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A Classical and a Bayesian Approach to Linear Ill-Posed Inverse Problems

MSc Thesis

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Chapter 1

Introduction

In this thesis we present two approaches for the solution of Inverse Problems, the Classical and the Bayesian. Inverse Problems are introduced in Section 1.1. As a guide to our exposition we use a particular problem, the "Laplacian-like" inverse problem, which is defined in Section 1.2 and which simplifies the theory, enabling us to built intuition on the subject.

In Chapter 2 we give a brief presentation of the Classical Approach to Inverse Problems. The majority of the theory presented in this chapter is taken from [5], but is supplemented by material from [22] and [16]. In order to keep the presentation concise and simple, the proofs of the theorems presented in this chapter are our simplified versions of the proofs presented in [5] and [16]. Our proofs hold for the framework defined in Section 1.2 and simplifications arise from the self-adjointness of the forward operator. At the same time, in order to give a better picture of the theory, we make additional remarks and present additional examples.

First, we introduce the Moore-Penrose generalized inverse, which provides a way of inverting ill-posed linear equations and then we give a short presentation of the Classical Regularization Theory and particularly the Tikhonov Regularization method, [21]. The chapter concludes with a presentation of the method of Tikhonov Regularization in Hilbert Scales, [18], [17], [9]. The reason that we are mainly focused on the Tikhonov Regularization method and its generalizations, is that it is related to the Bayesian Approach to Inverse Problems, [6].

In Chapter 3 we provide some original results for a generalized version of the method of Tikhonov Regularization in Hilbert Scales applied to the "Laplacian-like" inverse problem. We first supply sufficient conditions for convergence and then proceed to give convergence rates under additional assumptions.

Finally, in Chapter 4 we examine the Bayesian Approach to Inverse Problems. In Section 4.1 we give a brief presentation of the theory, without supplying any proofs. The theorems used for the presentation can be found in Section 4.3. In Section 4.2 we first present some posterior consistency results in finite dimensions and then prove original posterior consistency results for the "Laplacian-like" inverse problem. The posterior consistency results presented are of two kingds: frequentist Bayesian and subjectivistic Bayesian, [4]. We also give necessary and sufficient conditions for the equivalence (in the sense of measures) of the posterior and the prior in the "Laplacian-like" inverse problem. In our work, we simplify the calculation of the posterior distribution by the use of a class of conjugate priors with respect to the data likelihood, in particular we examine the case where both the prior and the noise is *Gaussian*, [1]. The presentation in this chapter is based primarily on [20] and secondarily on [15] and [13].

The work in this thesis hence contains orginal work of two kinds: firstly the proofs of all the results concerning classical regularization have been developed independently from the original sources, in the particular case where the forward operator is self-adjoint; secondly the work on the "Laplacian-like" inverse problem, Bayesian and Classical, presented in Chapters 3 and 4, is entirely new.

1.1 Inverse Problems

In this section we introduce the concept of an Inverse Problem. According to [5], "Inverse Problems are concerned with determining causes for a desired or an observed effect".

Suppose $F : X \to Y$ is a function between the spaces X and Y and consider the equation

$$y = F(u). \tag{1.1}$$

To give some intuition, suppose F is the solution of a differential equation. If we consider y^δ to be a (possibly noisy) observation of the solution

$$y^\delta = F(u) + \eta, \tag{1.2}$$

where we know that the size of the noise is less than δ (classical approach) or we have some information on the statistical behaviour of the noise (Bayesian approach), then in Inverse Problems we are trying to determine the parameter u which is the cause for the observed effect y^δ. If we consider y to be a desired solution of the differential equation, then in Inverse Problems we are trying to determine the parameter u which steers the solution to the desired value. In other words we are trying to invert (1.1) or (1.2) in a general sense.

We will henceforth refer to y as the *exact data* or *exact observation*, to y^δ as the *noisy data* or the *noisy observation* and to u as the *solution* of the Inverse Problem.

Inverse Problems are in general ill-posed in the Hadamard sense, since usually at least one of the following holds:

i) The existence of solution is not guaranteed for all admissible data $y \in Y$: the function F might not be surjective, so there might exist $y \in Y$ for which there is no $u \in X$ such that $y = F(u)$. Even if F is surjective, in practice we have noisy observations of the data y which might not be in the space Y, so we need a way to assign a solution u to these observations.

ii) The uniqueness of solution is not guaranteed for all admissible data $y \in Y$: the function F might not be injective, so there might exist data $y \in Y$ for which there are multiple u's such that $y = F(u)$. More generally, in the case of noisy observations we need to be able to assign a unique solution to every noisy measurement.

iii) The solution u does not depend continuously on the data y: this is currently a vague statement since we haven't introduced any topologies in the spaces X and Y, but we will make it concrete later. It is apparent that this is very important since in practice we have noisy observations of the data y, so we want small errors in y to give rise to small errors in the solution u.

In Chapter 2, we give some classical tools to deal with the ill-posedness of inverse problems and in Section 1 of Chapter 4, we give a description of the Bayesian approach to inverse problems. In Chapter 3 and Section 2 of Chapter 4, we give results related to a particular inverse problem, which we call the "Laplacian-like" inverse problem and which we define in the next section.

1.2 Statement of the Problem

1.2.1 General Framework

In this thesis we are concerned with Inverse Problems of the following general form: suppose that in (1.1) and (1.2), X and Y are Hilbert spaces and F is a compact, linear operator $K : X \to Y$. We are trying to invert

$$y = Ku, \quad u \in X, \, y \in Y, \tag{1.3}$$

assuming that we have observations of (1.3) polluted by an additive noise of known magnitude (classical approach)

$$y^{\delta} = Ku + \eta, \quad ||\eta|| = \delta, \tag{1.4}$$

or known statistics (Bayesian approach).

For a compact operator K, it is well known that its range, $\mathcal{R}(K)$, is closed if and only if it is finite-dimensional [5]. This immediately shows that in the interesting cases where the range of the operator is infinite-dimensional there is no guarantee of solution of the inverse problem (1.3) for all $y \in Y$. Indeed, since Y is a Hilbert space therefore closed, in these cases we have $\mathcal{R}(K) \subsetneq Y$. Furthermore, if the nullspace of K is non-trivial, then K is not injective and therefore even if we do have existence of solution, we don't have uniqueness. More importantly, as we will see in the next section, even if we can invert (1.3), the inverse is not continuous so we cannot calculate the solution u in a stable way. Therefore, in the general case, the inverse problem (1.3) is ill-posed.

Later in the thesis we will additionally assume that K is injective and self-adjoint.

When K is compact and self-adjoint it possesses an eigensystem $\{\phi_k, \nu_k\}_{k\in\mathbb{N}}$ where the eignevectors $\{\phi_k\}_{k\in\mathbb{N}}$ form an orthonormal basis of X. The assumption that K is injective, implies that $\nu_k \neq 0$, $\forall k \in \mathbb{N}$.

The space X can be identified with the space $\mathcal{H}$

$$\mathcal{H} = \left\{ u = \sum_{k=1}^{\infty} u_k \phi_k : \sum_{k=1}^{\infty} u_k^2 < \infty \right\}$$

which is a Hilbert space with the ℓ^2-norm and the ℓ^2-inner product.

For any $u \in \mathcal{H}$, we can write

$$u = \sum_{k=1}^{\infty} \langle u, \phi_k \rangle \phi_k = \sum_{k=1}^{\infty} u_k \phi_k, \quad \text{where} \;\; u_k := \langle u, \phi_k \rangle.$$

We can then define fractional powers of K by

$$K^s u = \sum \nu_k^s u_k \phi_k, \quad s \in \mathbb{R}.$$

For every $s \in \mathbb{R}$ we define the separable Hilbert spaces $\mathcal{X}_s$ by

$$\mathcal{X}_s = \left\{ u : \sum_{k=1}^{\infty} \nu_k^{-4s} u_k^2 < \infty \right\},$$

with the inner product

$$\langle u, x \rangle_{\mathcal{X}_s} := \langle K^{-2s} u, K^{-2s} x \rangle$$

and the norm

$$||u||_{\mathcal{X}_s} := \left|\left|K^{-2s}u\right|\right| = \left(\sum_{k=1}^{\infty} \nu_k^{-4s} u_k^2\right)^{\frac{1}{2}}.$$

Notation. Throughout this thesis we use the following notation:

$$u_k = \langle u, \phi_k \rangle,$$

$$y_k = \langle y, \phi_k \rangle,$$

$$y_k^\delta = \langle y^\delta, \phi_k \rangle,$$

and

$$q_k = \langle u^\dagger, \phi_k \rangle.$$

1.2.2 The Laplacian-like Inverse Problem

In particular in this thesis, we will study the inverse problem (1.3) with K defined as a negative power of an operator $\mathcal{A}$ satisfying the following:

Assumption 1.2.1. *$\mathcal{A} : \mathcal{D}(\mathcal{A}) \to \mathcal{H}$ is a linear bounded operator, satisfying the following properties:*

i) $\mathcal{A}$ is positive-definite, self-adjoint and invertible;

ii) $\mathcal{A}$ possesses an eigensystem $\{\phi_k, \mu_k\}_{k\in\mathbb{N}}$, where the eigenvectors $\{\phi_k\}_{k\in\mathbb{N}}$ form an orthonormal basis for $\mathcal{H}$;

iii) there exist $c_1^+, c_1^- > 0$ such that

$$c_1^- \leq \frac{\mu_k}{k^2} \leq c_1^+, \quad \forall k \in \mathbb{N}.$$

In other words the operator $\mathcal{A}$ is "*Laplacian-like*" in the sense that it's eigenvalues $\{\mu_k\}$ grow like the eigenvalues of the Laplacian in the real line, or as the eigenvalues of the Laplacian to the d-power in d-dimensions.

As before, we can define fractional powers of $\mathcal{A}$ by

$$\mathcal{A}^s u = \sum \mu_k^s u_k \phi_k, \quad s \in \mathbb{R},$$

and for any $s \in \mathbb{R}$ we define the separable Hilbert spaces $\mathcal{H}^s$ by

$$\mathcal{H}^s = \left\{ u : D \to \mathbb{R} : \sum_{k=1}^{\infty} \mu_k^s u_k^2 < \infty \right\},$$

with the inner product

$$\langle u, x \rangle_s := \langle \mathcal{A}^{\frac{s}{2}} u, \mathcal{A}^{\frac{s}{2}} x \rangle$$

and the norm

$$||u||_s := \left|\left|\mathcal{A}^{\frac{s}{2}} u\right|\right| = \left(\sum_{k=1}^{\infty} \mu_k^s u_k^2 \right)^{\frac{1}{2}}.$$

Note that $\mathcal{H}^s$ is a subspace of $\mathcal{H}$ if $s > 0$, while if $s < 0$ then $\mathcal{H}^s$ contains $\mathcal{H}$. The family of spaces $(\mathcal{H}^s)_{s\in\mathbb{R}}$ is called the *Hilbert Scale induced by* $\mathcal{A}$. Furthermore, note that for $K = \mathcal{A}^{-\ell}$ the spaces $\mathcal{H}^s$ and $\mathcal{X}_s$ are connected through scaling: indeed, since $\nu_k = \mu_k^{-\ell}$, we have that $\mathcal{H}^s = \mathcal{X}_{\frac{s}{4\ell}}$.

We consider (1.3) for operators $K : \mathcal{H}^\gamma \to \mathcal{H}^\beta$ of the form $K = \mathcal{A}^{-\ell},\ \ell > 0$ for appropriate values of $\gamma, \beta \in \mathbb{R}$.

Lemma 1.2.2. *If $K = \mathcal{A}^{-\ell}, \ell > 0$, then $K : \mathcal{H}^\gamma \to \mathcal{H}^\beta$ is well defined and bounded if and only if $\beta - 2\ell \leq \gamma$.*

Proof. First we show that if $\beta - 2\ell \leq \gamma$ then K is well defined. Indeed, let $u \in \mathcal{H}^\gamma$, then

$$||Ku||_\beta^2 = \sum_{k=1}^{\infty} \mu_k^{\beta-2\ell} u_k^2$$

which is finite if and only if $u \in \mathcal{H}^{\beta-2\ell}$, i.e. $\gamma \geq \beta - 2\ell$. To show that K is bounded, note that

$$||K||_{\mathcal{L}(\mathcal{H}^\gamma,\mathcal{H}^\beta)}^2 = \sup_{u\in\mathcal{H}^\gamma, u\neq 0} \frac{||Ku||_\beta^2}{||u||_\gamma^2} = \sup_{u\in\mathcal{H}^\gamma, u\neq 0} \frac{\sum_{k=1}^{\infty} \mu_k^{\beta-2\ell} u_k^2}{\sum_{k=1}^{\infty} \mu_k^{\gamma} u_k^2}.$$

If $\beta - 2\ell \leq \gamma$, then by Assumption 1.2.1(iii), $\exists c > 0$ such that

$$\mu_k^{\beta-2\ell} \leq c\mu_k^{\gamma},\ \forall k \in \mathbb{N},$$

thus

$$||K||_{\mathcal{L}(\mathcal{H}^\gamma,\mathcal{H}^\beta)}^2 \leq c.$$

Suppose that $\beta - 2\ell > \gamma$ and $\forall n \in \mathbb{N}$ define $u^{(n)} = \phi_n \neq 0$. Then $u^{(n)} \in \mathcal{H}^\gamma$, $\forall n \in \mathbb{N}$ and

$$\frac{\left|\left|Ku^{(n)}\right|\right|_\beta^2}{\left|\left|u^{(n)}\right|\right|_\gamma^2} = \mu_n^{\beta-2\ell-\gamma} \to \infty, \quad \text{as} \quad n \to \infty,$$

since $\mu_n \to \infty$ by Assumption 1.2.1(iii). Hence

$$||K||_{\mathcal{L}(\mathcal{H}^\gamma,\mathcal{H}^\beta)} = \infty$$

and K is unbounded. □

We can similarly prove the following:

Lemma 1.2.3. *Suppose $K = \mathcal{A}^{-\ell} : \mathcal{H}^\gamma \to \mathcal{H}^\beta, \ell > 0$, where $\beta - 2\ell \leq \gamma$. Then $K^{-1} = \mathcal{A}^\ell : \mathcal{H}^\beta \to \mathcal{H}^\gamma$ is well defined and bounded if and only if $\beta = \gamma + 2\ell$.*

Remark 1.2.4. The last lemma may cause some confusion. As mentioned earlier, the range of a compact operator is closed if and only if it is finite dimensional. The last lemma implies that the range of $K = \mathcal{A}^{-\ell} : \mathcal{H}^\gamma \to \mathcal{H}^{\gamma+2\ell}$ is the space $\mathcal{H}^{\gamma+2\ell}$, which is closed as a Hilbert space and at the same time infinite dimensional. To resolve this subtlety, note that $\mathcal{A}^{-\ell}, \ell > 0$, when viewed as an operator acting between $\mathcal{H}^\gamma$ and $\mathcal{H}^{\gamma+2\ell}$ is not compact, but it is compact when viewed as an operator acting between $\mathcal{H}^\gamma$ and $\mathcal{H}^\beta$ for $\beta < \gamma + 2\ell$.

Indeed, let $u \in \mathcal{B}_{\mathcal{H}^\gamma}$. Then

$$||Ku||_{\gamma+2\ell}^2 = \sum_{k=1}^{\infty} \mu_k^{-2\ell} \mu_k^{\gamma+2\ell} u_k^2 = ||u||_\gamma^2 \leq 1,$$

thus $K(\mathcal{B}_{\mathcal{H}_\gamma}) \subset \mathcal{B}_{\mathcal{H}^{\gamma+2\ell}}$. Conversely suppose $y \in \mathcal{B}_{\mathcal{H}^{\gamma+2\ell}}$. Then $u := \mathcal{A}^\ell y$ is an element of $\mathcal{B}_{\mathcal{H}^\gamma}$ since

$$||u||_\gamma^2 = \sum_{k=1}^{\infty} \mu_k^\gamma \mu_k^{2\ell} y_k^2 = ||y||_{\gamma+2\ell}^2$$

and $Ku = y$. Hence $\mathcal{B}_{\mathcal{H}^{\gamma+2\ell}} \subset K(\mathcal{B}_{\mathcal{H}_\gamma})$ and

$$K(\mathcal{B}_{\mathcal{H}_\gamma}) = \mathcal{B}_{\mathcal{H}^{\gamma+2\ell}}.$$

Since $\mathcal{H}^{\gamma+2\ell}$ is infinite dimensional, its unit ball $\mathcal{B}_{\mathcal{H}^{\gamma+2\ell}}$ is not compact thus $K : \mathcal{H}^\gamma \to \mathcal{H}^{\gamma+2\ell}$ is not compact.

On the other hand, the ball $\mathcal{B}_{\mathcal{H}^{\gamma+2\ell}}$ is a compact subset of $\mathcal{H}^\beta$ for $\beta < \gamma + 2\ell$, thus $K : \mathcal{H}^\gamma \to \mathcal{H}^\beta$, $\beta < \gamma + 2\ell$ is compact, [9]. In this case, by the last lemma, K is no longer invertible.

Observe here the importance of the noise. If there was not any noise, we could restrict the data to $\mathcal{H}^{\gamma+2\ell}$ and then we would have the existence and continuity of the inverse of K. The existence of the noise, may cause the observations to be in a larger space $\mathcal{H}^\beta$, $\beta < \gamma + 2\ell$ and in that case we no longer have the existence, nor the continuity of the inverse of K.

Chapter 2

The Classical Approach

In this chapter we give a brief, however self-contained, presentation of the Classical Approach to Inverse Problems. We are mainly focused on the Tihkonov Regularization method and its generalizations. Note that while in this thesis we are only interested in additive noise models, the theory developed in this chapter applies to general noise models.

2.1 The Moore-Penrose Generalized Inverse

In this section we deal with the first and second types of ill-posedness of the inverse problem (1.3), namely the possible lack of existence of solution and the possible lack of uniqueness of the solution. For the presentation of this theory, which is taken from [5], we consider $K : X \to Y$ to be a compact, linear operator.

Definition 2.1.1. *[5, Definition 2.1] We define the following two generalized notions of solution of* (1.3)*:*

i) $u \in X$ *is called a* least squares solution *of* (1.3)*, if*

$$||Ku - y|| = \inf \{||Kz - y|| : z \in X\}$$

ii) $u \in X$ *is called the* best-approximate solution *of* (1.3)*, if* u *is a least squares solution of* (1.3) *and*

$$||u|| = \inf \{||z|| : z \text{ is a least squares solution of } (1.3)\}$$

The notion of least squares solution enforces existence of solution, while the notion of the best-approximate solution provides a way of choosing one of the possibly multiple least squares solutions, enforcing in this way the uniqueness of the solution.

