T_{8,N}$	0	0	0	0.1	1.0	1.7
	T_N^*	0.5	5.0	14.7	0.8	9.8	23.1
1.4	$T_{3,N}$	0	0.7	2.9	0	3.0	10.8
	$T_{6,N}$	0	0.2	1.2	0	0.6	2.8
	$T_{8,N}$	0	0	0	0	0.3	2.1
	T_N^*	1.6	16.8	36.8	12.8	46.1	67.6
1.6	$T_{3,N}$	0	0.6	1.8	0	5.9	22.3
	$T_{6,N}$	0	0	1.4	0	0.6	2.0
	$T_{8,N}$	0	0	0	0	0.4	1.3
	T_N^*	4.7	33.8	61.2	37.0	79.6	90.3
1.8	$T_{3,N}$	0.1	0.5	2.6	0.6	16.7	48.7
	$T_{6,N}$	0	0	0.6	0	0	1.0
	$T_{8,N}$	0	0	0	0	0	0.9
	T_N^*	10.4	55.7	82.3	61.2	91.5	96.6
2.0	$T_{3,N}$	0	0.5	3.0	1.0	42.4	80.8
	$T_{6,N}$	0	0.1	0.4	0	0.1	1.2
	$T_{8,N}$	0	0	0	0	0	0.3
	T_N^*	17.7	66.6	89.2	74.2	93.7	97.7

Table I. *Empirical size and power (%) of $T_{p,N}$ (p =3, 6, 8) and T_N^* for the equality of two covariance operators.*

2·2. LOOKING AT DEVIATIONS IN ONE DIRECTION

We investigate the empirical size and power behaviour of the tests $T_{p,N}$ and T_N^* when the difference between the covariance operators in the two populations is only along one eigendirection. For this, we modified the simulation set-up of Section 3.1 in the paper, and generated non-Gaussian curves X_1 and X_2, via

$$X_1(t) = \sum_{k=1}^{10} \{2^{1/2} k^{-1/2} \sin(\pi k t) V_{1,k} + k^{-1/2} \cos(2\pi k t) W_{1,k}\}, \quad t \in \mathcal{I},$$

and

$$X_2(t) = \sum_{k=1}^{10} \gamma_k \{2^{1/2} k^{-1/2} \sin(\pi k t) V_{1,k} + k^{-1/2} \cos(2\pi k t) W_{1,k}\}, \quad t \in \mathcal{I},$$

where $V_{i,k}$ and $W_{i,k}$ ($i = 1, 2$, $k = 1, 2, \ldots, 10$) are independent t_5-distributed random variables, and $\gamma_k = \gamma\, I(k = r)$ for $r \in \{1, 3, 5\}$ and selected values of γ.

The results are summarized in Table III. The test based on T_N^* has, overall, a better size behaviour than that based on $T_{p,N}$. The test based on T_N^* has also a better power performance even for large values of r. The reason for this behaviour lies, probably, in the poor chi-square approx-

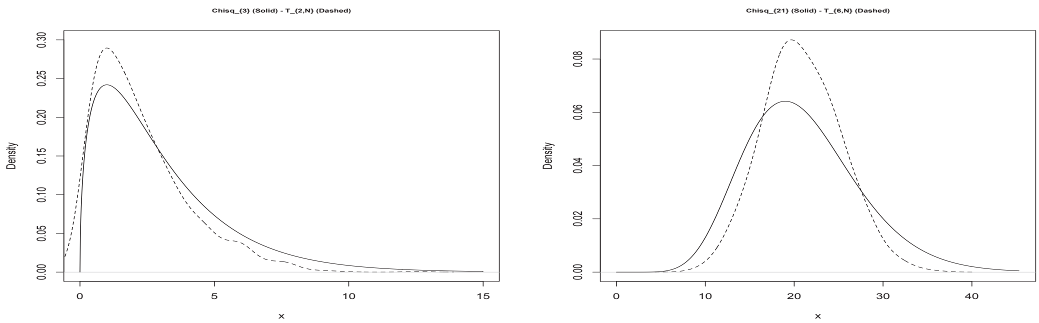

Fig. I. Density estimates, $T_{p,N}$: sample sizes $n_1 = n_2 = 25$, for $p = 2$ (left) and $p = 6$ (right). Estimated exact densities (dashed lines), and χ^2_3 and χ^2_{21} densities (solid lines).

imation of the distribution of $T_{p,N}$ under the null and the fact that this approximation becomes worse for larger values of p.