Definition 2.1.2. *[5, Definition 2.2] The Moore-Penrose generalized inverse, $K^\dagger$, of K is the unique linear extension of $\tilde{K}^{-1}$ to*

$$\mathcal{D}(K^\dagger) = \mathcal{R}(K) \dotplus \mathcal{R}(K)^\perp,$$

such that

$$\mathcal{N}(K^\dagger) = \mathcal{R}(K)^\perp,$$

where

$$\tilde{K} = K|_{\mathcal{N}(K)^\perp} : \mathcal{N}(K)^\perp \to \mathcal{R}(K)$$

Note that the Moore-Penrose generalized inverse is defined only for $y \in \mathcal{D}(K^\dagger)$. Let Q be the orthogonal projector onto $\overline{\mathcal{R}(K)}$. If the range of K is infinite dimensional and therefore non-closed, $\mathcal{D}(K^\dagger)$ is a proper subset of Y. If $y \in \mathcal{D}(K^\dagger)$ the Moore-Penrose generalized inverse projects y onto $\mathcal{R}(K)$ and then gives the unique element of $\mathcal{N}(K)^\perp$, symbolized by $u^\dagger$, for which $Ku^\dagger = Qy$.

The following theorem tells us that the Moore-Penrose generalized inverse $K^\dagger$ is the solution operator mapping y onto the best-approximate solution of (1.3), when such a solution exists.

Theorem 2.1.3. *[5, Theorem 2.5] Let $y \in \mathcal{D}(K^\dagger)$. Then* (1.3) *has a unique best-approximate solution, which is given by*

$$u^\dagger = K^\dagger y.$$

The set of all least squares solutions is $u^\dagger + \mathcal{N}(K)$.

We can characterize a least squares solution of (1.3) by the *normal equation*:

Theorem 2.1.4. *[5, Theorem 2.6] Let $y \in \mathcal{D}(K^\dagger)$. Then $u \in X$ is a least squares solution of (1.3) if and only if the normal equation*

$$K^*Ku = K^*y \tag{2.1}$$

holds.

Using the metric properties of Q we can show that u is a least squares solution of (1.3) if and only if $Ku = Qy$ [5]. This implies that if $y \notin \mathcal{D}(K^\dagger)$, then there is no least squares solution of (1.3) since in that case $Qy \notin \mathcal{R}(K)$ therefore there exists no $u \in X$ such that $Ku = Qy$. Hence, $\mathcal{D}(K^\dagger)$ is the natural domain for the definition of the Moore-Penrose generalized inverse.

Moreover, note that if K is injective, then there is a unique least squares solution for every $y \in \mathcal{D}(K^\dagger)$, since $Ku = Qy$ has a unique solution $u \in X$. The unique least squares solution is obviously the best-approximate solution. In this case $\mathcal{N}(K)^\perp = X$ holds, so we have that $\tilde{K} = K$ and $K^{-1} : \mathcal{R}(K) \to X$ is well defined, therefore

$$K^\dagger y = K^{-1} Qy, \; \forall y \in \mathcal{D}(K^\dagger).$$

The next proposition shows that even though the Moore-Penrose generalized inverse enforces existence and uniqueness of the solution by weakening the notion of solution, it doesn't deal with the instability of the inverse problem (1.3).

Proposition 2.1.5. *[5, Proposition 2.7] Let $K : X \to Y$ be compact, $\dim \mathcal{R}(K) = \infty$. Then $K^\dagger$ is a densely defined unbounded linear operator with closed graph.*

Definition 2.1.6. *Let $K : X \to Y$ be a compact, linear operator. A singular system $(\sigma_k; v_k, w_k)$ for K is defined as follows: $\{\sigma_k^2\}_{k\in\mathbb{N}}$ are the nonzero eigenvalues of the selfadjoint operator K^*K, written down in decreasing order with multiplicity, $\sigma_k > 0$ and $\{v_k\}_{k\in\mathbb{N}}$ is a corresponding complete orthonormal system of eigenvectors of K^*K, spanning $\overline{\mathcal{R}(K^*K)}$. The $\{w_k\}_{k\in\mathbb{N}}$ are defined as*

$$w_k := \frac{Kv_k}{||Kv_k||}$$

and they form a complete orthonormal system of eigenvectors of KK^*, spanning $\overline{\mathcal{R}(KK^*)}$.

The following formulas, called *singular value decomposition* of K and

K^* respectively, hold:

$$Ku = \sum_{k=1}^{\infty} \sigma_k \langle u, v_k \rangle w_k, \ \forall u \in X$$

and

$$K^* y = \sum_{k=1}^{\infty} \sigma_k \langle y, w_k \rangle v_k, \ \forall y \in Y.$$

Using the *Singular Value Decomposition* one can find a condition for the existence of the best-approximate solution, as well as an expression for it, if it does exist.

Theorem 2.1.7. *[5, Theorem 2.8] Let $(\sigma_k; v_k, w_k)$ be a singular system for the compact linear operator K, $y \in Y$. Then we have:*

i)

$$y \in \mathcal{D}(K^\dagger) \iff \sum_{k=1}^{\infty} \frac{|\langle y, w_k \rangle|^2}{\sigma_k^2} < \infty \tag{2.2}$$

ii) For $y \in \mathcal{D}(K^\dagger)$

$$K^\dagger y = \sum_{k=1}^{\infty} \frac{\langle y, w_k \rangle}{\sigma_k} v_k. \tag{2.3}$$

The condition (2.2) for the existence of a best-approximate solution is called *Picard's criterion.* It says that "a best-approximate solution exists if and only if the coefficients of the observation y with respect to the singular functions w_k decay fast enough related to the singular values σ_k" [5].

One can readily see in expression (2.3) the instability of the problem (1.3). Indeed, in the case where the range of K is infinite dimensional, we know from Operator Theory that the singular values σ_k of K accumulate at 0, so errors in data of a fixed size can be amplified by an arbitrarily large factor.

Remark 2.1.8. In the "Laplacian-like" problem with which we are concerned in this thesis, we have an orthonormal basis of $\mathcal{H}$ consisting of eigenfunctions of K, so we don't need to use the Singular Value Decomposition, since $v_k \equiv w_k \equiv \phi_k$ and $\sigma_k \equiv \mu_k^{-\ell}$, for all $k \in \mathbb{N}$. The Hilbert scale $\mathcal{H}^s, s \in \mathbb{R}$ is induced by $\mathcal{A}$ which is diagonalizable in the same orthonormal basis of eigenfunctions as K, hence the above theory is simplified:

$K : \mathcal{H}^\gamma \to \mathcal{H}^\gamma$ is compact, self-adjoint and injective, therefore

$$\mathcal{R}(K)^\perp = \mathcal{N}(K) = \{0\}$$

and

$$\mathcal{D}(K^\dagger) = \mathcal{R}(K).$$

This implies that $\tilde{K} \equiv K$ and that the orthogonal projector Q, of Y onto $\overline{R(K)}$, is the identity thus it no longer appears in the formulae. We hence have

$$u^\dagger = K^{-1}y, \ \forall y \in \mathcal{R}(K).$$

By Lemma 1.2.3, we have that $\mathcal{R}(K) = \mathcal{H}^{\gamma+2\ell}$, thus

$$\mathcal{D}(K^\dagger) = \mathcal{H}^{\gamma+2\ell}.$$

Suppose $y \in \mathcal{H}^\gamma$. Then by the preceding observations:

i)

$$y \in \mathcal{D}(K^\dagger) \iff \sum_{k=1}^{\infty} \mu_k^{\gamma+2\ell} y_k^2 < \infty \tag{2.4}$$

ii) If $y \in \mathcal{D}(K^\dagger)$,

$$u^\dagger = K^{-1}y = \sum_{k=1}^{\infty} \mu_k^\ell y_k \phi_k. \tag{2.5}$$

The criterion (2.4) for the existence of a solution is a decay condition on the coefficients of y similar to the Picard's criterion in (2.2). From (2.5) we can again see that we have instability, since $\mu_k \overset{k\to\infty}{\longrightarrow} \infty$, therefore for $\ell > 0$, $\mu_k^\ell \overset{k\to\infty}{\longrightarrow} \infty$ and so small errors in y can be amplified by an arbitrarily large factor and become large errors in the solution. Note that instability here refers to the norm topology of $\mathcal{H}^\gamma$, not to the norm topology of $\mathcal{H}^{\gamma+2\ell}$ where by Lemma 1.2.3 the inverse K^{-1} is continuous.

In the next section we introduce Tikhonov Regularization, which is a method to deal with the instability of inverse problems and in particular the problem (1.3).

2.2 Regularization of Inverse Problems

In this section we deal with the third type of ill-posedness of the inverse problem (1.3), namely the instability of the solution. Since our observations are polluted by noise and therefore they are inexact, we need an approximation u^δ of the (best-approximate) solution $u^\dagger$, which for small data errors is guaranteed to be close to $u^\dagger$. As we've already explained in the previous section, the use of the best-approximate solution $K^\dagger y^\delta$ is not good for two reasons:

a) First, the additive noise can be such that $y^\delta \notin \mathcal{D}(K^\dagger)$ and so there exists no least squares solution at all.

b) Second and most importantly, even if $y^\delta \in \mathcal{D}(K^\dagger)$, in all the interesting cases $K^\dagger$ is unbounded and so even if the error level δ in the data is very low, we can have a very large solution error $\left|\left|u^\dagger - u^\delta\right|\right|$.

In Subsections 2.2.1 and 2.2.2 we develop some basic general regularization theory and then in the rest of the section we discuss *Tikhonov Regularization*.

2.2.1 Basic Definitions and Results of Regularization Theory

According to [5] "Regularization is the approximation of an ill-posed problem by a family of neighbouring well-posed problems".

Here, we want to approximate the best-approximate solution $u^\dagger = K^\dagger y$ of (1.3), for the specific right hand side $y \in \mathcal{D}(K^\dagger)$ where we have polluted data $y^\delta \in Y$ known, such that

$$\left|\left|y - y^\delta\right|\right| \leq \delta.$$

To do this, we construct a family of approximations u^δ_λ parametrized by the *regularization parameter* λ, which we want to depend continuously on the data y^δ and as $\delta \to 0$ for the appropriate choice of the regularization parameter $\lambda = \lambda(\delta)$

$$u^\delta_\lambda \to u^\dagger.$$

A choice of the regularization paramater, depending only on the error level δ, and not on the data y^δ, is called an *a-priori parameter choice rule.*

For $y \in \mathcal{D}(K^\dagger)$ we have the expression (2.3) for the best-approximate solution of (1.3). Problems are caused by the factor $\frac{1}{\sigma_k}$ in (2.3) which tends to infinity in the ill-posed case.

The idea, according to [22], [16], is to multiply each factor $\frac{1}{\sigma_k}$ in (2.3) by the value of a filter function $f : (0, \infty) \times (0, ||K||^2] \to \mathbb{R}$, $f_\lambda(\sigma_k^2)$ which behaves in general terms in the following way:

i) for each $\lambda > 0$

$$\frac{f_\lambda(\sigma^2)}{\sigma} \xrightarrow{\sigma \to 0} 0$$

ii) for each $\sigma \in (0, ||K||]$

$$f_\lambda(\sigma^2) \xrightarrow{\lambda \to 0} 1$$

i.e. it filters out the components which correspond to small singular values which cause the instability, and as the regularization parameter vanishes the effect of this filter function fades away.

The regularized version of (2.3) is then defined by

$$R_\lambda y = \sum_{k=1}^{\infty} \frac{f(\sigma_k^2)}{\sigma_k} \langle y, w_k \rangle \, v_k. \tag{2.6}$$

Example 2.2.1. One example of such a filter function is the Truncated SVD filter function (TSVD) given by

$$f_\lambda(\sigma^2) = \begin{cases} 1, & if \quad \sigma^2 > \lambda, \\ 0, & if \quad \sigma^2 \leq \lambda, \end{cases}$$

for all $\sigma \in (0, ||K||]$. The regularized version of (2.3) becomes

$$R_\lambda^{TSVD} y = \sum_{\sigma_k^2 > \lambda} \frac{\langle y, w_k \rangle}{\sigma_k} v_k.$$

In [5] the same idea is applied in a more operator-theoretic context. They introduce the theory of Functional Calculus, in order to define functions of operators [3], [11]. In our particular problem though, since we can diagonalize the operator K in the basis $\{\phi_k\}$, we don't need to use the tools of Functional Calculus.

The theory developed in the rest of Chapter 1 is primarily taken from [5] where it is developed in greater generality for $K : X \to Y$, bounded linear operator. In order to simplify the proofs of this theory, we will henceforward assume that $K : X \to X$ is compact, self-adjoint and injective, with an eigensystem $\{\phi_k, \nu_k\}_{k\in\mathbb{N}}$, as stated in Subsection 1.2.1. All of the proofs provided in the rest of the chapter are our simplifications of the proofs given in [5]. For illustrative purposes we will retain the statements of the theorems as they are in the more general theory developed in [5].

From the normal equation, if K^*K is invertible, for $y \in \mathcal{D}(K^\dagger)$ we have

$$u^\dagger = (K^*K)^{-1}K^*y \tag{2.7}$$

In the ill-posed case where $R(K)$ is non-closed $K^*K : X \to X$ is not invertible. Indeed, $\mathcal{R}(K^*)$ is determined by the K^* image of $\mathcal{N}(K^*)^\perp = \overline{\mathcal{R}(K)}$, therefore, since $\mathcal{R}(K) \subsetneq \overline{\mathcal{R}(K)}$, we have that $\mathcal{R}(K^*K) \subsetneq X$, thus K^*K is not invertible.

The regularized version of (2.7) for $y \in \mathcal{D}(K^\dagger)$ is then defined by replacing $(K^*K)^{-1}$ by the values of a parameter depending family of functions $g : (0, \infty) \times (0, ||K||^2] \to \mathbb{R}$

$$u_\lambda = g_\lambda(K^*K)K^*y. \tag{2.8}$$

For inexact data $y^\delta \in Y$ we then define the regularized approximation

$$u_\lambda^\delta = g_\lambda(K^*K)K^*y^\delta. \tag{2.9}$$

Using the diagonalization of K in the eigensystem $\{\phi_k, \nu_k\}_{k\in\mathbb{N}}$ we have

$$u_\lambda = \sum_{k=1}^{\infty} g_\lambda(\nu_k^2)\nu_k y_k \phi_k \tag{2.10}$$

and

$$u_\lambda^\delta = \sum_{k=1}^{\infty} g_\lambda(\nu_k^2)\nu_k y_k^\delta \phi_k. \tag{2.11}$$

The following theorem confirms that for a properly chosen family of functions g_λ, u_λ tends to the best approximate solution as the regularization fades away.

Theorem 2.2.2. *[5, Theorem 4.1] Suppose that for every $\lambda > 0$, $g_\lambda : [0, ||K||^2] \to \mathbb{R}$ fulfills the following assumptions: g_λ is piecewise continuous and there exists $C > 0$ such that*

$$|\sigma g_\lambda(\sigma)| \le C, \tag{2.12a}$$

and

$$\lim_{\lambda \to 0} g_\lambda(\sigma) = \frac{1}{\sigma}, \tag{2.12b}$$

for all $\sigma \in (0, ||K||^2]$.

Then for all $y \in \mathcal{D}(K^\dagger)$,

$$\lim_{\lambda \to 0} g_\lambda(K^*K)K^*y = u^\dagger.$$

If $y \notin \mathcal{D}(K^\dagger)$, then

$$\lim_{\lambda \to 0} ||g_\lambda(K^*K)K^*y|| = \infty.$$

Proof. Suppose $y \in \mathcal{D}(K^\dagger)$, so that $u^\dagger$ is well defined,

$$u^\dagger = \sum_{k=1}^{\infty} \frac{y_k}{\nu_k}\phi_k \in X. \tag{i}$$

Then by (2.10)

$$\left|\left|u^\dagger - u_\lambda\right|\right|^2 = \sum_{k=1}^{\infty} \left(\frac{1 - g_\lambda(\nu_k^2)\nu_k^2}{\nu_k}\right)^2 y_k^2. \tag{ii}$$

By (2.12a) we have

$$|1 - \sigma g_\lambda(\sigma)| \le 1 + |\sigma g_\lambda(\sigma)| \le 1 + C,\ \forall \sigma \in [0, ||K||^2], \lambda > 0,$$

hence

$$\left(\frac{1-g_\lambda(\nu_k^2)\nu_k^2}{\nu_k}\right)^2 y_k^2 \leq (1+C)^2\frac{y_k^2}{\nu_k^2}, \ \ \forall\lambda>0,\ \forall k\in\mathbb{N},$$

where by (i)

$$\sum_{k=1}^{\infty}\frac{y_k^2}{\nu_k^2} = \left|\left|u^\dagger\right|\right|^2 < \infty.$$

Furthermore by (2.12b) we have that for all $k \in \mathbb{N}$

$$1-g_\lambda(\nu_k^2)\nu_k^2 \to 0, \quad \text{as} \quad \lambda\to 0$$

and so by applying the Dominated Convergence Theorem to (ii) we get that

$$\lim_{\lambda\to 0}\left|\left|u^\dagger - u_\lambda\right|\right| = 0.$$

Suppose now that $y \notin \mathcal{D}(K^\dagger)$ and assume that there exists a sequence $\lambda_n \to 0$ such that $||u_{\lambda_n}||$ is bounded. Since X is a Hilbert space therefore reflexive, there exists a subsequence (denoted again by u_{λ_n}) which converges weakly to some $u \in X$. Since K is compact we have that

$$Ku_{\lambda_n} \to Ku. \tag{iii}$$

On the other hand again by the Dominated Convergence Theorem using (2.12a), (2.12b) we have

$$Ku_{\lambda_n} = \sum_{k=1}^{\infty} g_{\lambda_n}(\nu_k^2)\nu_k^2 y_k\phi_k \overset{n\to\infty}{\longrightarrow} \sum_{k=1}^{\infty} y_k\phi_k = y. \tag{iv}$$

By (iii) and (iv) we get $Ku = y$, therefore $y \in \mathcal{D}(K^\dagger)$, contradiction. □

The next theorem shows the effect of the regularization to the stability problems, i.e. it shows that the regularized approximation u_λ is continuous with respect to the data.

Theorem 2.2.3. *[5, Theorem 4.2] Let g_λ and C be as in Theorem 2.2.2, u_λ and u_λ^δ be defined by* (2.8) *and* (2.9) *respectively. For $\lambda > 0$, let*

$$G_\lambda := \sup\left\{|g_\lambda(\sigma)| : \sigma \in [0, ||K||^2]\right\}.$$

Then

$$\left|\left|Ku_\lambda - Ku_\lambda^\delta\right|\right| \le C\delta \tag{2.13a}$$

and

$$\left|\left|u_\lambda - u_\lambda^\delta\right|\right| \le \delta\sqrt{CG_\lambda} \tag{2.13b}$$

hold.

Proof. By (2.10), (2.11) and (2.12a) we have

$$\left|\left|Ku_\lambda - Ku_\lambda^\delta\right|\right|^2 = \sum_{k=1}^{\infty} \left(\nu_k^2 g_\lambda(\nu_k^2)(y_k - y_k^\delta)\right)^2$$

$$\le C^2 \sum_{k=1}^{\infty} (y_k - y_k^\delta)^2 = C^2 \left|\left|y - y^\delta\right|\right|^2 \le C^2\delta^2$$

hence (2.13a) holds.

By (2.10), (2.11), (2.12a) and Cauchy-Schwarz inequality we have

$$\left|\left|u_\lambda - u_\lambda^\delta\right|\right|^2 = \sum_{k=1}^{\infty} \left(\nu_k g_\lambda(\nu_k^2)(y_k - y_k^\delta)\right)^2 \tag{i}$$

$$\le \left(\sum_{k=1}^{\infty} \nu_k^4 g_\lambda^2(\nu_k^2)(y_k - y_k^\delta)^2\right)^{\frac{1}{2}} \left(\sum_{k=1}^{\infty} g_\lambda^2(\nu_k^2)(y_k - y_k^\delta)^2\right)^{\frac{1}{2}}$$

$$\overset{(i)}{=} \left|\left|Ku_\lambda - Ku_\lambda^\delta\right|\right| \left(\sum_{k=1}^{\infty} g_\lambda^2(\nu_k^2)(y_k - y_k^\delta)^2\right)^{\frac{1}{2}}$$

$$\overset{(2.13a)}{\le} C\delta G_\lambda \left(\sum_{k=1}^{\infty} (y_k - y_k^\delta)^2\right)^{\frac{1}{2}} = CG_\lambda\delta^2$$

hence (2.13b) holds. □

The quantity $\left|\left|u_\lambda - u_\lambda^\delta\right|\right|$ is called *propagated data error* since it measures the difference between the regularized approximations, u_λ and u_λ^δ, which is caused by the error in the data. By the last theorem we can split the total solution error in two components

$$\left|\left|u^\dagger - u_\lambda^\delta\right|\right| \le \left|\left|u^\dagger - u_\lambda\right|\right| + \delta\sqrt{CG_\lambda} \tag{2.14}$$

By Theorem 2.2.2 the first term vanishes as the regularization parameter tends to 0, but by the assumptions on g, for fixed error level δ, as $\lambda \to 0$ the value of G_λ explodes. It is apparent from this observation that in order to achieve convergence of the regularized approximation u_λ^δ to $u^\dagger$, as the error level tends to 0, we need to let the regularization parameter λ also go to 0, but in a carefully chosen δ-dependent way.

Example 2.2.4. The Truncated SVD can also be expressed in terms of the notation and theory developed in [5]. Indeed, for

$$g_\lambda^{TSVD}(\sigma^2) = \begin{cases} \frac{1}{\sigma^2}, & if \quad \sigma^2 > \lambda, \\ 0, & if \quad \sigma^2 \leq \lambda, \end{cases}$$

we get using (2.10)

$$u_\lambda^{TSVD} = \sum_{\nu_k^2 > \lambda} \frac{y_k}{\nu_k} \phi_k.$$

Note that for every $\lambda > 0$, g_λ^{TSVD} is piecewise continuous and that the other two conditions of Theorem 2.2.2 are satisfied trivially (for C=1). Furthermore, observe that $G_\lambda^{TSVD} = \frac{1}{\lambda}$, for every $\lambda > 0$.

2.2.2 Order Optimality - The Worst-Case Error

We now give some results regarding convergence rates of regularization methods satisfying the assumptions of Theorem 2.2.2.

For $\mu, \rho > 0$ define

$$X_\mu = \mathcal{R}((K^*K)^\mu)$$

with the norm

$$||x||_{X_\mu} = \left|\left|(K^*K)^{-\mu}x\right|\right|, \ \forall x \in X_\mu$$

and

$$X_{\mu,\rho} = \left\{u \in X : u = (K^*K)^\mu w, \ ||w|| \leq \rho\right\}.$$

Note, that for K compact, self-adjoint and injective as we will always assume in the proofs of the following theory, X_μ is identified with $\mathcal{X}_\mu$ and $||.||_{X_\mu}$ is identified with $||.||_{\mathcal{X}_\mu}$.

Definition 2.2.5. *[16, Definition 1.18] Let Z be a subset of X and suppose $||.||_Z$ is a stronger norm on Z, i.e. $\exists c > 0$ such that $||x|| \le c\,||x||_Z$, $\forall x \in Z$. Then we define*

$$\mathcal{F}(\delta, \rho, ||.||_Z) := \sup\left\{||x|| : x \in Z, ||Kx|| \le \delta, ||x||_Z \le \rho\right\}, \tag{2.15}$$

and call $\mathcal{F}(\delta, \rho, ||.||_Z)$ the worst-case error for data-error δ and a-priori information $||x||_Z \le \rho$.

The worst-case error depends on the operator K and the norms in X, Z. It is desirable that the worst-case error not only converges to zero as the noise-level δ tends to 0 but that is of order as close to δ as possible. To explain the reason for this we first give the following proposition:

Proposition 2.2.6. *[5, Proposition 3.10] Let $\mu, \rho, \delta > 0$ and $R : Y \to X$ be an arbitrary map with $R(0) = 0$. Define*

$$\Delta(\delta, X_{\mu,\rho}, R) = \sup\left\{\left|\left|R(y^\delta) - u\right|\right| : u \in X_{\mu,\rho},\ y^\delta \in Y,\ \left|\left|Ku - y^\delta\right|\right| \le \delta\right\}.$$

Then

$$\Delta(\delta, X_{\mu,\rho}, R) \ge \mathcal{F}(\delta, \rho, ||.||_{X_\mu}). \tag{2.16}$$

Proof. Let $u \in X_{\mu,\rho}$ with $||Ku|| \le \delta$ be arbitrary. Then for $y^\delta = 0 \in Y$, $R(y^\delta) = R(0) = 0$ we have

$$\left|\left|R(y^\delta) - u\right|\right| = ||u||$$

hence

$$\Delta(\delta, X_{\mu,\rho}, R) \ge ||R(0) - u|| = ||u||.$$

By taking the supremum over all $u \in X_{\mu,\rho}$ with $||Ku|| \le \delta$ we obtain (2.16). □

Since we can view u_λ^δ as

$$u_\lambda^\delta = R_\lambda(y^\delta)$$

for a map $R_\lambda : Y \to X$ such that $R_\lambda(0) = 0$, the last result implies that under the *a-priori* assumption $u^\dagger \in X_{\mu,\rho}$, no regularization method can converge faster than the worst-case error $\mathcal{F}(\delta, \rho, ||.||_{X_\mu})$ as $\delta \to 0$. The worst-case error is a lower bound for the solution error of a regularization method under the *a-priori* information $u^\dagger \in X_{\mu,\rho}$.

This motivates the following definition of optimality for a regularization method [16]:

Definition 2.2.7. *We say that a regularization method* (R_λ, λ) *is* asymptotically optimal in $X_{\mu,\rho}$, *if there exists* $c > 0$ *such that under the a-priori information* $u^\dagger \in X_{\mu,\rho}$ *and* $||y - y^\delta|| \leq \delta$, *the regularized approximation* $u_\lambda^\delta = R_\lambda y^\delta$ *is such that*

$$||u_\lambda^\delta - u^\dagger|| \leq c\mathcal{F}(\delta, \rho, ||.||_{X_\mu}), \quad \forall \delta > 0.$$

By the last definition, a regularization method (R_λ, λ) is optimal if under the *a-priori* information $u^\dagger \in X_{\mu,\rho}$, as $\delta \to 0$ it achieves the same convergence rate as the worst-case error, which by the last proposition is the best possible convergence rate.

We now give an interpolation inequality necessary for the analysis which follows.

Lemma 2.2.8. *For* $q \geq r \geq 0$

$$||K^{2r}x|| \leq ||K^{2q}x||^{\frac{r}{q}} ||x||^{1-\frac{r}{q}} \tag{2.17}$$

Proof. By Hölder inequality we have

$$||K^{2r}x||^2 = \sum_{k=1}^{\infty} \nu_k^{4r} x_k^2 = \sum_{k=1}^{\infty} \nu_k^{4r} x_k^{\frac{2r}{q}} x_k^{2-\frac{2r}{q}}$$

$$\leq \left(\sum_{k=1}^{\infty} \nu_k^{4q} x_k^2 \right)^{\frac{r}{q}} \left(\sum_{k=1}^{\infty} x_k^2 \right)^{1-\frac{r}{q}} = ||K^{2q}x||^{\frac{2r}{q}} ||x||^{2(1-\frac{r}{q})}.$$

□

The next theorem which is a generalization of [16, Theorem 1.21] based on a combination of [5, Proposition 3.14] and [5, Proposition 3.15], provides an estimate of the worst-case error.

Theorem 2.2.9. *Let $\mu, \rho > 0$. Under the a-priori information that $x \in X_\mu$ and $||x||_\mu \leq \rho$ we have the following estimate for the worst-case error:*

$$\mathcal{F}(\delta, \rho, ||.||_{X_\mu}) \leq \delta^{\frac{2\mu}{2\mu+1}} \rho^{\frac{1}{2\mu+1}}$$

Furthermore, there exists a sequence $\delta_k \stackrel{k\to\infty}{\longrightarrow} 0$ such that

$$\mathcal{F}(\delta_k, \rho, ||.||_{X_\mu}) = \delta_k^{\frac{2\mu}{2\mu+1}} \rho^{\frac{1}{2\mu+1}},$$

i.e. the estimate is asymptotically sharp.

Proof. Let $x = K^{2\mu} z \in X_\mu$ with $||Kx|| \leq \delta$ and $||x||_{X_\mu} \leq \rho$
i.e. $||K^{2\mu+1} z|| \leq \delta$ and $||z|| \leq \rho$.
Then using the interpolation inequality (2.17) for $r = \mu$, $q = \mu + \frac{1}{2}$, we have

$$||x|| = \left|\left|K^{2\mu} z\right|\right| \leq \left|\left|K^{2\mu+1} z\right|\right|^{\frac{2\mu}{2\mu+1}} ||z||^{\frac{1}{2\mu+1}} \leq \delta^{\frac{2\mu}{2\mu+1}} \rho^{\frac{1}{2\mu+1}},$$

hence the desired estimate on the worst-case error.

For the sharpness assertion, it suffices to determine $\{\delta_k\}$ and $\{x_k\} \subset X_\mu$ such that

i) $\delta_k \stackrel{k\to\infty}{\longrightarrow} 0$,

ii) $||x_k||_{X_\mu} \leq \rho$,

iii) $||Kx_k|| \leq \delta_k$ and

iv) $||x_k|| = \delta^{\frac{2\mu}{2\mu+1}} \rho^{\frac{1}{2\mu+1}}$.

Define $\delta_k = \rho \nu_k^{2\mu+1}$. Since $\nu_k \stackrel{k\to\infty}{\longrightarrow} 0$ we have that $\delta_k \stackrel{k\to\infty}{\longrightarrow} 0$.
Observe that

$$\left(\frac{\delta_k}{\rho}\right)^{\frac{2}{2\mu+1}} = \nu_k^2. \tag{$*$}$$

Let $x_k = \rho K^{2\mu} \phi_k$. Obviously $x_k \in X_\mu$ and $||x_k||_{X_\mu} \leq \rho$.

Furthermore,

$$x_k = \rho \nu_k^{2\mu} \phi_k = \rho \left(\frac{\delta_k}{\rho} \right)^{\frac{2\mu}{2\mu+1}} \phi_k = \delta_k^{\frac{2\mu}{2\mu+1}} \rho^{\frac{1}{2\mu+1}} \phi_k,$$

hence

$$||x_k|| = \delta_k^{\frac{2\mu}{2\mu+1}} \rho^{\frac{1}{2\mu+1}}$$

and

$$K^2 x_k = \delta_k^{\frac{2\mu}{2\mu+1}} \rho^{\frac{1}{2\mu+1}} \nu_k^2 \phi_k \overset{(*)}{=} \delta_k^{\frac{2\mu+2}{2\mu+1}} \rho^{\frac{-1}{2\mu+1}} \phi_k,$$

hence

$$||Kx_k||^2 = \langle Kx_k, Kx_k \rangle = \left\langle K^2 x_k, x_k \right\rangle = \left\langle \delta_k^{\frac{2\mu+2}{2\mu+1}} \rho^{\frac{-1}{2\mu+1}} \phi_k, \delta_k^{\frac{2\mu}{2\mu+1}} \rho^{\frac{1}{2\mu+1}} \phi_k \right\rangle = \delta_k^2.$$

□

Observe that for $\mu \to 0$ the convergence rates become very slow.

The next theorem in conjunction with the succeeding corollary, provide sufficient conditions which secure the optimality of a regularization method.

Theorem 2.2.10. *[5, Theorem 4.3] Let g_λ fulfill the assumptions of Theorem 2.2.2 and define*

$$r_\lambda(\sigma) := 1 - \sigma g_\lambda(\sigma), \ \ \forall \lambda > 0, \ \sigma \in [0, ||K||^2].$$

Suppose $\mu, \rho > 0$ and let $\omega_\mu : (0, \lambda_0) \to \mathbb{R}$ be such that for all $\lambda \in (0, \lambda_0)$ and $\sigma \in [0, ||K||^2]$

$$\sigma^\mu |r_\lambda(\sigma)| \leq \omega_\mu(\lambda). \tag{2.18}$$

Then for $u^\dagger \in X_{\mu,\rho}$,

$$\left|\left|u_\lambda - u^\dagger\right|\right| \leq \omega_\mu(\lambda)\rho$$

and

$$\left|\left|Ku_\lambda - Ku^\dagger\right|\right| \leq \omega_{\mu+\frac{1}{2}}(\lambda)\rho.$$

Proof. Let $w \in X$ be such that $u^\dagger = (K^*K)^\mu w, \ ||w|| \leq \rho$.
By the normal equation (2.1) we have

$$K^* y = (K^*K)^{\mu+1} w$$

or equivalently

$$\nu_k y_k = \nu_k^{2+2\mu} w_k, \ \forall k \in \mathbb{N},$$

thus

$$u^\dagger - u_\lambda = \sum_{k=1}^{\infty} \nu_k^{2\mu} w_k \phi_k - \sum_{k=1}^{\infty} g_\lambda(\nu_k^2)\nu_k y_k \phi_k$$

$$= \sum_{k=1}^{\infty} \nu_k^{2\mu} w_k \phi_k - \sum_{k=1}^{\infty} g_\lambda(\nu_k^2)\nu_k^{2+2\mu} w_k \phi_k = \sum_{k=1}^{\infty} \nu_k^{2\mu}(1 - g_\lambda(\nu_k^2)\nu_k^2) w_k \phi_k$$

$$= \sum_{k=1}^{\infty} \nu_k^{2\mu} r_\lambda(\nu_k^2) w_k \phi_k$$

and similarly

$$Ku^\dagger - Ku_\lambda = \sum_{k=1}^{\infty} \nu_k^{2\mu+1} r_\lambda(\nu_k^2) w_k \phi_k.$$

By (2.18)

$$\left|\left|u^\dagger - u_\lambda\right|\right|^2 = \sum_{k=1}^{\infty} (\nu_k^{2\mu} r_\lambda(\nu_k^2))^2 w_k^2$$

$$\leq \sum_{k=1}^{\infty} \omega_\mu^2(\lambda) w_k^2 = \omega_\mu^2(\lambda) \left|\left|w\right|\right|^2 \leq \omega_\mu^2(\lambda)\rho^2$$

and

$$\left|\left|Ku^\dagger - Ku_\lambda\right|\right|^2 = \sum_{k=1}^{\infty} (\nu_k^{2\mu+1} r_\lambda(\nu_k^2))^2 w_k^2$$

$$\leq \sum_{k=1}^{\infty} \omega_{\mu+\frac{1}{2}}^2(\lambda) w_k^2 = \omega_{\mu+\frac{1}{2}}^2(\lambda) \left|\left|w\right|\right|^2 \leq \omega_{\mu+\frac{1}{2}}^2(\lambda)\rho^2.$$

□

Corollary 2.2.11. *[5, Corollary 4.4] Let g_λ satisfy the assumptions of Theorem 2.2.10 with*

$$\omega_\mu(\lambda) = c\lambda^\mu, \tag{2.19}$$

for some $c > 0$ and assume that $G_\lambda = O(\frac{1}{\lambda})$, as $\lambda \to 0$, where G_λ is defined in Theorem 2.2.3. Then, with the parameter choice rule

$$\lambda \sim \left(\frac{\delta}{\rho}\right)^{\frac{2}{2\mu+1}}, \tag{2.20}$$

the regularization method (R_λ, λ), defined by g_λ, is of optimal order in $X_{\mu,\rho}$.

Proof. By (2.14), (2.19) and Theorem 2.2.10 we have

$$||u_\lambda^\delta - u^\dagger|| \le ||u_\lambda - u^\dagger|| + \delta\sqrt{CG_\lambda}$$
$$\le c\lambda^\mu \rho + \delta\sqrt{\frac{C}{\lambda}}.$$

By (2.20) the right hand side is bounded by

$$c'\left(\rho\left(\frac{\delta}{\rho}\right)^{\frac{2\mu}{2\mu+1}} + \delta\left(\frac{\rho}{\delta}\right)^{\frac{1}{2\mu+1}}\right) = c'\delta^{\frac{2\mu}{2\mu+1}}\rho^{\frac{1}{2\mu+1}},$$

where $c' > 0$ is a constant. Hence by the definition of order optimality and Theorem 2.2.9, the method (R_λ, λ) is of optimal order in $X_{\mu,\rho}$.

□

Remark 2.2.12. Note that (2.19) may only be true for $\mu \in (0, \mu_0]$ for some $\mu_0 > 0$ which according to [5] is called *the qualification* of the regularization method. If (2.19) holds for all $\mu > 0$, then the last corollary implies that for larger values of μ, the convergence rate of the solution error to 0, as $\delta \to 0$, under the *a-priori* information $u^\dagger \in X_\mu$, is guaranteed to improve eventually getting arbitrarily close to $O(\delta)$ for large enough values of μ. In that case, we say that the regularization method does not saturate. However, for finite values of μ_0 this is not always the case; the regularization method may saturate at some rate, as we will see later is the case for the *Tikhonov Regularization* method.

Example 2.2.13. We revisit Example 2.2.4 to show that the Truncated SVD method satisfies the assumptions of Theorem 2.2.10 and Corollary 2.2.11, which secure that it is an optimal method in $X_{\mu,\rho}$. We have already seen that $G_\lambda^{TSVD} = \frac{1}{\lambda}$ for all $\lambda > 0$, thus it suffices to show that there exist $c, \lambda_0, \mu_0 > 0$ such that

$$\sigma^\mu |r_\lambda(\sigma)| \le c\lambda^\mu,\ \forall \sigma \in [0, ||K||^2],\ \forall \lambda \in (0, \lambda_0),\ \forall \mu(0, \mu_0). \tag{2.21}$$

First we observe that

$$r_\lambda^{TSVD}(\sigma) = \begin{cases} 0, & if \quad \sigma > \lambda, \\ 1, & if \quad \sigma \le \lambda, \end{cases}$$

thus

$$\sigma^\mu |r_\lambda^{TSVD}(\sigma)| = \begin{cases} 0, & if \quad \sigma > \lambda, \\ \sigma^\mu, & if \quad \sigma \leq \lambda. \end{cases}$$

By the last calculation we have that the condition (2.21) is satisfied for $c = 1$ for all $\lambda_0 > 0$ and most importantly for all $\mu_0 > 0$. By the last remark, we thus have that the Truncated SVD regularization method does not saturate.