2·3. Bootstrap versus asymptotic for the same projection-based test

We additionally demonstrate the benefits of the bootstrap approximation versus the asymptotic approximation for the same projection-based test statistics for testing the null hypothesis of equality of two covariance operators. For this, we compare the empirical size and power of $T_{p,N}$ and $T^*_{p,N}$ applied using critical values obtained from the asymptotic distribution and from the bootstrap approximation, respectively. The asymptotic distribution of $T_{p,N}$ for the non-Gaussian case has been derived in Fremdt *et al.* (2012), while a theoretical justification of using $T^*_{p,N}$ has been given in Paparoditis & Sapatinas (2015).

Hence, we adopted the simulation setup of Fremdt *et al.* (2012) and generated non-Gaussian curves X_1 and X_2, via

$$X_i(t) = A\sin(\pi t) + B\sin(2\pi t) + C\sin(4\pi t), \quad t \in \mathcal{I}, \;\; i \in \{1, 2\}, \tag{1}$$

where $A = 7Y_1$, $B = 3Y_2$, $C = Y_3$ with Y_1, Y_2 and Y_3 being independent t_5-distributed random variables. All curves were simulated at 500 equidistant points in the unit interval $\mathcal{I}$, and transformed into functional objects using the Fourier basis with 49 basis functions. For each data generating process, we considered 2000 replications, for different sample sizes and the three most common nominal levels α. All bootstrap calculations were based on 1000 bootstrap replications.

We present in Table IV empirical size results for $T_{p,N}$ and $T^*_{p,N}$, using $p = 2$ and 3 functional principal components. It is evident that $T^*_{p,N}$ have good size behavior and do not suffer from the under-rejection problems of $T_{p,N}$. Table V shows empirical power results for $T_{2,N}$ and $T^*_{2,N}$. For this, the curves in the first sample were generated according to (1) while the curves in the second sample were generated according to a scaled version, i.e., $X_2(t) = \gamma X_1(t)$, $t \in \mathcal{I}$, for selected values of the scaling parameter γ. As Table V shows, $T^*_{2,N}$ has higher power than $T_{2,N}$. The lower power of $T_{2,N}$ is due to the overestimation of the right-tail of the true density, see Figures 6.1 and 6.2 in Paparoditis & Sapatinas (2015). Notice that, while this overestimation leads to a conservative test under the null, it leads at the same time to a loss of power under the alternative. As our empirical evidence shows, the tests based on bootstrap approximations not only have better size behavior under the null but they also have higher power under the alternative.