2.2.3 Tikhonov Regularization

In this subsection we present a particular regularization method, that is a particular choice of the family of functions g_λ introduced in the previous subsection. This method is called *Tikhonov Regularization* [21], [5], [16].

Define the Tikhonov filter function

$$g_\lambda^T(\sigma) = \frac{1}{\sigma + \lambda}, \quad \lambda > 0, \quad \sigma \in [0, ||K||^2],$$

and observe that

$$|\sigma g_\lambda^T(\sigma)| \leq 1, \ \forall \lambda > 0, \ \forall \sigma \in [0, ||K||^2] \tag{2.22}$$

and that for $\sigma \neq 0$

$$\lim_{\lambda \to 0} g_\lambda^T(\sigma) = \frac{1}{\sigma} \tag{2.23}$$

i.e. g^T satisfies the hypotheses of Theorem 2.2.2.

With this choice of filter function, the regularized approximation for inexact data $y^\delta \in Y (\equiv X)$ is given by

$$u_\lambda^\delta = (K^* K + \lambda I)^{-1} K^* y^\delta \tag{2.24}$$

or using the diagonalization of K, by

$$u_\lambda^\delta = \sum_{k=1}^{\infty} \frac{\nu_k}{\nu_k^2 + \lambda} y_k^\delta \phi_k. \tag{2.25}$$

Note that u_λ^δ is well defined because for all $k \in \mathbb{N}$ we have $\frac{\nu_k}{\nu_k^2+\lambda} \leq \frac{\nu_k}{\lambda}$ which for all $\lambda > 0$ is bounded, since K is assumed to be compact.

The following theorem provides a Variational Characterization for the Thikhonov regularization.

Theorem 2.2.14. *[5, Theorem 5.1] Let* $u_\lambda^\delta = (K^*K + \lambda I)^{-1} K^* y^\delta$. *Then* u_λ^δ *is the unique minimizer of the Tikhonov functional*

$$I_\lambda(u) = \frac{1}{2} \left|\left|Ku - y^\delta\right|\right|^2 + \frac{\lambda}{2} \left|\left|u\right|\right|^2 \tag{2.26}$$

over the space $\mathcal{H}$.

Proof. In the orthonormal basis $\{\phi_k\}$, the Tikhonov functional is given by

$$I_\lambda(u) = \frac{1}{2} \sum_{k=1}^{\infty} (\nu_k u_k - y_k^\delta)^2 + \frac{\lambda}{2} \sum_{k=1}^{\infty} u_k^2.$$

We minimize $I_\lambda(u)$ by minimizing each term of the series over u_k. We differentiate each term of the series with respect to u_k to get

$$\nu_k^2 u_k - \nu_k y_k^\delta + \lambda u_k = 0,$$

therefore the minimum of I_λ is attained for

$$u_k = \frac{\nu_k y_k^\delta}{\nu_k^2 + \lambda}$$

Hence

$$\operatorname*{argmin}_{u \in X} I_\lambda(u) = \sum_{k=1}^{\infty} \frac{\nu_k y_k^\delta}{\nu_k^2 + \lambda} \phi_k = u_\lambda^\delta.$$

□

The next theorem provides a sufficient condition on the *a-priori* parameter choice rule $\lambda = \lambda(\delta)$, for the convergence of the Tikhonov regularized approximation u_λ^δ to $u^\dagger$:

Theorem 2.2.15. *[5, Theorem 5.2] [16, Theorem 2.12] Suppose* $y \in \mathcal{R}(K)$ *and* $\left|\left|y - y^\delta\right|\right| \leq \delta$.

If $\lambda = \lambda(\delta)$ is such that

$$\lim_{\delta \to 0} \lambda(\delta) = 0 \tag{2.27a}$$

and

$$\lim_{\delta \to 0} \frac{\delta^2}{\lambda(\delta)} = 0 \tag{2.27b}$$

then

$$\lim_{\delta \to 0} u^{\delta}_{\lambda(\delta)} = u^{\dagger}.$$

Proof. By the assumption $y \in \mathcal{R}(K)$, so there exists a unique $u^{\dagger}$ in $\mathcal{N}(K)^{\perp} = \mathcal{H}$, such that $y = Ku^{\dagger}$.

In the orthonormal basis $\{\phi_k\}$ we have the expressions

$$y = \sum_{k=1}^{\infty} y_k \phi_k = \sum_{k=1}^{\infty} \nu_k q_k \phi_k, \quad \text{where} \quad y_k = \langle y, \phi_k \rangle \quad \text{and} \quad q_k = \left\langle u^{\dagger}, \phi_k \right\rangle.$$

By (2.25) we have

$$\begin{aligned} \left|\left|u^{\delta}_{\lambda} - u^{\dagger}\right|\right|^2 &= \sum_{k=1}^{\infty} \left(\frac{\nu_k y^{\delta}_k}{\nu_k^2 + \lambda} - q_k \right)^2 = \sum_{k=1}^{\infty} \left(\frac{\nu_k y^{\delta}_k - q_k \nu_k^2 - \lambda q_k}{\nu_k^2 + \lambda} \right)^2 \\ &\leq 2 \sum_{k=1}^{\infty} \left(\frac{\nu_k y^{\delta}_k - q_k \nu_k^2}{\nu_k^2 + \lambda} \right)^2 + 2 \sum_{k=1}^{\infty} \left(\frac{\lambda q_k}{\nu_k^2 + \lambda} \right)^2 \\ &= 2 \sum_{k=1}^{\infty} \frac{\nu_k^2 (y^{\delta}_k - y_k)^2}{(\nu_k^2 + \lambda)^2} + 2\lambda^2 \sum_{k=1}^{\infty} \frac{q_k^2}{(\nu_k^2 + \lambda)^2} = A + B. \end{aligned}$$

We show that both A and B tend to 0 for $\lambda = \lambda(\delta)$ satisfying (2.27a) and (2.27b) to conclude the desired result.

For the term A we have

$$A \leq \sum_{k=1}^{\infty} \frac{(y^{\delta}_k - y_k)^2}{\lambda} = \frac{\delta^2}{\lambda}$$

so for $\lambda = \lambda(\delta)$ satisfying (2.27b) it vanishes.

For $\lambda = \lambda(\delta)$ satisfying (2.27a) term B also vanishes, by the Dominated

Convergence Theorem since for all $k \in \mathbb{N}$

$$\frac{2\lambda^2 q_k^2}{(\nu_k^2+\lambda)^2} \to 0, \quad \text{as} \quad \lambda \to 0$$

and for each $\lambda > 0$

$$\frac{2\lambda^2 q_k^2}{(\nu_k^2+\lambda)^2} \le \frac{2\lambda^2 q_k^2}{\lambda^2} = 2q_k^2$$

which is summable since $u^\dagger \in \mathcal{H}$.

□

We now apply the results of the previous subsection to Tikhonov regularization to obtain convergence rate results [5, Example 4.15]. First note that

$$\begin{aligned} G_\lambda^T &= \sup\left\{|g_\lambda^T(\sigma)| : \sigma \in [0, ||K||^2]\right\} \\ &= \sup\left\{\left|\frac{1}{\sigma+\lambda}\right| : \sigma \in [0, ||K||^2]\right\} = \frac{1}{\lambda} \end{aligned} \tag{2.28}$$

therefore, since we have already seen that g^T satisfies the hypotheses of Theorem 2.2.2, by Theorem 2.2.3 we have the stability estimate

$$||u_\lambda - u_\lambda^\delta|| \le \frac{\delta}{\sqrt{\lambda}}.$$

Observe that

$$r_\lambda^T(\sigma) = 1 - \sigma g_\lambda^T(\sigma) = \frac{\lambda}{\sigma+\lambda}.$$

We need to compute for $\mu > 0$, a function $\omega_\mu : (0, \lambda_0) \to \mathbb{R}$ such that for all $\lambda \in (0, \lambda_0),\ \sigma \in [0, ||K||^2]$

$$\sigma^\mu |r_\lambda^T(\sigma)| \le \omega_\mu(\lambda).$$

Define

$$h_\mu(\sigma) := \sigma^\mu |r_\lambda^T(\sigma)| = \sigma^\mu \frac{\lambda}{\sigma+\lambda}.$$

For $0 \le \mu < 1$ this function attains its maximum for $\sigma = \frac{\lambda\mu}{1-\mu}$, therefore

$$h_\mu(\sigma) \le \frac{\lambda^\mu \mu^\mu}{(1-\mu)^\mu} \frac{\lambda}{\frac{\lambda\mu}{1-\mu}+\lambda} = \lambda^\mu \mu^\mu (1-\mu)^{1-\mu} \le \lambda^\mu$$

while for $\mu \ge 1,\ \ h_\mu$ is strictly increasing, thus it attains its largest value in $[0, ||K||^2]$ at the right end of the interval

$$h_\mu(\sigma) \leq ||K||^{2\mu} \frac{\lambda}{||K||^2 + \lambda} \leq ||K||^{2\mu-2} \lambda.$$

This means we can take

$$\omega_\mu(\lambda) = \begin{cases} \lambda^\mu, & if \quad \mu \leq 1 \\ c\lambda, & if \quad \mu > 1 \end{cases}$$

with $c = ||K||^{2\mu-2}$, thus, for $\mu \in [0, 1]$, we have

$$\sigma^\mu |r_\lambda^T(\sigma)| \leq \lambda^\mu. \tag{2.29}$$

By Corollary 2.2.11, for $\mu \in (0, 1]$ and $\rho > 0$, for the parameter choice rule

$$\lambda \sim \left(\frac{\delta}{\rho}\right)^{\frac{2}{2\mu+1}},$$

the Tikhonov Regularization is of optimal order in $X_{\mu,\rho}$, since

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right| = O(\delta^{\frac{2\mu}{2\mu+1}} \rho^{\frac{1}{2\mu+1}}),$$

which is of the same order of magnitude as the worst-case error corresponding to the *a-priori* assumption $u^\dagger \in X_{\mu,\rho}$.

For $\mu = 1$ we obtain the best possible convergence rate, with the above choice, that is for regularization parameter

$$\lambda \sim \left(\frac{\delta}{\rho}\right)^{\frac{2}{3}},$$

we have

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right| = O(\delta^{\frac{2}{3}}),$$

for $u^\dagger \in X_{1,\rho}$.

The next theorem shows that the Tikhonov regularization never yields a convergence rate which is faster than $O(\delta^{\frac{2}{3}})$. We say that the Tikhonov regularization method saturates at this rate.

Theorem 2.2.16. *[5, Proposition 5.3] [16, Theorem 2.13] Let $u^\dagger \in X$ and assume that there exists a parameter choice rule $\lambda(\delta, y^\delta)$ such that*

$$\lim_{\delta\to 0} \left|\left|u_\lambda^\delta - u^\dagger\right|\right| \delta^{-\frac{2}{3}} = 0, \;\; \forall y^\delta \in Y \textit{ with } \quad \left|\left|y^\delta - y\right|\right| \leq \delta,$$

where u_λ^δ is the Tikhonov regularized approximation.
Then $u^\dagger = 0$.

Proof. We prove this by contradiction. Assume that $u^\dagger \neq 0$.
We show first that $\lambda(\delta)\delta^{-\frac{2}{3}} \to 0$, as $\delta \to 0$.

By the definition (2.24) of the Tikhonov regularization and the normal equation (2.1) we have that

$$(\lambda(\delta)I + K^*K)(u_{\lambda(\delta)}^\delta - u^\dagger) = K^*y^\delta - \lambda(\delta)u^\dagger - K^*Ku^\dagger = K^*(y^\delta - y) - \lambda(\delta)u^\dagger,$$

hence we have the estimate

$$|\lambda(\delta)|\,||x|| \leq ||K||\,\delta + (\lambda(\delta + ||K||^2)\left|\left|u_{\lambda(\delta)}^\delta - u^\dagger\right|\right|$$

and by multiplying both sides with $\delta^{-\frac{2}{3}}$ we get

$$|\lambda(\delta)|\delta^{-\frac{2}{3}}\,||x|| \leq ||K||\,\delta^{\frac{1}{3}} + (\lambda(\delta + ||K||^2)\left|\left|u_{\lambda(\delta)}^\delta - u^\dagger\right|\right|\delta^{-\frac{2}{3}}.$$

By the assumption this yields that

$$\frac{\lambda(\delta)}{\delta^{\frac{2}{3}}} \to 0 \quad \text{as} \quad \delta \to 0. \qquad (a)$$

Define $\delta_k = \nu_k^3$ and $y^{\delta_k} = y + \delta_k\phi_k$, for every $k \in \mathbb{N}$. Then $\delta_k \overset{k\to\infty}{\longrightarrow} 0$ and for $\lambda_k = \lambda(\delta_k)$

$$\begin{aligned} u_{\lambda_k}^{\delta_k} - u^\dagger &= (u_{\lambda_k}^{\delta_k} - u_{\lambda_k}) + (u_{\lambda_k} - u^\dagger) \\ &= \frac{\delta_k\nu_k}{\lambda_k + \nu_k^2}\phi_k + (u_{\lambda_k} - u^\dagger). \end{aligned} \qquad (b)$$

Since the assumption implies that also

$$\left|\left|u_{\lambda_k} - u^\dagger\right|\right|\delta_k^{-\frac{2}{3}} \overset{k\to\infty}{\longrightarrow} 0,$$

we conclude by multiplying both sides of (b) by $\delta^{-\frac{2}{3}}$, that

$$\frac{\delta_k^{\frac{1}{3}}\nu_k}{\lambda_k+\nu_k^2}\to 0, \quad \text{as} \quad k\to\infty. \tag{c}$$

The contradiction follows using (a), since

$$\frac{\delta_k^{\frac{1}{3}}\nu_k}{\lambda_k+\nu_k^2}=\frac{\nu_k^2}{\lambda_k+\nu_k^2}=(1+\lambda_k\delta_k^{-\frac{2}{3}})^{-1}\to 1, \quad \text{as} \quad k\to\infty,$$

which contradicts (c).

□

This result says that the Tikhonov regularization method is not optimal under stronger assumptions on the solution $u^\dagger$, i.e. under the assumption $u^\dagger \in X_\mu,\ \mu > 1$. As mentioned earlier we say that the Tikhonov regularization method saturates for $\mu = 1$. Note that the assertion in the last theorem is proved not only for *a-priori* parameter choice rules, but for general parameter choice rules.

2.3 Tikhonov Regularization in Hilbert Scales

We now give a brief introduction to the method of Tikhonov Regularization in Hilbert Scales, first introduced by Natterer [18].

We develop this theory directly for the problem that we will examine in the next chapter, that is for $K = \mathcal{A}^{-\ell},\ \ell > 0$ where the form of $\mathcal{A}$ is given in Section 1.2.2 and for the Hilbert Scale $\mathcal{H}^s$ introduced in Section 1.2.2. In this section we consider K to be a map $\mathcal{H} \to \mathcal{H}$. The theory is taken from [5] with some modifications, but again the proofs presented below are simplifications for the particular case examined in this thesis. In [5] the theory is developed in greater generality, not only for Tikhonov Regularization, for more general operators K and for a Hilbert Scale induced by an operator L which has a weaker relation to K than the relation between $\mathcal{A}$ and K in our case. In this context L is called the *regularizing operator*. A detailed discussion in the case of the Tikhonov Regularization in Hilbert Scales, for different choices of the regularizing operator L, can be found in

[17].

Natterer's idea was to regularize the problem $Ku = y$ by minimizing the slightly generalized Tikhonov functional

$$I_{\lambda,\alpha}(u) = \frac{1}{2}\left|\left|Ku - y^\delta\right|\right|^2 + \frac{\lambda}{2}\left|\left|u\right|\right|_\alpha^2, \tag{2.30}$$

where $\alpha \geq 0$, thus he defined the regularized approximation as

$$u_{\lambda,\alpha}^\delta = \operatorname*{argmin}_{u \in \mathcal{H}^\alpha} I_{\lambda,\alpha}(u).$$

This time we start with a variational characterization for the Tikhonov Regularization in Hilbert Scales, so we would like to have an operator-theoretic characterization for it, provided by the following proposition:

Proposition 2.3.1. *Let* $u_{\lambda,\alpha}^\delta = \operatorname*{argmin}_{u \in \mathcal{H}^\alpha} I_{\lambda,\alpha}(u)$.
Then $u_{\lambda,\alpha}^\delta$ *is given by*

$$u_{\lambda,\alpha}^\delta = g_\lambda^T(\mathcal{A}^{-\alpha}K^*K)\mathcal{A}^{-\alpha}K^*y^\delta \tag{2.31}$$

or using the diagonalization of $\mathcal{A}$ *by*

$$u_{\lambda,\alpha}^\delta = \sum_{k=1}^\infty \frac{\mu_k^{-\ell}y_k^\delta}{\mu_k^{-2\ell} + \lambda\mu_k^\alpha}\phi_k \tag{2.32}$$

Proof. In the orthonormal basis $\{\phi_k\}$, the generalized Tikhonov functional is given by

$$I_{\lambda,\alpha}(u) = \frac{1}{2}\sum_{k=1}^\infty(\mu_k^{-\ell}u_k - y_k^\delta)^2 + \frac{\lambda}{2}\sum_{k=1}^\infty \mu_k^\alpha u_k^2.$$

We minimize $I_{\lambda,\alpha}(u)$ by minimizing each term of the series over u_k. We differentiate each term of the series with respect to u_k to get

$$\mu_k^{-2\ell}u_k - \mu_k^{-\ell}y_k^\delta + \lambda\mu_k^\alpha u_k = 0,$$

therefore the minimum of $I_{\lambda,\alpha}$ is attained for

$$u_k = \frac{\mu_k^{-\ell}y_k^\delta}{\mu_k^{-2\ell} + \lambda\mu_k^\alpha}$$

and

$$\operatorname*{argmin}_{u\in\mathcal{H}^\alpha} I_{\lambda,\alpha}(u) = \sum_{k=1}^{\infty} \frac{\mu_k^{-\ell} y_k^\delta}{\mu_k^{-2\ell} + \lambda\mu_k^\alpha}\phi_k = u_{\lambda,\alpha}^\delta.$$

On the other hand, using the diagonalization of $\mathcal{A}$ in the orthonormal basis $\{\phi_k\}$ we have

$$g_\lambda^T(\mathcal{A}^{-\alpha}K^*K)\mathcal{A}^{-\alpha}K^*y^\delta = \sum_{k=1}^{\infty} \frac{1}{\mu_k^{-\alpha}\mu_k^{-2\ell} + \lambda}\mu_k^{-\alpha}\mu_k^{-\ell}y_k^\delta\phi_k$$

$$= \sum_{k=1}^{\infty} \frac{\mu_k^{-\ell} y_k^\delta}{\mu_k^{-2\ell} + \lambda\mu_k^\alpha}\phi_k = u_{\lambda,\alpha}^\delta.$$

□

Remark 2.3.2. Consider the restriction of K to $\mathcal{H}^\alpha$, $K_\alpha : \mathcal{H}^\alpha \to \mathcal{H}$. Then for every $u \in \mathcal{H}^\alpha$ and $x \in \mathcal{H}$ we have on the one hand that

$$\langle K_\alpha u, x\rangle = \langle Ku, x\rangle = \langle u, K^*x\rangle$$

and on the other hand that

$$\langle K_\alpha u, x\rangle = \langle u, (K_\alpha)^*x\rangle_\alpha = \langle u, \mathcal{A}^\alpha(K_\alpha)^*x\rangle .$$

This implies that $(K_\alpha)^* = \mathcal{A}^{-\alpha}K^*$. After this observation, we can rewrite (2.31) as

$$u_{\lambda,\alpha}^\delta = g_\lambda^T((K_\alpha)^*K_\alpha)K_\alpha^*y^\delta \tag{2.33}$$

and since this expression has the form of (2.9), all the convergence results with respect to the norm in space $\mathcal{H}^\alpha$, of Section 2.2 hold.

For Natterer's theory to work, he required operators K satisfying the inequality

$$\underline{m}\,||u||_{-p} \leq ||Ku|| \leq \overline{m}\,||u||_{-p}, \quad \forall u \in \mathcal{H}, \tag{2.34}$$

for some $p > 0$ and $0 < \underline{m} \leq \overline{m} < \infty$ in order for an essential for his analysis inequality by *Heinz* [10], to hold [5].

In our case inequality (2.34) holds trivially for $p = 2\ell$, since

$$||Ku|| = \left|\left|\mathcal{A}^{-\ell}u\right|\right| = ||u||_{-2\ell}\,,\ \forall u \in \mathcal{H}$$

and Heinz's inequality degenerates to equation (2.35) given below:

Proposition 2.3.3. *Let $\alpha \geq 0$ and define $B = \mathcal{A}^{-\alpha}K^*K$. Then for every $\nu \in \mathbb{R}$*

$$\left|\left|B^{\frac{\nu}{2}}u\right|\right| = ||u||_{-\nu(2\ell+\alpha)}\,,\ \forall u \in \mathcal{D}\left(B^{\frac{\nu}{2}}\right) \tag{2.35}$$

and

$$\mathcal{R}(B^{\nu}) = \mathcal{H}^{2\nu(2\ell+\alpha)}. \tag{2.36}$$

Proof. For $u \in \mathcal{D}(B^{\frac{\nu}{2}})$ we have

$$\left|\left|B^{\frac{\nu}{2}}u\right|\right| = \left|\left|K^{\nu}\mathcal{A}^{-\frac{\alpha\nu}{2}}u\right|\right| = \left|\left|\mathcal{A}^{-\nu\ell}\mathcal{A}^{-\frac{\alpha\nu}{2}}u\right|\right| = ||u||_{-\nu(2\ell+\alpha)}\,.$$

For the second assertion, we have $y \in \mathcal{R}(B^{\nu})$ if and only if $B^{-\nu}y \in \mathcal{H}$, which is equivalent to $\sum_{k=1}^{\infty} \mu_k^{2\nu(2\ell+\alpha)} y_k^2 < \infty$, i.e. $y \in \mathcal{H}^{2\nu(2\ell+\alpha)}$. □

By a similar calculation as in the previous section, we can verify that the Tikhonov filter function $g_\lambda^T(\sigma) = \frac{1}{\sigma+\lambda}$, satisfies the following conditions for every $\sigma \in [0, ||B||]$ and $\lambda > 0$:

$$|\sigma g_\lambda^T(\sigma)| \leq 1 \tag{2.37}$$

$$\lim_{\lambda \to 0} g_\lambda^T(\sigma) = \frac{1}{\sigma},\ \ \sigma \neq 0, \tag{2.38}$$

$$|g_\lambda^T(\sigma)| \leq \frac{1}{\lambda},\ \ \forall \lambda > 0 \tag{2.39}$$

and

$$\sigma^{\mu}|r_\lambda^T(\sigma)| \leq \lambda^{\mu},\ \forall \mu \in [0, \mu_0],\ \forall \lambda > 0. \tag{2.40}$$

Lemma 2.3.4. *For every $0 \leq t \leq 1$*

$$\sigma^t|g_\lambda^T(\sigma)| \leq 2\lambda^{t-1},\ \forall \sigma \in [0, ||B||]\,, \lambda > 0. \tag{2.41}$$

Proof. Fix $0 \leq t \leq 1$, $\sigma \in [0, ||B||]$ and $\lambda > 0$.

By (2.40) for $\mu = 0$ we have

$$|r_\lambda^T(\sigma)| \leq 1,$$

so since

$$\sigma|g_\lambda^T(\sigma)| - 1 \leq |r_\lambda^T(\sigma)|,$$

we get

$$\sigma|g_\lambda^T(\sigma)| \leq 2,$$

thus, since $0 \leq t \leq 1$, we deduce that

$$\sigma^t|g_\lambda^T(\sigma)|^t \leq 2^t. \tag{i}$$

By (2.39) we have

$$|g_\lambda^T(\sigma)|^{1-t} \leq \lambda^{t-1}$$

therefore

$$|g_\lambda^T(\sigma)|^t \geq |g_\lambda^T(\sigma)|\lambda^{1-t}. \tag{ii}$$

Combining (i) and (ii) we get

$$\sigma^t|g_\lambda^T(\sigma)|\lambda^{1-t} \overset{(ii)}{\leq} \sigma^t|g_\lambda^T(\sigma)|^t \overset{(i)}{\leq} 2^t \leq 2,$$

hence

$$\sigma^t|g_\lambda^T(\sigma)| \leq 2\lambda^{t-1}.$$

□

The following theorem which is a modified version of [5, Theorem 8.23], establishes convergence results with respect to weaker norms, even when $u^\dagger \notin \mathcal{H}^\alpha$.

Theorem 2.3.5. *Let $\alpha \geq 0$ and let $u^\delta_{\lambda,\alpha}$ be the generalized Tikhonov regularized approximation given by* (2.31)*. Then under the a-priori information $u^\dagger \in \mathcal{H}^m$, for the parameter choice*

$$\lambda = \left(\frac{\delta}{||u^\dagger||_m}\right)^{\frac{2(2\ell+\alpha)}{2\ell+m}} \tag{2.42}$$

and if $m \leq 4\ell + 2\alpha$, we obtain the estimate

$$\left|\left|u^\delta_{\lambda,\alpha} - u^\dagger\right|\right| \leq C\delta^{\frac{m}{2\ell+m}} \left|\left|u^\dagger\right|\right|_m^{\frac{2\ell}{2\ell+m}},$$

for some constant $C > 0$.

Proof. By the triangular inequality we have

$$\left|\left|u^\delta_{\lambda,\alpha} - u^\dagger\right|\right| \leq \left|\left|u^\delta_{\lambda,\alpha} - u_{\lambda,\alpha}\right|\right| + \left|\left|u_{\lambda,\alpha} - u^\dagger\right|\right|.$$

We first estimate the propagated data error $\left|\left|u^\delta_{\lambda,\alpha} - u_{\lambda,\alpha}\right|\right|$.

Since the operators $g^T_\lambda(B), K$ and $\mathcal{A}$ are diagonalizable in the same orthonormal basis $\{\phi_k\}$, they commute, therefore

$$\left|\left|u^\delta_{\lambda,\alpha} - u_{\lambda,\alpha}\right|\right| = \left|\left|g^T_\lambda(B)\mathcal{A}^{-\alpha}K^*(y^\delta - y)\right|\right| = \left|\left|\mathcal{A}^{-\frac{\alpha}{2}}g^T_\lambda(B)\mathcal{A}^{-\frac{\alpha}{2}}K^*(y^\delta - y)\right|\right|$$

$$= \left|\left|g^T_\lambda(B)\mathcal{A}^{-\frac{\alpha}{2}}K^*(y^\delta - y)\right|\right|_{-\alpha} = \left|\left|g^T_\lambda(B)B^{\frac{1}{2}}(y^\delta - y)\right|\right|_{-\alpha},$$

hence by (2.35) for $\nu = \frac{\alpha}{2\ell+\alpha}$

$$\left|\left|u^\delta_{\lambda,\alpha} - u_{\lambda,\alpha}\right|\right| = \left|\left|B^{\frac{\alpha}{4\ell+2\alpha}}g^T_\lambda(B)B^{\frac{1}{2}}(y^\delta - y)\right|\right| = \left|\left|B^{\frac{\ell+\alpha}{2\ell+\alpha}}g^T_\lambda(B)(y^\delta - y)\right|\right|.$$

By (2.41) for $t = \frac{\ell+\alpha}{2\ell+\alpha}$, we can deduce using the expansion in the orthonormal basis $\{\phi_k\}$, that

$$\left|\left|u^\delta_{\lambda,\alpha} - u_{\lambda,\alpha}\right|\right| \leq c\lambda^{-\frac{\ell}{2\ell+\alpha}}\delta, \tag{i}$$

where $c > 0$ is a generic constant.

Since $u^\dagger \in \mathcal{H}^m$, by (2.36) we have that

$$\mathcal{A}^{\frac{\alpha}{2}}u^\dagger \in \mathcal{H}^{m-\alpha} = \mathcal{R}\left(B^{\frac{m-\alpha}{2(2\ell+\alpha)}}\right),$$

thus there exists $v \in \mathcal{H}$ such that

$$\mathcal{A}^{\frac{\alpha}{2}}u^\dagger = B^{\frac{m-\alpha}{2(2\ell+\alpha)}}v. \tag{ii}$$

This along with the normal equation (2.1), imply that

$$\left|\left|u_{\lambda,\alpha}-u^{\dagger}\right|\right| = \left|\left|g_{\lambda}^{T}(B)\mathcal{A}^{-\alpha}K^{*}y-u^{\dagger}\right|\right| = \left|\left|(g_{\lambda}^{T}(B)\mathcal{A}^{-\alpha}K^{*}K-\mathcal{A}^{-\frac{\alpha}{2}}\mathcal{A}^{\frac{\alpha}{2}})u^{\dagger}\right|\right|$$

$$= \left|\left|\mathcal{A}^{-\frac{\alpha}{2}}(g_{\lambda}^{T}(B)B-I)\mathcal{A}^{\frac{\alpha}{2}}u^{\dagger}\right|\right| = \left|\left|r_{\lambda}^{T}(B)B^{\frac{m-\alpha}{2(2\ell+\alpha)}}v\right|\right|_{-\alpha},$$

so that by (2.35) for $\nu=\frac{\alpha}{2\ell+\alpha}$ we have

$$\left|\left|u_{\lambda,\alpha}-u^{\dagger}\right|\right| = \left|\left|B^{\frac{\alpha}{2(2\ell+\alpha)}}r_{\lambda}^{T}(B)B^{\frac{m-\alpha}{2(2\ell+\alpha)}}v\right|\right| = \left|\left|B^{\frac{m}{2(2\ell+\alpha)}}r_{\lambda}^{T}(B)v\right|\right|,$$

thus by (2.40) for $\mu=\frac{m}{2(2\ell+\alpha)}\leq 1$ since $m\leq 4\ell+2\alpha$,

$$\left|\left|u_{\lambda,\alpha}-u^{\dagger}\right|\right| \leq c\lambda^{\frac{m}{2(2\ell+\alpha)}}\left|\left|v\right|\right|.$$

Again by (2.35) for $\nu=\frac{\alpha-m}{2\ell+\alpha}$, using (ii) we have

$$\left|\left|v\right|\right| = \left|\left|B^{\frac{\alpha-m}{2(2\ell+\alpha)}}\mathcal{A}^{\frac{\alpha}{2}}u^{\dagger}\right|\right| = \left|\left|\mathcal{A}^{\frac{\alpha}{2}}u^{\dagger}\right|\right|_{m-\alpha} = \left|\left|u^{\dagger}\right|\right|_{m},$$

so that

$$\left|\left|u_{\lambda,\alpha}-u^{\dagger}\right|\right| \leq c\lambda^{\frac{m}{2(2\ell+\alpha)}}\left|\left|u^{\dagger}\right|\right|_{m}. \qquad (iii)$$

By (2.42), (i) and (iii) we have the desired assertion.

□

Remark 2.3.6.

i) Observe that the restriction on the allowed values of m

$$m\leq 4\ell+2\alpha,$$

comes from the qualification of the Tikhonov Regularization method, i.e. from the fact that (2.40) holds for $0\leq\mu\leq 1$.

ii) For $0<m\leq\alpha$, we still have convergence, i.e. we may have convergence even if $u^{\dagger}\notin\mathcal{H}^{\alpha}$ as mentioned earlier.

iii) For fixed $\alpha\geq 0$, the last theorem provides the best possible rate for the *a-priori* information that $u^\dagger \in \mathcal{H}^{4\ell+2\alpha}$, in which case it gives that

$$||u^\delta_{\lambda,\alpha} - u^\dagger|| = O(\delta^{\frac{4\ell+2\alpha}{6\ell+2\alpha}}).$$

This means that the Tikhonov Regularization in Hilbert Scales method saturates at a faster rate than

$$O(\delta^{\frac{4\ell+2\alpha}{6\ell+2\alpha}}),$$

which, for $\alpha > 0$, is already better than the saturation rate of the Tikhonov regularization, $O(\delta^{\frac{2}{3}})$.

For $\alpha = 0$, we get that for $u^\dagger \in \mathcal{H}^{4\ell}$ the convergence rate is $O(\delta^{\frac{2}{3}})$. The *a-priori* information $u^\dagger \in \mathcal{H}^{4\ell}$ is equivalent to $u^\dagger \in X_1$, so this rate agrees with the rate provided by Corollary 2.2.11 in Section 2.2.3.

Moreover, for large enough values of α we have that for the *a-priori* information $u^\dagger \in \mathcal{H}^{4\ell+2\alpha}$ the convergence rate provided by the last theorem can get arbitrarily close to $O(\delta)$, thus for large enough values of α the saturation rate of the Tikhonov Regularization in Hilbert Scales method can get arbitrarily close to $O(\delta)$. This shows that Natterer's idea to regularize in a stronger norm, does provide better convergence rates under sufficient *a-priori* assumptions on the smoothness of the solution.

iv) For no *a-priori* information, i.e. for $m = 0$ the theorem does not directly secure convergence for any value of $\alpha \geq 0$. However, we can see from the proof that we have at least convergence, as long as

$$\lambda(\delta) \to 0 \quad \text{and} \quad \delta\lambda(\delta)^{-\frac{\ell}{2\ell+\alpha}} \to 0$$

as $\delta \to 0$. Indeed, by (i) we have that

$$||u^\delta_{\lambda,\alpha} - u_{\lambda,\alpha}|| \to 0$$

and by Remark 2.3.2, (2.37), (2.38) and Theorem 2.2.2

$$||u_{\lambda,\alpha} - u^\dagger||_\alpha \to 0, \quad \text{as} \quad \lambda \to 0$$

thus, since $\alpha \geq 0$

$$\left|\left|u_{\lambda,\alpha} - u^{\dagger}\right|\right| \to 0, \quad \text{as} \quad \lambda \to 0.$$

For $\alpha = 0$ we have convergence provided $\lambda \to 0$ and $\delta\lambda(\delta^{\frac{1}{2}}) \to 0$ which is in agreement with Theorem 2.2.15.

v) Finally, by slightly modifying the proof of Theorem 2.3.5, we have that for the parameter choice $\lambda \sim \delta^{\frac{2(2\ell+\alpha)}{2\ell+m}}$, we obtain a rate for convergence in other norms, namely

$$\left|\left|u_{\lambda}^{\delta} - u^{\dagger}\right|\right|_{\gamma} = O(\delta^{\frac{m-\gamma}{2\ell+m}})$$

for $\max\{-2\ell, m-4\ell-2\alpha\} \leq \gamma \leq \min\{2\ell+2\alpha, m\}$.
Indeed, like the proof of Theorem 2.3.5 we have that

$$\left|\left|u_{\lambda,\alpha}^{\delta} - u_{\lambda,\alpha}\right|\right|_{\gamma} = \left|\left|g_{\lambda}^{T}(B)B^{\frac{1}{2}}(y^{\delta} - y)\right|\right|_{\gamma-\alpha},$$

hence by (2.35) for $\nu = \frac{\alpha-\gamma}{2\ell+\alpha}$

$$\left|\left|u_{\lambda,\alpha}^{\delta} - u_{\lambda,\alpha}\right|\right|_{\gamma} = \left|\left|B^{\frac{2\ell+2\alpha-\gamma}{2(2\ell+\alpha)}} g_{\lambda}^{T}(B)(y^{\delta} - y)\right|\right|.$$

Set $t = \frac{2\ell+2\alpha-\gamma}{2(2\ell+\alpha)}$ and note that by the assumption on γ we have that $t \in [0,1]$. Then by (2.41) we have that

$$\left|\left|u_{\lambda,\alpha}^{\delta} - u_{\lambda,\alpha}\right|\right|_{\gamma} \leq c\lambda^{-\frac{\gamma+2\ell}{2(2\ell+\alpha)}}\delta, \quad c > 0. \qquad (i')$$

Like before, there exists $v \in \mathcal{H}$ such that (ii) holds and

$$\left|\left|u_{\lambda,\alpha} - u^{\dagger}\right|\right|_{\gamma} = \left|\left|r_{\lambda}^{T}(B)B^{\frac{m-\alpha}{2(2\ell+\alpha)}}v\right|\right|_{\gamma-\alpha},$$

hence by (2.35) for $\nu = \frac{\alpha-\gamma}{2\ell+\alpha}$

$$\left|\left|u_{\lambda,\alpha} - u^{\dagger}\right|\right|_{\gamma} = \left|\left|B^{\frac{m-\gamma}{2(2\ell+\alpha)}}r_{\lambda}^{T}(B)v\right|\right|.$$

Set $\mu = \frac{m-\gamma}{2(2\ell+\alpha)}$ and note that by the assumption on γ we have that $\mu \in$

$[0,1]$. Then by (2.40) we have that

$$\left|\left|u_{\lambda,\alpha} - u^{\dagger}\right|\right|_{\gamma} \leq c\lambda^{\frac{m-\gamma}{2(2\ell+\alpha)}} \left|\left|v\right|\right|. \qquad (iii')$$

Combining (i') and (iii') we get the desired result for $\lambda \sim \delta^{\frac{2(2\ell+\alpha)}{2\ell+m}}$.

Note that for fixed m, if we allow convergence in weaker norms, i.e. $\gamma < 0$, we get faster rates, as one would expect. Likewise, if we require convergence in stronger norms, i.e. $\gamma > 0$, we get slower convergence rates.

Chapter 3

The Laplacian-like Inverse Problem

In this chapter we examine the "Laplacian-like" inverse problem, defined in Section 1.2.2, using a generalization of the Tikhonov Regularization in Hilbert Scales method. In Section 3.1, we first define the generalized regularized approximation which is motivated by the Bayesian approach to Inverse Problems and then provide sufficient conditions for the convergence of it to the best-approximate solution, as the noise disappears. Note that in the "Laplacian-like" problem the best-approximate solution is the true solution, as we have seen in Remark 2.1.8. In Section 3.2, we make additional assumptions on the algebraic structure of the noise and the true solution and provide convergence rates for the proposed method.