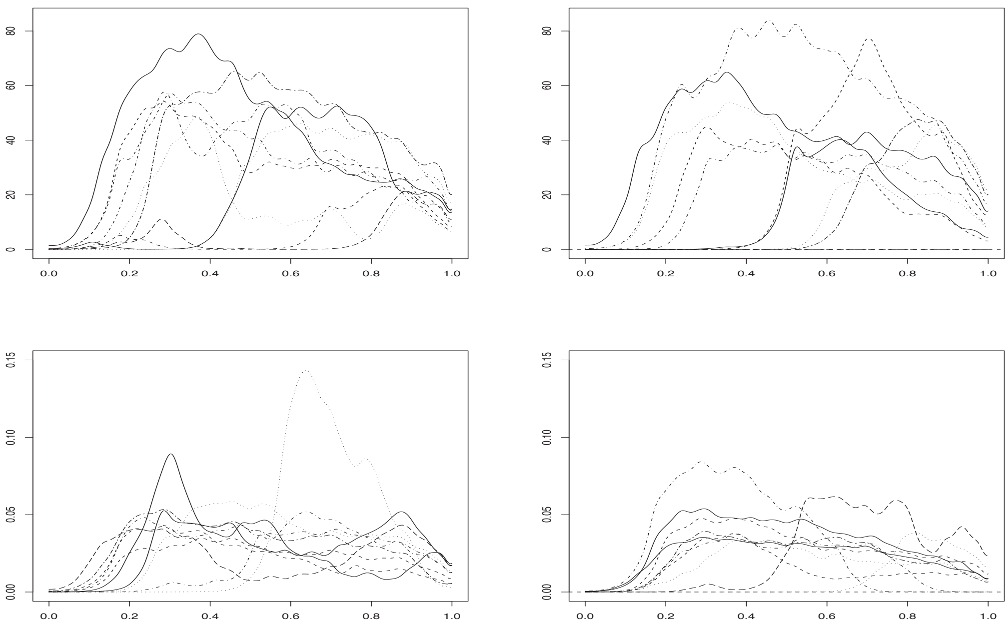

Fig. II. 10 randomly selected smoothed curves of short-lived (left panel) and 10 randomly selected smoothed curves of long-lived flies (right panel). Top: absolute curves; x-axis: days rescaled on [0,1], y-axis: number of eggs laid. Bottom: relative curves; x-axis: days rescaled on [0,1], y-axis: number of eggs relative to the eggs laid in the fly's lifetime.

2·4. MEDITERRANEAN FRUIT FLIES

In our analysis, we consider $N = 534$ egg-laying curves of medflies who lived at least 43 days, but, as in, e.g., Horváth & Kokoszka (2012, Chapter 5) and Fremdt et al. (2013), we only consider the egg-laying activities on the first 30 days. Two versions of these egg-laying curves are considered and are scaled such that the corresponding curves in either version are defined on the interval $\mathcal{I} = [0, 1]$. The curves in version 1 are denoted by $X_i(t)$ and represent the absolute counts of eggs laid by fly i on day $\lfloor 30t \rfloor$. The curves in version 2 are denoted by $Y_i(t)$ and represent the counts of eggs laid by fly i on day $\lfloor 30t \rfloor$ relative to the total number of eggs laid in the lifetime of fly i. Furthermore, the 534 flies are classified into short-lived flies, those who died before the end of the 43rd day after birth, and long-lived flies, those who lived 44 days or longer. In this particular data set analyzed, there are $n_1 = 256$ short-lived flies and $n_2 = 278$ long-lived flies. Based on the above classification, we consider two samples. Sample 1 represents the absolute egg-laying curves $\{X_{1,i}(t) : \ t \in \mathcal{I}, \ i = 1, 2, \ldots, 256\}$ or the relative egg-laid curves $\{Y_{1,i}(t) : \ t \in \mathcal{I}, \ i = 1, 2, \ldots, 256\}$ of the short-lived flies. Sample 2 represents the absolute egg-laying curves $\{X_{2,i}(t) : \ t \in \mathcal{I}, \ i = 1, 2, \ldots, 278\}$ or the relative egg-laid curves $\{Y_{2,i}(t) : \ t \in \mathcal{I}, \ i = 1, 2, \ldots, 278\}$ of the long-lived flies; see Figure II.