3.1 A Generalized Tikhonov Regularization

Let $\alpha, \beta, \gamma \in \mathbb{R}$ and $\ell > 0$. Consider the inverse problem

$$y = Ku^{\dagger}, \tag{3.1}$$

where $u^{\dagger}, y \in \mathcal{H}^{\gamma}$ and $K = \mathcal{A}^{-\ell} : \mathcal{H}^{\gamma} \to \mathcal{H}^{\gamma}$, with $\mathcal{A}$ satisfying the Assumption 1.2.1.

Assume we have observations of (3.1), polluted by some additive noise

$\eta \in \mathcal{H}^\beta$, of known magnitude

$$y^\delta = Ku^\dagger + \eta, \quad ||\eta||_\beta = c\delta. \tag{3.2}$$

Assumption 3.1.1. *We concatenate the assumptions we have in this section:*

A1) The observations are polluted by additive noise, $\eta \in \mathcal{H}^\beta$, such that $||\eta||_\beta = c\delta$.

A2) The operator K is a negative power of $\mathcal{A}$,

$$K = \mathcal{A}^{-\ell}, \ell > 0,$$

where $\mathcal{A}$ is "Laplacian-like", as defined in the Assumption 1.2.1 in Section 1.2.2.

A3) We have the a-priori *information that $u^\dagger \in \mathcal{H}^\gamma$, thus we view K as $K : \mathcal{H}^\gamma \to \mathcal{H}^\gamma$.*

We generalize Naterer's idea as described in Section 2.2.3 by allowing weighted norms in the least squares term too and approximate $u^\dagger$ by the minimizer, $u^{\lambda,\delta}$, of the generalized Tikhonov Functional:

$$I_{\lambda,\alpha,\beta}(u) = \frac{1}{2}\left|\left|y^\delta - Ku\right|\right|_\beta^2 + \frac{\lambda}{2}\left|\left|u\right|\right|_\alpha^2, \quad \lambda > 0, \tag{3.3}$$

$$u_\lambda^\delta = \arg\min_{u \in \mathcal{H}^\alpha} I_{\lambda,\alpha,\beta}(u).$$

Our proposition is based on the fact that in the Bayesian approach (cf. (4.13) in Section 4.1), the posterior mean is the minimizer of Tikhonov functionals of this form, i.e. which involve weighted norms in both the least squares term and the penalty term.

Remark 3.1.2. Note that in order for the Tikhonov Functional $I_{\lambda,\alpha,\beta}$ to be finite, it is necessary that $y^\delta \in \mathcal{H}^\beta$. This imposes the condition $\beta \leq \gamma + 2\ell$. Indeed, since by the Assumption 3.1.1(A1) we have that $\eta \in \mathcal{H}^\beta$, the requirement $y^\delta \in \mathcal{H}^\beta$ is equivalent to $Ku^\dagger \in \mathcal{H}^\beta$. We have, for $q_k = \langle u^\dagger, \phi_k \rangle$, that

$$\left|\left|Ku^\dagger\right|\right|_\beta < \infty,$$

if and only if

$$\sum_{k=1}^{\infty} \mu_k^{-2\ell+\beta} q_k^2 < \infty,$$

which by the Assumption 3.1.1(A3), is equivalent to $\gamma \geq -2\ell+\beta$. However, in the case where $\beta > \gamma + 2\ell$, or even $\beta \geq \gamma + 2\ell$, we have that the noise has the same or even better regularity than $Ku^\dagger$, thus the polluted data y^δ stay in the range of K, which by Lemma 1.2.3 is $\mathcal{H}^{\gamma+2\ell}$. Since by Lemma 1.2.3 $K : \mathcal{H}^\gamma \to \mathcal{H}^{\gamma+2\ell}$ is invertible with bounded inverse, the problem in this case is not ill-posed and no regularization is needed.

Thus, we henceforward have the assumption:

A4) $\beta < \gamma + 2\ell$.

Lemma 3.1.3. *The minimizer of the Tikhonov Functional*

$$I_{\lambda,\alpha,\beta}(u) = \frac{1}{2} \left|\left|y^\delta - Ku\right|\right|_\beta^2 + \frac{\lambda}{2} \left|\left|u\right|\right|_\alpha^2, \quad \lambda > 0,$$

over the space $\mathcal{H}^\alpha$, *is*

$$u_\lambda^\delta = \sum_{k=1}^{\infty} \frac{\mu_k^{-\ell+\beta} y_k^\delta}{\mu_k^{-2\ell+\beta} + \lambda\mu_k^\alpha} \phi_k \tag{3.4}$$

and for the additive noise model (3.2) considered here,

$$u_\lambda^\delta = \sum_{k=1}^{\infty} \frac{\mu_k^{-2\ell+\beta} q_k + \mu_k^{-\ell+\beta} \eta_k}{\mu_k^{-2\ell+\beta} + \lambda\mu_k^\alpha} \phi_k. \tag{3.5}$$

Proof. Using the diagonalization of $\mathcal{A}$, we can express the Tikhonov functional, $I_{\lambda,\alpha,\beta}$, as

$$I_{\lambda,\alpha,\beta}(u) = \frac{1}{2} \sum_{k=1}^{\infty} \left\{ (y_k^\delta - \mu_k^{-\ell} u_k)^2 \mu_k^\beta + \lambda \mu_k^\alpha u_k^2 \right\}$$

$$= \frac{1}{2} \sum_{k=1}^{\infty} \left\{ \mu_k^\beta (y_k^\delta)^2 + \mu_k^{-2\ell+\beta} u_k^2 - 2\mu_k^{-\ell+\beta} y_k^\delta u_k + \lambda \mu_k^\alpha u_k^2 \right\}.$$

We minimize $I_{\lambda,\alpha,\beta}(u)$ by minimizing each term of the series over u_k. We differentiate with respect to u_k to get

$$2\mu_k^{-2\ell+\beta}u_k^{\lambda,\delta} - 2\mu_k^{-\ell+\beta}y_k^\delta + 2\lambda\mu_k^\alpha u_k^{\lambda,\delta} = 0,$$

therefore the minimum of $I_{\lambda,\alpha,\beta}$ is attained for

$$u_k^{\lambda,\delta} = \frac{\mu_k^{-\ell+\beta}y_k^\delta}{\mu_k^{-2\ell+\beta} + \lambda\mu_k^\alpha}.$$

Since $y_k^\delta = \mu_k^{-\ell}q_k + \eta_k$, we have

$$u_k^{\lambda,\delta} = \frac{\mu_k^{-2\ell+\beta}q_k + \mu_k^{-\ell+\beta}\eta_k}{\mu_k^{-2\ell+\beta} + \lambda\mu_k^\alpha}\phi_k.$$

□

Remark 3.1.4. Note that under the assumptions (A1)-(A4), u_λ^δ as defined in the last lemma, is always well defined and lives in $\mathcal{H}^\alpha$. Indeed, since $\mu_k > 0$, $\forall k \in \mathbb{N}$ and $\lambda > 0$, we have

$$||u_\lambda^\delta||_\alpha^2 = \sum_{k=1}^\infty \frac{\mu_k^{-2\ell+2\beta+\alpha}(y_k^\delta)^2}{\mu_k^{-4\ell+2\beta} + 2\lambda\mu_k^{-2\ell+\beta+\alpha} + \lambda^2\mu_k^{2\alpha}}$$

$$\leq \sum_{k=1}^\infty \frac{\mu_k^{-2\ell+2\beta+\alpha}(y_k^\delta)^2}{2\lambda\mu_k^{-2\ell+\beta+\alpha}} = \sum_{k=1}^\infty \frac{\mu_k^\beta(y_k^\delta)^2}{2\lambda} < \infty,$$

since by (A4) we have that $y^\delta \in \mathcal{H}^\beta$.

We now provide a sufficient condition on α and γ and an *a-priori* parameter choice rule, $\lambda = \lambda(\delta)$, for the convergence of u_λ^δ to $u^\dagger$, in $\mathcal{H}^\gamma$, as $\delta \to 0$.

Theorem 3.1.5. *Suppose $\gamma \leq \alpha$. If $\lambda = \lambda(\delta)$ is such that*

$$\lambda(\delta) \to 0 \quad \textit{and} \quad \frac{\delta^2}{\lambda(\delta)} \to 0, \quad \textit{as} \quad \delta \to 0,$$

then

$$||u_\lambda^\delta \to u^\dagger||_\gamma \to 0, \quad \textit{as} \quad \delta \to 0.$$

Proof. By Lemma 3.1.3

$$||u_\lambda^\delta - u^\dagger||_\gamma^2 = \sum_{k=1}^\infty \left(\frac{\mu_k^{-2\ell+\beta} q_k + \mu_k^{-\ell+\beta} \eta_k}{\mu_k^{-2\ell+\beta} + \lambda \mu_k^\alpha} - q_k \right)^2 \mu_k^\gamma$$

$$= \sum_{k=1}^\infty \left(\frac{\mu_k^{-\ell+\beta} \eta_k - \lambda \mu_k^\alpha q_k}{\mu_k^{-2\ell+\beta} + \lambda \mu_k^\alpha} \right)^2 \mu_k^\gamma \leq \sum_{k=1}^\infty \frac{2\mu_k^{-2\ell+2\beta+\gamma} \eta_k^2 + 2\lambda^2 \mu_k^{2\alpha+\gamma} q_k^2}{\mu_k^{-4\ell+2\beta} + \lambda^2 \mu_k^{2\alpha} + 2\lambda \mu_k^{-2\ell+\beta+\alpha}}$$

$$= \sum_{k=1}^\infty \frac{2\mu_k^{-2\ell+2\beta+\gamma} \eta_k^2}{\mu_k^{-4\ell+2\beta} + \lambda^2 \mu_k^{2\alpha} + 2\lambda \mu_k^{-2\ell+\beta+\alpha}} + \sum_{k=1}^\infty \frac{2\lambda^2 \mu_k^{2\alpha+\gamma} q_k^2}{\mu_k^{-4\ell+2\beta} + \lambda^2 \mu_k^{2\alpha} + 2\lambda \mu_k^{-2\ell+\beta+\alpha}}$$

The first term can be bounded in the following way:

$$\sum_{k=1}^\infty \frac{2\mu_k^{-2\ell+2\beta+\gamma} \eta_k^2}{\mu_k^{-4\ell+2\beta} + \lambda^2 \mu_k^{2\alpha} + 2\lambda \mu_k^{-2\ell+\beta+\alpha}} \leq \sum_{k=1}^\infty \frac{2\mu_k^{-2\ell+2\beta+\gamma} \eta_k^2}{2\lambda \mu_k^{-2\ell+\beta+\alpha}} = \sum_{k=1}^\infty \frac{\mu_k^{\beta+\gamma-\alpha} \eta_k^2}{\lambda}$$

since $\forall k \in \mathbb{N},\ \mu_k > 0$ and $\lambda > 0$.

If $\beta + \gamma - \alpha \leq \beta$, i.e. $\gamma \leq \alpha$, by Assumption 1.2.1(iii), we have

$$\sum_{k=1}^\infty \frac{\mu_k^{\beta+\gamma-\alpha} \eta_k^2}{\lambda} \leq C \sum_{k=1}^\infty \frac{\mu_k^\beta \eta_k^2}{\lambda} = C \frac{\delta^2}{\lambda}.$$

The second term for $\lambda \to 0$ vanishes by the Dominated Convergence Theorem, since for each $k \in \mathbb{N}$

$$\frac{2\lambda^2 \mu_k^{2\alpha+\gamma} q_k^2}{\mu_k^{-4\ell+2\beta} + \lambda^2 \mu_k^{2\alpha} + 2\lambda \mu_k^{-2\ell+\beta+\alpha}} \to 0$$

as $\lambda \to 0$ and for each $\lambda > 0$, since $\mu_k > 0,\ \forall k \in \mathbb{N}$ we have

$$\frac{2\lambda^2 \mu_k^{2\alpha+\gamma} q_k^2}{\mu_k^{-4\ell+2\beta} + \lambda^2 \mu_k^{2\alpha} + 2\lambda \mu_k^{-2\ell+\beta+\alpha}} \leq \frac{2\lambda^2 \mu_k^{2\alpha+\gamma} q_k^2}{\lambda^2 \mu_k^{2\alpha}} = 2\mu_k^\gamma q_k^2,$$

which is summable because $u^\dagger \in \mathcal{H}^\gamma$.

So for $\gamma \leq \alpha$ and $\lambda = \lambda(\delta)$ such that

$$\lambda(\delta) \to 0$$

and

$$\frac{\delta^2}{\lambda(\delta)} \to 0$$

as $\delta \to 0$, we have the desired convergence of u_λ^δ to $u^\dagger$ in $\mathcal{H}^\gamma$.

□

Remark 3.1.6.

i) The condition $\gamma \leq \alpha$ in the last theorem, means that we regularize in a stronger norm than the norm in which we seek convergence, i.e. the norm in the penalty term, $\frac{\lambda}{2}\,||u||_\alpha^2$, of the Tikhonov functional $I_{\lambda,\alpha,\beta}$, is stronger than the norm considered as the reference norm. This, as the theorem states, secures that the penalty term is indeed regularizing and we have convergence.

ii) The condition $\gamma \leq \alpha$ is a generalization of the condition in Remark 2.3.6(iv), of Section 2.3, which concerns the case $\gamma = \beta = 0$. In Remark 2.3.6(iv), for no *a-priori* information, we require $\alpha \geq 0$, $\lambda(\delta) \to 0$ and $\delta\lambda(\delta)^{-\frac{\ell}{2\ell+\alpha}} \to 0$, as $\delta \to 0$. Since $\delta^2\lambda(\delta)^{-1} \to 0$ implies that $\delta\lambda(\delta)^{-\frac{\ell}{2\ell+\alpha}} \to 0$ for every $\alpha \geq 0$, we have the agreement of the two conditions.

iii) For $\alpha = \gamma = \beta = 0$, the result of the last theorem agrees with Theorem 2.2.15.

3.2 Convergence Rates

In Chapter 2, in order to deduce convergence rates for the Tikhonov Regularization, we make *a-priori* assumptions on the regularity of the true solution, $u^\dagger$ (cf. e.g. Theorem 2.3.5). Here, we model these *a-priori* assumptions, by imposing additional assumptions on the algebraic structure of the true solution. Furthermore, we make additional assumptions on the algebraic structure of the noise, η:

Assumption 3.2.1. *Assume that there exist constants* $c_2^-, c_2^+, c_3^-, c_3^+ > 0$ *and* $r, h \in \mathbb{R}$ *such that the Fourier coefficients of the noise* η *and the true solution* $u^\dagger$*,* η_k *and* q_k *respectively, satisfy* $\forall k \in \mathbb{N}$

A5)

$$c_2^- \leq \frac{|\eta_k|}{\delta k^r} \leq c_2^+$$

A6)

$$c_3^- \leq \frac{|q_k|}{k^h} \leq c_3^+$$

The following theorem exhaustively answers the question whether or not we have convergence of u_λ^δ to $u^\dagger$ as $\delta \to 0$ in $\mathcal{H}^\gamma$ and provides convergence rates. Define

$$a = -2\ell + 2\beta + r + \gamma - 2\alpha,$$

$$b = h + \gamma,$$

$$c = -4\ell + 2\beta - 2\alpha.$$

Note that under the assumptions $(A1) - (A6)$ we necessarily have that

$$2b < -1$$

(cf. Lemma 3.2.4). This observation is of interest in order to secure that in the following theorem we exhaust all the possible combinations of $\alpha \neq \beta$. Define

$$\epsilon_\gamma = \left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma.$$

Theorem 3.2.2. *Assume $a \neq b$*

1) Suppose $c \geq 0$:

- *A) If $2a - 2c \geq -1$ then ϵ^γ does not converge to 0 as $\delta \to 0$.*
- *B) If $2a - 2c < -1$ we have convergence and for the parameter choice $\lambda = \delta$*

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta).$$

2) Suppose $c < 0$:

- *A) If $2a - 2c > -1$ and $2b - 2c > -1$:*
  - *i) If $2a \geq -1$ then ϵ^γ does not converge to 0 as $\delta \to 0$.*
  - *ii) If $2a < -1$ we have convergence and for the parameter choice $\lambda = \delta^{\frac{c}{b+c-a}}$*

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta^{\frac{2b+1}{2(b+c-a)}}).$$

B) If $2a-2c=-1$ and $2b-2c=-1$ we have convergence and for the parameter choice $\lambda=\delta$

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta\sqrt{|\log\delta|}).$$

C) If $2a-2c<-1$ and $2b-2c<-1$, we have convergence and for the parameter choice $\lambda=\delta$

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta).$$

D) If $2a-2c<-1$ and $2b-2c>-1$, we have convergence and for the parameter choice $\lambda=\delta^{\frac{2c}{2b+1}}$

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta).$$

E) If $2a-2c<-1$ and $2b-2c=-1$ we have convergence and for the parameter choice $\lambda=\delta$

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta\sqrt{|\log\delta|}).$$

F) If $2a-2c>-1$ and $2b-2c<-1$:

i) If $2a\geq -1$, then ϵ^γ does not converge to 0 as $\delta\to 0$.

ii) If $2a<-1$, we have convergence and for the parameter choice $\lambda=\delta^{\frac{2c}{4c-2a-1}}$

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta^{\frac{2c}{4c-2a-1}}).$$

G) If $2a-2c>-1$ and $2b-2c=-1$:

i) If $2a\geq -1$ then ϵ^γ does not converge to 0 as $\delta\to 0$.

ii) If $2a<-1$ we have convergence and for the parameter choice $\lambda=\delta^{\frac{2c}{4c-2a-1}}$

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta^{\frac{2c}{4c-2a-1}}\sqrt{|\log\delta|}).$$

H) If $2a-2c=-1$ and $2b-2c<-1$ we have convergence and for *the parameter choice* $\lambda = \delta$

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta\sqrt{|\log\delta|}).$$

I) If $2a - 2c = -1$ *and* $2b - 2c > -1$, *we have convergence and for the parameter choice* $\lambda = \delta^{\frac{2c}{2b+1}}$

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta\sqrt{|\log\delta|}).$$

The rates of convergence are sharp and the a-priori *parameter choice rules in cases* $(1B), (2Aii), (2C), (2Dii)$ *and* $(2Fii)$ *are chosen in an optimal way.*

The different cases distinguished above regarding the values of a, b, c, *exhaust all the possibilities for* $a \neq b$.

Before proving Theorem 3.2.2, we state and prove the following lemmas:

Lemma 3.2.3. *Let* $\lambda \in (0, 1)$. *Then we have the following estimates for the integral*

$$I := \int_1^\infty \frac{x^e}{(x^c + \lambda)^2} dx$$

i) If $e - 2c < -1$, *then* $I = \kappa$, *where* κ *is independent of* λ.

ii) If $e - 2c \geq -1$, *and* $e \geq -1$ *then* $I = +\infty$.

iii) If $e - 2c > -1$, $e < -1$ *and* $c < 0$, *then*

$$I \in \left[\frac{M}{4}, M\right],$$

where

$$M := \frac{2c\lambda^{\frac{e-2c+1}{c}}}{(e+1)(e-2c+1)} - \frac{1}{e-2c+1}.$$

iv) If $e - 2c = -1$, $e < -1$ *and* $c < 0$ *then*

$$I \in \left[\frac{N}{4}, N\right],$$

where

$$N := \frac{\log \lambda}{c} - \frac{1}{e+1}.$$

Proof. Fix $\lambda \in (0,1)$.

i) $\lambda > 0$, therefore

$$I \leq \int_1^\infty \frac{x^e}{x^{2c}} dx = \int_1^\infty x^{e-2c} dx,$$

where the last integral is independent of λ and finite since $e-2c < -1$.

ii) If $c < 0$, then since $\lambda < 1$ and $x \in (1, +\infty)$, we have that $(x^c + \lambda) \leq 2$, therefore

$$I \geq \int_1^\infty \frac{x^e}{4} dx = +\infty,$$

since $e \geq -1$.

If $c \geq 0$, then since $\lambda < 1$ and $x > 1$, we have $(x^c + \lambda) \leq 2x^c$, therefore

$$I \geq C \int_1^\infty x^{e-2c} dx = +\infty,$$

since $e - 2c \geq -1$.

iii) By the change of variables $y = \lambda^{-\frac{1}{c}} x, \; dx = \lambda^{\frac{1}{c}} dy$, we have

$$\int_1^\infty \frac{x^e}{(x^c+\lambda)^2} dx = \int_{\lambda^{-\frac{1}{c}}}^\infty \frac{\lambda^{\frac{e}{c}} y^e}{(\lambda y^c + \lambda)^2} \lambda^{\frac{1}{c}} dy = \lambda^{\frac{e+1-2c}{c}} \int_{\lambda^{-\frac{1}{c}}}^\infty \frac{y^e}{(y^c+1)^2} dy$$

$$= \lambda^{\frac{e+1-2c}{c}} \left(\int_{\lambda^{-\frac{1}{c}}}^1 \frac{y^e}{(y^c+1)^2} dy + \int_1^\infty \frac{y^e}{(y^c+1)^2} dy \right),$$

where $\lambda^{-\frac{1}{c}} < 1$, since $c < 0$.

We now estimate the two integrals separately:

a) For $\lambda^{-\frac{1}{c}} \leq y \leq 1$, $c < 0$, we have $1 \leq y^c \leq \lambda^{-1}$

$$\Rightarrow y^{2c} \leq (y^c+1)^2 \leq 4y^{2c}$$

$$\Rightarrow \frac{1}{y^{2c}} \geq \frac{1}{(y^c+1)^2} \geq \frac{1}{4y^{2c}}.$$

Hence, since $e-2c > -1$

$$\int_{\lambda^{-\frac{1}{c}}}^{1} \frac{y^e}{(y^c+1)^2}dy \leq \int_{\lambda^{-\frac{1}{c}}}^{1} y^{e-2c}dy = \frac{(1-\lambda^{-\frac{e-2c+1}{c}})}{(e-2c+1)} \qquad (Ai)$$

and

$$\int_{\lambda^{-\frac{1}{c}}}^{1} \frac{y^e}{(y^c+1)^2}dy \geq \frac{1}{4}\int_{\lambda^{-\frac{1}{c}}}^{1} y^{e-2c}dy = \frac{(1-\lambda^{-\frac{e-2c+1}{c}})}{4(e-2c+1)}. \qquad (Aii)$$

b) For $y \geq 1$, $c < 0$, we have $0 \leq y^c \leq 1$

$$\Rightarrow 1 \leq (y^c+1)^2 \leq 4$$

$$\Rightarrow 1 \geq \frac{1}{(y^c+1)^2} \geq \frac{1}{4}.$$

Hence, since $e < -1$

$$\int_{1}^{\infty} \frac{y^e}{(y^c+1)^2}dy \leq \int_{1}^{\infty} y^e dy = -\frac{1}{e+1} \qquad (Bi)$$

and

$$\int_{1}^{\infty} \frac{y^e}{(y^c+1)^2}dy \geq \frac{1}{4}\int_{1}^{\infty} y^e dy = -\frac{1}{4(e+1)}. \qquad (Bii)$$

Combining (Ai) and (Bi), we get

$$\int_{1}^{\infty} \frac{x^e}{(x^c+\lambda)^2}dx \leq \lambda^{\frac{e-2c+1}{c}}\left(\frac{1-\lambda^{-\frac{e-2c+1}{c}}}{e-2c+1} - \frac{1}{e+1}\right)$$

$$= \frac{2c\lambda^{\frac{e-2c+1}{c}}}{(e+1)(e-2c+1)} - \frac{1}{e-2c+1} = M$$

and combining (Aii) and (Bii) we get

$$\int_1^\infty \frac{x^e}{(x^c+\lambda)^2}dx \geq \frac{\lambda^{\frac{e-2c+1}{c}}}{4}\left(\frac{1-\lambda^{-\frac{e-2c+1}{c}}}{e-2c+1} - \frac{1}{e+1}\right)$$

$$= \frac{c\lambda^{\frac{e-2c+1}{c}}}{2(e+1)(e-2c+1)} - \frac{1}{4(e-2c+1)} = \frac{M}{4}.$$

iv) Just like in case (iii), we have:

$$\int_1^\infty \frac{x^e}{(x^c+\lambda)^2}dx = \int_{\lambda^{-\frac{1}{c}}}^1 \frac{y^e}{(y^c+1)^2}dy + \int_1^\infty \frac{y^e}{(y^c+1)^2}dy,$$

where $\lambda^{\frac{e+1-2c}{c}} = 1$, since $e-2c = -1$. Again like $(Ai), (Aii), (Bi), (Bii)$ of (iii) we have:

$$\int_{\lambda^{-\frac{1}{c}}}^1 \frac{y^e}{(y^c+1)^2}dy \leq \int_{\lambda^{-\frac{1}{c}}}^1 y^{e-2c}dy = \int_{\lambda^{-\frac{1}{c}}}^1 y^{-1}dy = \frac{\log\lambda}{c} \qquad (Ai')$$

and

$$\int_{\lambda^{-\frac{1}{c}}}^1 \frac{y^e}{(y^c+1)^2}dy \geq \frac{1}{4}\int_{\lambda^{-\frac{1}{c}}}^1 y^{e-2c}dy = \frac{1}{4}\int_{\lambda^{-\frac{1}{c}}}^1 y^{-1}dy = \frac{\log\lambda}{4c}, \quad (Aii')$$

$$\int_1^\infty \frac{y^e}{(y^c+1)^2}dy \leq \int_1^\infty y^e dy = -\frac{1}{e+1} \qquad (Bi)$$

and

$$\int_1^\infty \frac{y^e}{(y^c+1)^2}dy \geq \frac{1}{4}\int_1^\infty y^e dy = -\frac{1}{4(e+1)}. \qquad (Bii)$$

Combining (Ai') with (Bi) and (Aii') with (Bii) we get the desired result.

□

Lemma 3.2.4. *Under the assumptions (A1)-(A6) we have the following conditions on r and h:*

$$r < -\frac{1}{2} - \beta, \tag{3.6a}$$

$$h < -\frac{1}{2} - \gamma. \tag{3.6b}$$

Proof. By Assumption 3.1.1(A1) we have that $\eta \in \mathcal{H}^\beta$, therefore:

i) If $\beta \geq 0$, from (A2) we have $\mu_k^\beta \geq (c_1^- k^2)^\beta$ and from (A5) we have $\eta_k^2 \geq (c_2^- \delta k^r)^2$. Thus, there exists $c_{21}^- > 0$ such that

$$||\eta||_\beta^2 = \sum_{k=1}^\infty \mu_k^\beta \eta_k^2 \geq \delta^2 \sum_{k=1}^\infty c_{21}^- k^{2\beta+2r}$$

$$\Rightarrow \quad \delta^2 c_{21}^- \sum_{k=1}^\infty k^{2\beta+2r} \leq ||\eta||_\beta^2 < +\infty$$

$$\Rightarrow \quad 2\beta + 2r < -1 \quad \Rightarrow \quad r < -\frac{1}{2} - \beta.$$

ii) If $\beta < 0$, from (A2) we have $\mu_k^\beta \geq (c_1^+ k^2)^\beta$ and from (A5) we have $\eta_k^2 \geq (c_2^- \delta k^r)^2$. Thus, there exists $c_{21}^+ > 0$ such that

$$||\eta||_\beta^2 = \sum_{k=1}^\infty \mu_k^\beta \eta_k^2 \geq \delta^2 \sum_{k=1}^\infty c_{21}^+ k^{2\beta+2r},$$

hence, like in case (i) we have

$$r < -\frac{1}{2} - \beta.$$

By Assumption 3.1.1(A3) we have that $u^\dagger \in \mathcal{H}^\gamma$, therefore:

i) If $\gamma \geq 0$, from (A2) we have $\mu_k^\gamma \geq (c_1^- k^2)^\gamma$ and from (A6) we have $q_k^2 \geq (c_3^- k^h)^2$. Thus, there exists c_{31}^- such that

$$\left|\left|u^\dagger\right|\right|_\gamma^2 = \sum_{k=1}^\infty \mu_k^\gamma q_k^2 \geq \sum_{k=1}^\infty c_{31}^- k^{2\gamma+2h}$$

$$\Rightarrow \quad c_{31}^- \sum_{k=1}^\infty k^{2\gamma+2h} \leq \left|\left|u^\dagger\right|\right|_\gamma^2 < +\infty$$

$$\Rightarrow \quad 2\gamma + 2h < -1 \quad \Rightarrow \quad h < -\frac{1}{2} - \gamma.$$

ii) If $\gamma < 0$, from (A2) we have $\mu_k^\gamma \geq (c_1^+ k^2)^\gamma$ and from (A6) we have $q_k^2 \geq (c_3^- k^h)^2$. Thus, there exists $c_{31}^+ > 0$ such that

$$\left|\left|u^\dagger\right|\right|_\gamma^2 = \sum_{k=1}^{\infty} \mu_k^\gamma q_k^2 \geq \sum_{k=1}^{\infty} c_{31}^+ k^{2\gamma+2h},$$

hence, like in case (i) we have

$$h < -\frac{1}{2} - \gamma.$$

□

Lemma 3.2.5. *We can split the solution error in three parts:*

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma^2 = I_1 + I_2 + I_3$$

where

$$I_1 = \sum_{k=1}^{\infty} \frac{\mu_k^{-2\ell+2\beta+\gamma-2\alpha} \eta_k^2}{(\mu_k^{-2\ell+\beta-\alpha} + \lambda)^2},$$

$$I_2 = \lambda^2 \sum_{k=1}^{\infty} \frac{\mu_k^\gamma q_k^2}{(\mu_k^{-2\ell+\beta-\alpha} + \lambda)^2}$$

and

$$I_3 = -2\lambda \sum_{k=1}^{\infty} \frac{\mu_k^{-\ell+\beta+\gamma-\alpha} \eta_k q_k}{(\mu_k^{-2\ell+\beta-\alpha} + \lambda)^2}.$$

Proof. By Lemma 3.1.3

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma^2 = \sum_{k=1}^{\infty} \left(\frac{\mu_k^{-2\ell+\beta} q_k + \mu_k^{-\ell+\beta} \eta_k}{\mu_k^{-2\ell+\beta} + \lambda \mu_k^\alpha} - q_k \right)^2 \mu_k^\gamma$$

$$= \sum_{k=1}^{\infty} \left(\frac{\mu_k^{-\ell+\beta} \eta_k - \lambda \mu_k^\alpha q_k}{\mu_k^{-2\ell+\beta} + \lambda \mu_k^\alpha} \right)^2 \mu_k^\gamma = \sum_{k=1}^{\infty} \left(\frac{\mu_k^{-\ell+\beta-\alpha} \eta_k - \lambda q_k}{\mu_k^{-2\ell+\beta-\alpha} + \lambda} \right)^2 \mu_k^\gamma$$

$$\sum_{k=1}^{\infty} \left(\frac{\mu_k^{-\ell+\beta+\frac{\gamma}{2}-\alpha} \eta_k - \lambda \mu_k^{\frac{\gamma}{2}} q_k}{\mu_k^{-2\ell+\beta-\alpha} + \lambda} \right)^2 = I_1 + I_2 + I_3$$

□

Lemma 3.2.6. *Let $\lambda \in (0,1)$. There exist $M_1^-, M_1^+, M_2^-, M_2^+, M_3^-, M_3^+ > 0$, such that*

$$M_1^- \delta^2 \sum_{k=1}^{\infty} \frac{k^{2a}}{(k^c+\lambda)^2} \leq I_1 \leq M_1^+ \delta^2 \sum_{k=1}^{\infty} \frac{k^{2a}}{(k^c+\lambda)^2} \tag{3.7}$$

$$M_2^- \lambda^2 \sum_{k=1}^{\infty} \frac{k^{2b}}{(k^c+\lambda)^2} \leq I_2 \leq M_2^+ \lambda^2 \sum_{k=1}^{\infty} \frac{k^{2b}}{(k^c+\lambda)^2} \tag{3.8}$$

$$M_3^- \lambda\delta \sum_{k=1}^{\infty} \frac{k^{a+b}}{(k^c+\lambda)^2} \leq |I_3| \leq M_3^+ \lambda\delta \sum_{k=1}^{\infty} \frac{k^{a+b}}{(k^c+\lambda)^2} \tag{3.9}$$

where a, b and c are defined before the statement of Theorem 3.2.2.

Proof. By (A2) we have $\forall k \in \mathbb{N}$:

$$\mu_k^{-2\ell+2\beta+\gamma-2\alpha} \leq \begin{cases} (c_1^+)^{-2\ell+2\beta+\gamma-2\alpha} k^{-4\ell+4\beta+2\gamma-4\alpha}, & if \quad -2\ell+2\beta+\gamma-2\alpha \geq 0 \\ (c_1^-)^{-2\ell+2\beta+\gamma-2\alpha} k^{-4\ell+4\beta+2\gamma-4\alpha}, & if \quad -2\ell+2\beta+\gamma-2\alpha < 0 \end{cases}$$

$$\mu_k^{-2\ell+2\beta+\gamma-2\alpha} \geq \begin{cases} (c_1^-)^{-2\ell+2\beta+\gamma-2\alpha} k^{-4\ell+4\beta+2\gamma-4\alpha}, & if \quad -2\ell+2\beta+\gamma-2\alpha \geq 0 \\ (c_1^+)^{-2\ell+2\beta+\gamma-2\alpha} k^{-4\ell+4\beta+2\gamma-4\alpha}, & if \quad -2\ell+2\beta+\gamma-2\alpha < 0 \end{cases}$$

$$\mu_k^{\gamma} \leq \begin{cases} (c_1^+)^{\gamma} k^{2\gamma}, & if \quad \gamma \geq 0 \\ (c_1^-)^{\gamma} k^{2\gamma}, & if \quad \gamma < 0 \end{cases}$$

$$\mu_k^{\gamma} \geq \begin{cases} (c_1^-)^{\gamma} k^{2\gamma}, & if \quad \gamma \geq 0 \\ (c_1^+)^{\gamma} k^{2\gamma}, & if \quad \gamma < 0 \end{cases}$$

$$\mu_k^{-\ell+\beta+\gamma-\alpha} \leq \begin{cases} (c_1^+)^{-\ell+\beta+\gamma-\alpha} k^{-2\ell+2\beta+2\gamma-2\alpha}, & if \quad -\ell+\beta+\gamma-\alpha \geq 0 \\ (c_1^-)^{-\ell+\beta+\gamma-\alpha} k^{-2\ell+2\beta+2\gamma-2\alpha}, & if \quad -\ell+\beta+\gamma-\alpha < 0 \end{cases}$$

$$\mu_k^{-\ell+\beta+\gamma-\alpha} \geq \begin{cases} (c_1^-)^{-\ell+\beta+\gamma-\alpha} k^{-2\ell+2\beta+2\gamma-2\alpha}, & if \quad -\ell+\beta+\gamma-\alpha \geq 0 \\ (c_1^+)^{-\ell+\beta+\gamma-\alpha} k^{-2\ell+2\beta+2\gamma-2\alpha}, & if \quad -\ell+\beta+\gamma-\alpha < 0 \end{cases}$$

and

$$\mu_k^{-2\ell+\beta-\alpha} \leq \begin{cases} (c_1^+)^{-2\ell+\beta-\alpha} k^{-4\ell+2\beta-2\alpha}, & if \quad -2\ell+\beta-\alpha \geq 0 \\ (c_1^-)^{-2\ell+\beta-\alpha} k^{-4\ell+2\beta-2\alpha}, & if \quad -2\ell+\beta-\alpha < 0 \end{cases}$$

$$\mu_k^{-2\ell+\beta-\alpha} \geq \begin{cases} (c_1^-)^{-2\ell+\beta-\alpha} k^{-4\ell+2\beta-2\alpha}, & if \quad -2\ell+\beta-\alpha \geq 0 \\ (c_1^+)^{-2\ell+\beta-\alpha} k^{-4\ell+2\beta-2\alpha}, & if \quad -2\ell+\beta-\alpha < 0. \end{cases}$$

Therefore there exist $c_4^+, c_4^-, c_5^+, c_5^-, c_6^+, c_6^-, c_7^+, c_7^- > 0$, such that $\forall k \in \mathbb{N}$:

$$c_4^- k^{-4\ell+4\beta+2\gamma-4\alpha} \leq \mu_k^{-2\ell+2\beta+\gamma-2\alpha} \leq c_4^+ k^{-4\ell+4\beta+2\gamma-4\alpha},$$

$$c_5^- k^{2\gamma} \leq \mu_k^{\gamma} \leq c_5^+ k^{2\gamma},$$

$$c_6^- k^{-2\ell+2\beta+2\gamma-2\alpha} \leq \mu_k^{-\ell+\beta+\gamma-\alpha} \leq c_6^+ k^{-2\ell+2\beta+2\gamma-2\alpha},$$

$$c_7^- k^{-4\ell+2\beta-2\alpha} \leq \mu_k^{-2\ell+\beta-\alpha} \leq c_7^+ k^{-4\ell+2\beta-2\alpha}$$

and since $k, \lambda > 0$ for $c_8^+ = \max\left\{c_7^+, 1\right\}$ and $c_8^- = \min\left\{c_7^-, 1\right\}$ we have

$$c_8^- (k^{-4\ell+2\beta-2\alpha} + \lambda) \leq \mu_k^{-2\ell+\beta-\alpha} + \lambda \leq c_8^+ (k^{-4\ell+2\beta-2\alpha} + \lambda).$$

By (A5) and (A6) we have

$$\delta^2 (c_2^-)^2 k^{2r} \leq \eta_k^2 \leq \delta^2 (c_2^+)^2 k^{2r},$$

$$(c_3^-)^2 k^{2h} \leq q_k^2 \leq (c_3^+)^2 k^{2h}$$

and

$$\delta c_2^- c_3^- k^{h+r} \leq |q_k||\eta_k| \leq \delta c_2^+ c_3^+ k^{h+r},$$

$\forall k \in \mathbb{N}$.