δ	Test	$n_1 = n_2 = 25$ $\alpha = 1\%$	5%	10%	$n_1 = n_2 = 50$ $\alpha = 1\%$	5%	10%
0.0	$S_{3,N}^{(1)}$	0.7	3.3	7.2	0.6	4.5	8.5
	$S_{3,N}^{(2)}$	0.7	3.4	6.6	0.9	4.3	8.7
	$S_{6,N}^{(1)}$	2.0	7.5	13.4	1.4	5.9	11.2
	$S_{6,N}^{(2)}$	0.3	4.4	9.8	0.8	5.1	8.4
	$S_{8,N}^{(1)}$	3.7	11.0	17.4	2.2	8.2	13.1
	$S_{8,N}^{(2)}$	0.8	4.7	10.6	1.0	4.5	10.3
	S_N^+	1.2	5.8	11.8	0.7	4.4	8.5
0.2	$S_{3,N}^{(1)}$	1.3	5.9	12.1	2.6	8.9	16.9
	$S_{3,N}^{(2)}$	1.8	6.2	12.1	2.4	9.9	17.0
	$S_{6,N}^{(1)}$	3.4	10.0	16.8	4.4	12.6	21.0
	$S_{6,N}^{(2)}$	2.5	7.6	14.0	3.7	11.4	20.3
	$S_{8,N}^{(1)}$	5.0	13.4	21.0	4.3	12.9	20.6
	$S_{8,N}^{(2)}$	1.8	7.2	14.8	3.5	12.7	19.1
	S_N^+	2.4	8.8	15.8	4.0	13.0	20.6
0.4	$S_{3,N}^{(1)}$	6.4	15.2	23.6	14.7	33.7	46.1
	$S_{3,N}^{(2)}$	5.0	14.5	23.0	12.6	31.1	42.9
	$S_{6,N}^{(1)}$	8.8	22.7	32.0	19.8	39.2	53.0
	$S_{6,N}^{(2)}$	4.6	19.0	29.8	17.6	39.7	52.3
	$S_{8,N}^{(1)}$	12.7	25.7	35.3	18.9	38.0	50.4
	$S_{8,N}^{(2)}$	6.3	19.2	30.4	19.1	38.9	51.4
	S_N^+	5.7	20.6	32.0	18.2	40.6	54.0
0.6	$S_{3,N}^{(1)}$	17.0	34.8	47.0	47.7	67.6	78.7
	$S_{3,N}^{(2)}$	14.7	31.7	45.7	43.1	66.5	78.3
	$S_{6,N}^{(1)}$	25.6	45.5	57.6	52.2	74.2	82.0
	$S_{6,N}^{(2)}$	19.8	44.0	55.7	53.3	75.6	82.8
	$S_{8,N}^{(1)}$	29.9	49.1	60.2	50.9	74.9	84.5
	$S_{8,N}^{(2)}$	20.3	41.2	55.0	51.4	73.8	81.9
	S_N^+	21.6	44.8	59.3	54.2	77.9	86.8
0.8	$S_{3,N}^{(1)}$	37.8	59.9	72.2	76.6	89.5	93.7
	$S_{3,N}^{(2)}$	34.6	58.4	69.5	73.1	89.2	93.7
	$S_{6,N}^{(1)}$	48.4	70.0	79.3	87.0	95.7	97.8
	$S_{6,N}^{(2)}$	43.0	67.0	79.6	84.9	96.6	98.7
	$S_{8,N}^{(1)}$	52.6	72.7	80.4	85.6	94.0	97.2
	$S_{8,N}^{(2)}$	44.2	70.2	80.4	85.5	95.4	98.2
	S_N^+	47.3	71.2	81.3	86.1	96.1	98.2
1.0	$S_{3,N}^{(1)}$	58.0	76.1	83.5	94.2	98.5	99.4
	$S_{3,N}^{(2)}$	55.0	76.2	84.1	93.5	98.5	99.6
	$S_{6,N}^{(1)}$	72.6	87.4	93.1	98.0	99.8	100
	$S_{6,N}^{(2)}$	68.3	86.0	93.3	98.7	99.8	100
	$S_{8,N}^{(1)}$	77.4	89.8	94.2	97.8	99.4	99.9
	$S_{8,N}^{(2)}$	74.4	89.9	94.5	97.9	99.7	99.8
	S_N^+	73.7	90.7	95.2	98.0	99.5	99.7

Table II. *Empirical size and power (%) of $S_{p,N}^{(1)}$, $S_{p,N}^{(2)}$ (p =3, 6, 8) and S_N^+ for the equality of two mean functions.*

r	γ	Test	$n_1 = n_2 = 25$ $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$	$n_1 = n_2 = 50$ $\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$
1	0.0	$T_{2,N}$	0	2.1	5.9	0.2	1.8	5.1
		$T_{3,N}$	0	1.6	6.5	0.5	1.7	4.9
		$T_{6,N}$	0	0.1	2.9	0.1	2.1	4.4
		T_N^*	0.2	1.9	5.7	0.4	2.3	6.8
	1.6	$T_{2,N}$	0.2	2.0	6.9	0.7	10.6	28.0
		$T_{3,N}$	0.2	3.8	10.5	2.0	11.6	23.7
		$T_{6,N}$	0	0.4	2.8	0.1	6.0	13.8
		T_N^*	3.6	25.8	48.1	23.3	58.5	74.7
	2.0	$T_{2,N}$	0.1	2.6	12.6	3.3	39.8	68.2
		$T_{3,N}$	0.5	6.1	18.0	1.5	9.4	21.6
		$T_{6,N}$	0	7.0	8.0	1.4	14.5	28.6
		T_N^*	16.9	55.7	74.0	59.8	87.5	93.6
3	0.0	$T_{2,N}$	0	1.6	4.7	0.4	1.9	5.1
		$T_{3,N}$	0	1.6	3.9	0.2	2.2	6.6
		$T_{6,N}$	0	0.3	13.0	0	1.3	4.3
		T_N^*	0	1.9	6.4	0.2	2.4	6.3
	1.6	$T_{2,N}$	0.2	2.7	7.2	0.7	1.9	5.1
		$T_{3,N}$	0	2.6	7.1	0.2	4.8	11.8
		$T_{6,N}$	0	0.4	2.5	0.1	3.3	8.9
		T_N^*	0.1	2.8	12.6	0.8	9.4	22.5
	2.0	$T_{2,N}$	0.5	5.5	13.3	3.0	18.6	34.0
		$T_{3,N}$	0.2	3.5	10.8	2.4	14.3	26.3
		$T_{6,N}$	0	0.5	4.4	0.6	9.0	19.6
		T_N^*	1.4	14.3	31.5	9.3	36.0	57.5
5	0.0	$T_{2,N}$	0	1.8	4.0	0.1	1.9	4.9
		$T_{3,N}$	0	1.2	3.7	0.2	1.6	5.3
		$T_{6,N}$	0	0.1	1.4	0	1.3	4.0
		T_N^*	0	1.8	5.6	0.2	2.0	5.7
	1.6	$T_{2,N}$	0	1.6	4.9	0.3	4.0	7.9
		$T_{3,N}$	0.1	2.0	6.0	0.3	3.9	8.9
		$T_{6,N}$	0	0.1	2.1	0.2	2.7	7.8
		T_N^*	0.1	2.6	9.9	0.3	4.0	11.2
	2.0	$T_{2,N}$	0.4	3.4	8.7	2.2	9.0	21.1
		$T_{3,N}$	0.2	2.2	8.2	1.5	11.4	22.3
		$T_{6,N}$	0	0.4	2.4	0.4	5.0	15.6
		T_N^*	0.2	5.7	17.1	2.3	11.5	28.5

Table III. *Empirical size and power (%) of $T_{p,N}$ ($p = 2, 3, 6$) and T_N^* for the equality of two covariance operators, when the difference between the covariance operators in the two populations is only along the k eigendirection ($k = 1, 3, 5$).*

n_1	n_2	Test	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$
25	25	$T_{2,N}$	0.2	1.1	3.9
		$T^*_{2,N}$	0.8	4.9	10.5
25	25	$T_{3,N}$	0	1.0	3.5
		$T^*_{3,N}$	0.6	4.5	9.7
50	50	$T_{2,N}$	0	1.9	5.3
		$T^*_{2,N}$	1.0	4.3	8.3
50	50	$T_{3,N}$	0.2	1.2	5.1
		$T^*_{3,N}$	0.7	5.3	9.4
100	100	$T_{2,N}$	0	1.9	4.2
		$T^*_{2,N}$	0.4	4.6	10.0
100	100	$T_{3,N}$	0.5	1.9	4.5
		$T^*_{3,N}$	0.6	5.4	9.4

Table IV. *Empirical size (%) of $T_{p,N}$ and $T^*_{p,N}$ ($p = 2, 3$) for the equality of two covariance functions.*

		$n_1 = n_2 = 25$			$n_1 = n_2 = 50$		
γ	Test	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$	$\alpha = 1\%$	$\alpha = 5\%$	$\alpha = 10\%$
2.0	$T_{2,N}$	0	4.3	21.5	7.6	61.3	85.7
	$T^*_{2,N}$	0.8	19.1	44.7	39.6	83.0	94.1
2.2	$T_{2,N}$	0	6.1	32.1	19.7	77.6	93.3
	$T^*_{2,N}$	2.3	33.0	59.7	69.7	93.9	98.2
2.4	$T_{2,N}$	0	13.4	44.2	35.9	90.3	97.8
	$T^*_{2,N}$	4.5	46.3	73.2	81.9	97.7	99.1
2.6	$T_{2,N}$	0	16.4	53.3	50.1	94.6	98.9
	$T^*_{2,N}$	8.4	59.5	83.8	90.1	99.2	99.6
2.8	$T_{2,N}$	0	23.1	66.1	65.5	98.1	99.6
	$T^*_{2,N}$	12.5	66.9	90.7	94.6	99.1	99.8
3.0	$T_{2,N}$	0	31.6	75.1	74.5	98.7	99.7
	$T^*_{2,N}$	18.8	75.5	90.6	96.7	100	100

Table V. *Empirical power (%) of $T_{2,N}$ and $T^*_{2,N}$ for the equality of two covariance functions.*

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