We define

$$M_1^- = \frac{c_4^-(c_2^-)^2}{(c_8^+)^2}, \quad M_1^+ = \frac{c_4^+(c_2^+)^2}{(c_8^-)^2},$$

$$M_2^- = \frac{c_5^-(c_3^-)^2}{(c_8^+)^2}, \quad M_2^+ = \frac{c_5^+(c_3^+)^2}{(c_8^-)^2},$$

$$M_3^- = \frac{c_6^- c_2^- c_3^-}{(c_8^+)^2}, \quad M_3^+ = \frac{c_6^+ c_2^+ c_3^+}{(c_8^-)^2},$$

to get the inequalities (3.7), (3.8) and (3.9).

□

We now give the proof of Theorem 3.2.2:

Proof. By Lemma 3.2.5 we can split the solution error:

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma^2 = I_1 + I_2 + I_3$$

Since $a \neq b$ and $a + b < \max\{2a, 2b\}$, by Lemma 3.2.6 it follows that whether we have convergence or not of $u^{\lambda,\delta}$ to $u^\dagger$ in $\mathcal{H}^\gamma$, depends only on I_1 and I_2. The inequalities (3.7), (3.8) and (3.9), imply that in the cases where we do have convergence, the rate of convergence is the slowest rate of the rates of convergence of I_1 and I_2 to 0, since the rate of convergence of I_3 is always the fastest. Furthermore, they imply that the rates of convergence are sharp.

The following inequality holds:

$$\int_1^\infty \frac{x^e}{(x^c+\lambda)^2} \leq \sum_{k=1}^\infty \frac{k^e}{(k^c+\lambda)^2} dx \leq \frac{1}{(1+\lambda)^2} + \int_1^\infty \frac{x^e}{(x^c+\lambda)^2} dx. \qquad (*)$$

Define

$$J_1 = \delta^2 \int_1^\infty \frac{x^{2a}}{(x^c+\lambda)^2}, \quad J_2 = \lambda^2 \int_1^\infty \frac{x^{2b}}{(x^c+\lambda)^2}.$$

Then, by (3.7) and (3.8)

$$J_1 \leq I_1 \leq \frac{\delta^2}{(1+\lambda)^2} + J_1$$

and

$$J_2 \leq I_2 \leq \frac{\lambda^2}{(1+\lambda)^2} + J_2.$$

Since, in order to have convergence of both I_1 and I_2, as $\delta \to 0$, we need a parameter choice $\lambda = \lambda(\delta) \to 0$, we have that I_i converges to 0 if and only if J_i converges to 0. By Lemma 3.2.3, the integrals in J_1 and J_2 are in general λ-dependent and do not converge to 0 as $\lambda \to 0$. Thus, the convergence rate of J_1 is slower than $O(\delta^2)$ and the convergence rate of J_2 is slower than $O(\lambda^2)$. Hence, the term $\frac{1}{1+\lambda^2}$ on the right hand side of inequality $(*)$, doesn't affect the convergence rates, since it always converges faster than J_1 and J_2. Consequently, the rates of convergence of I_i and J_i are identical, $i = 1, 2$, therefore we can examine the rates of convergence of J_i.

Suppose $c \geq 0$.

Since $c \geq 0$, we have that $2b - 2c < -1$.

A) If $2a - 2c \geq -1$, then $2a \geq 2a - 2c \geq -1$ and so by Lemma 3.2.3 $J_1 = \infty$, therefore we do not have convergence.

B) If $2a - 2c < -1$, then since $2b - 2c < -1$, by Lemma 3.2.3 we have that $J_1 = \delta^2 C$ and $J_2 = \lambda^2 C$, so we do have convergence and for the parameter choice $\lambda = \delta$, we have

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta).$$

Suppose $c < 0$:

A) If $2a - 2c > -1$ and $2b - 2c > -1$:

   i) If $2a \geq -1$ then by Lemma 3.2.3 we have $J_1 = \infty$, therefore we do not have convergence.

   ii) If $2a < -1$, then since $2b < -1$, by Lemma 3.2.3 we have $J_1 \in \left[\frac{\delta^2 M_a}{4}, \delta^2 M_a\right]$, $J_2 \in \left[\frac{\lambda^2 M_b}{4}, \lambda^2 M_b\right]$ where

$$M_a = \frac{2c\lambda^{\frac{2a-2c+1}{c}}}{(2a+1)(2a-2c+1)} - \frac{1}{2a-2c+1},$$

so

$$\delta^2 M_a = \frac{2c\delta^2\lambda^{\frac{2a-2c+1}{c}}}{(2a+1)(2a-2c+1)} - \frac{\delta^2}{2a-2c+1}$$

and

$$M_b = \frac{2c\lambda^{\frac{2b-2c+1}{c}}}{(2b+1)(2b-2c+1)} - \frac{1}{2b-2c+1},$$

so

$$\lambda^2 M_b = \frac{2c\lambda^{\frac{2b+1}{c}}}{(2b+1)(2b-2c+1)} - \frac{\lambda^2}{2b-2c+1}.$$

Since $2a-2c>-1$ and $c<0$, we have $\frac{2a-2c+1}{c}<0$, therefore

$$J_1 = O(\delta^2\lambda^{\frac{2a-2c+1}{c}}).$$

Since $2b-2c>-1$ and $c<0$, we have $\frac{2b+1}{c}<2$, therefore

$$J_2 = O(\lambda^{\frac{2b+1}{c}}).$$

Define

$$\phi(\lambda) = \delta^2\lambda^{\frac{2a-2c+1}{c}} + \lambda^{\frac{2b+1}{c}}.$$

We want to choose $\lambda=\lambda(\delta)$, in order to minimize ϕ and get the optimal convergence rate:

$$\phi'(\lambda) = c_1\delta^2\lambda^{\frac{2a-3c+1}{c}} + c_2\lambda^{\frac{2b+1-c}{c}},$$

therefore the optimal choice is $\lambda = \delta^{\frac{c}{b+c-a}}$, which gives that $J_1, J_2 = O(\delta^{\frac{2b+1}{b+c-a}})$. Consequently for $\lambda = \delta^{\frac{c}{b+c-a}}$ we have

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta^{\frac{2b+1}{2(b+c-a)}}).$$

Note that since $2a-2c>-1$ and $2b<-1 \Rightarrow -2b>1$, we have that $2a-2c-2b>0$ therefore $b+c-a<0$. Consequently $\frac{2b+1}{2(b+c-a)}>0$ so we do have convergence, as $\delta\to 0$.

In addition, note that $2a-2c>-1 \Rightarrow 2c-2a<1 \Rightarrow 2b+2c-2a<1+2b$, thus since $2b+2c-2a<0$ we have $\frac{2b+1}{2c+2b-2a}<1$.

B) If $2a-2c=-1$ and $2b-2c=-1$, then since $c<0$ we have that $2a < -1$ and as always $2b < -1$. By Lemma 3.2.3 we have $J_1 \in \left[\frac{\delta^2 N_a}{4}, \delta^2 N_a\right]$, $J_2 \in \left[\frac{\lambda^2 N_b}{4}, \lambda^2 N_b\right]$ where

$$N_a = \frac{\log \lambda}{c} - \frac{1}{2a+1},$$

so

$$\delta^2 N_a = \frac{\delta^2 \log \lambda}{c} - \frac{\delta^2}{2a+1}$$

and

$$N_b = \frac{\log \lambda}{c} - \frac{1}{2b+1},$$

so

$$\lambda^2 N_b = \frac{\lambda^2 \log \lambda}{c} - \frac{\lambda^2}{e+1}.$$

We thus have that $J_1 = O(\delta^2 \log \delta)$ and $J_2 = O(\lambda^2 \log \lambda)$, therefore for the parameter choice $\lambda = \delta$, we have

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta\sqrt{|\log \delta|}).$$

Hence, we do have convergence since $\delta^2 \log \delta \to 0$, as $\delta \to 0$.

C) If $2a - 2c < -1$ and $2b - 2c < -1$, by Lemma 3.2.3 we have that $J_1 = \delta^2 \kappa, J_2 = \lambda^2 \kappa$, so we have convergence and for the parameter choice $\lambda = \delta$, we have

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta).$$

D) If $2a - 2c < -1$ and $2b - 2c > -1$, then since $2b < -1$, by Lemma 3.2.3 we have $J_1 = O(\delta^2)$ and $J_2 \in \left[\frac{\lambda^2 M}{4}, \lambda^2 M\right]$, where

$$M = \frac{2c\lambda^{\frac{2b-2c+1}{c}}}{(2b+1)(2b-2c+1)} - \frac{1}{2b-2c+1}$$

so

$$\lambda^2 M = \frac{2c\lambda^{\frac{2b+1}{c}}}{(2b+1)(2b-2c+1)} - \frac{\lambda^2}{2b-2c+1}.$$

Since $2b - 2c > -1$ and $c < 0$ we have $\frac{2b+1}{c} < 2$, therefore

$$J_2 = O(\lambda^{\frac{2b+1}{c}}).$$

Note that since $2b < -1$ it follows that $\frac{2b+1}{c} > 0$, hence we do have convergence.
For the parameter choice $\lambda = \delta^{\frac{2c}{2b+1}}$, we have $J_1, J_2 = O(\delta^2)$ therefore

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta).$$

E) If $2a - 2c < -1$ and $2b - 2c = -1$, then since $2b < -1$, by Lemma 3.2.3 we have $J_1 = O(\delta^2)$ and $J_2 \in \left[\frac{\lambda^2 N}{4}, \lambda^2 N\right]$, where

$$N = \frac{\log \lambda}{c} - \frac{1}{2b+1},$$

so

$$\lambda^2 N = \frac{\lambda^2 \log \lambda}{c} - \frac{\lambda^2}{2b+1}.$$

Thus

$$J_2 = O(\lambda^2 \log \lambda).$$

For the parameter choice $\lambda = \delta$ we have $J_1 = O(\delta^2), J_2 = O(\delta^2 \log \delta)$, therefore

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta\sqrt{|\log \delta|}),$$

hence we do have convergence since $\delta^2 \log \delta \to 0$, as $\delta \to 0$.

F) If $2a - 2c > -1$ and $2b - 2c < -1$:

i) If $2a \geq -1$, then by Lemma 3.2.3 we have $J_1 = \infty$, therefore we do not have convergence.

ii) If $2a < -1$, by Lemma 3.2.3 we have $J_2 = O(\lambda^2)$ and $J_1 \in \left[\frac{\delta^2 M}{4}, \delta^2 M\right]$, where

$$M = \frac{2c\lambda^{\frac{2a-2c+1}{c}}}{(2a+1)(2a-2c+1)} - \frac{1}{2a-2c+1},$$

so

$$\delta^2 M = \frac{2c\delta^2\lambda^{\frac{2a-2c+1}{c}}}{(2a+1)(2a-2c+1)} - \frac{\delta^2}{2a-2c+1}.$$

Since $2a-2c > -1$ and $c < 0$ we have $\frac{2a-2c+1}{c} < 0$, therefore

$$J_1 = O(\delta^2 \lambda^{\frac{2a-2c+1}{c}}).$$

Define

$$\phi(\lambda) = \delta^2 \lambda^{\frac{2a-2c+1}{c}} + \lambda^2.$$

We want to minimize ϕ in order to get the optimal choice for the regularization parameter λ:

$$\phi'(\lambda) = c_1 \delta^2 \lambda^{\frac{2a-3c+1}{c}} + c_2 \lambda,$$

therefore the optimal choice is $\lambda = \delta^{\frac{2c}{4c-2a-1}}$ which gives that $J_1, J_2 = O(\delta^{\frac{4c}{4c-2a-1}})$. Consequently, for $\lambda = \delta^{\frac{2c}{4c-2a-1}}$ we have

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta^{\frac{2c}{4c-2a-1}}).$$

Note that since $2a - 2c > -1$ and $c < 0$, we have that $2c - 2a - 1 < 0$ and $4c - 2a - 1 < 0$, therefore $\frac{2c}{4c-2a-1} > 0$, so we do have convergence as $\delta \to 0$.

In addition note that $4c - 2a - 1 < 2c$, so since $4c - 2a - 1$, $\frac{2c}{4c-2a-1} < 1$, hence the convergence is slower than $O(\delta)$.

G) If $2a - 2c > -1$ and $2b - 2c = -1$:

i) If $2a \geq -1$, then by Lemma 3.2.3 we have $J_1 = \infty$, therefore we do not have convergence.

ii) If $2a < -1$, then since $2b < -1$, by Lemma 3.2.3 we have that $J_1 \in \left[\frac{\delta^2 M}{4}, \delta^2 M\right]$ and $J_2 \in \left[\frac{\lambda^2 N}{4}, \lambda^2 N\right]$, where

$$M = \frac{2c\lambda^{\frac{2a-2c+1}{c}}}{(2a+1)(2a-2c+1)} - \frac{1}{2a-2c+1},$$

so

$$\delta^2 M = \frac{2c\delta^2 \lambda^{\frac{2a-2c+1}{c}}}{(2a+1)(2a-2c+1)} - \frac{\delta^2}{2a-2c+1}$$

and

$$N = \frac{\log \lambda}{c} - \frac{1}{2b+1},$$

so

$$\lambda^2 N = \frac{\lambda^2 \log \lambda}{c} - \frac{\lambda^2}{2b+1}.$$

Since $2a - 2c > -1$ and $c < 0$ we have that $\frac{2a-2c+1}{c} < 0$, thus $J_1 = O(\delta^2 \lambda^{\frac{2a-2c+1}{c}})$ and $J_2 = O(\lambda^2 \log \lambda)$.
For the parameter choice $\lambda = \delta^{\frac{2c}{4c-2a-1}}$, we have that $J_1 = O(\delta^{\frac{4c}{4c-2a-1}})$ and $J_2 = O(\delta^{\frac{4c}{4c-2a-1}} \log \delta)$, hence

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta^{\frac{2c}{4c-2a-1}} \sqrt{|\log \delta|}).$$

Note that since $2a - 2c > -1$ and $c < 0$, we have that $2c-2a-1 < 0$ and $4c-2a-1 < 0$, therefore $\frac{2c}{4c-2a-1} > 0$. Since $\delta^p \log \delta \to 0$, as $\delta \to 0$, we do have convergence.

In addition note that $4c-2a-1 < 2c$, so since $4c-2a-1$, $\frac{2c}{4c-2a-1} < 1$.

H) If $2a-2c = -1$ and $2b-2c < -1$, then since $c < 0$, we have $2a < -1$, thus by Lemma 3.2.3 we have $J_1 \in \left[\frac{\delta^2 N}{4}, \delta^2 N\right]$ and $J_2 = O(\lambda^2)$, where

$$N = \frac{\log \lambda}{c} - \frac{1}{2a+1},$$

so

$$\delta^2 N = \frac{\delta^2 \log \lambda}{c} - \frac{\delta^2}{2a+1}.$$

Thus

$$J_1 = O(\delta^2 \log \lambda).$$

For the parameter choice $\lambda = \delta$, we have $J_1 = O(\delta^2 \log \delta)$, $J_2 = O(\delta^2)$, therefore

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta \sqrt{|\log \delta|}),$$

hence we do have convergence since $\delta^2 \log \delta \to 0$, as $\delta \to 0$.

I) If $2a-2c=-1$ and $2b-2c>-1$, then since $c<0$, we have $2a<-1$ and as always $2b<-1$. By Lemma 3.2.3 we have $J_1 \in \left[\frac{\delta^2 N}{4}, \delta^2 N\right]$ and $J_2 \in \left[\frac{\lambda^2 M}{4}, \lambda^2 M\right]$, where

$$M = \frac{2c\lambda^{\frac{2b-2c+1}{c}}}{(2b+1)(2b-2c+1)} - \frac{1}{2b-2c+1}$$

so

$$\lambda^2 M = \frac{2c\lambda^{\frac{2b+1}{c}}}{(2b+1)(2b-2c+1)} - \frac{\lambda^2}{2b-2c+1},$$

and

$$N = \frac{\log\lambda}{c} - \frac{1}{2a+1},$$

so

$$\delta^2 N = \frac{\delta^2 \log\lambda}{c} - \frac{\delta^2}{2a+1}.$$

Since $\frac{2b+1}{c} < 2$ we have that $J_1 = O(\delta^2 \log\lambda)$ and $J_2 = O(\lambda^{\frac{2b+1}{c}})$. For the parameter choice $\lambda = \delta^{\frac{2c}{2b+1}}$, we have that $J_1 = O(\delta^2 \log\delta)$ and $J_2 = O(\delta^2)$. Thus,

$$\left|\left|u_\lambda^\delta - u^\dagger\right|\right|_\gamma = O(\delta\sqrt{|\log\delta|})$$

and we do have convergence since $\delta^2 \log\delta \to 0$, as $\delta \to 0$.

□

Remark 3.2.7. We replace $\alpha, \beta, \gamma, \ell, r, h$ in a, b, c to get:

$$2a = -4\ell + 4\beta + 2r + 2\gamma - 4\alpha,$$

$$2a - 2c = 4\ell + 2\gamma + 2r,$$

$$2b = 2h + 2\gamma < -1$$

and

$$2b - 2c = 8\ell - 4\beta + 4\alpha + 2h + 2\gamma.$$

i) Fix $\alpha, \beta, \gamma, \ell, h$, where by $(A4)$ $\beta < \gamma + 2\ell$. If the noise is smooth enough, i.e. r is small enough, so that $2a - 2c < -1$ or equivalently

$$4\ell + 2\gamma + 2r < -1$$

$$\iff r < -\frac{1}{2} - \gamma - 2\ell,$$

then we are in one of the cases $(1B), (2C), (2D)$ or $(2E)$, thus we always have convergence and depending on the values of c and $2b-2c$, we have a convergence rate $O(\delta)$ or $O(\delta\sqrt{|\log\delta|})$. This was expected, since the condition $r < -\frac{1}{2} - \gamma - 2\ell$ secures that $\eta \in \mathcal{H}^{\gamma+2\ell}$, therefore since $Ku^\dagger \in \mathcal{H}^{\gamma+2\ell}$ we have that $y^\delta \in \mathcal{H}^{\gamma+2\ell}$. As we have already observed in Remark 3.1.2, in this case, by Lemma 1.2.3 we have that $K : \mathcal{H}^\gamma \to \mathcal{H}^{\gamma+2\ell}$ is invertible with bounded inverse K^{-1}, thus the problem is not ill-posed and no regularization is needed. Since a linear bounded function is also *Lipschitz* continuous, we expected the convergence rate of $O(\delta)$.

ii) Fix $\alpha, \beta, \gamma, \ell, r$, where by $(A4)$ $\beta < \gamma + 2\ell$. If the true solution $u^\dagger$ is smooth enough to have $2b - 2c < -1$ or equivalently

$$8\ell - 4\beta + 4\alpha + 2h + 2\gamma < -1,$$

then we are in one of the cases $(1A), (1B), (2C), (2F)$ or $(2H)$. We do not always have convergence which was expected, since even if the true solution is very smooth, we need to regularize in a strong enough norm for this to help. Indeed, for α large enough, we have that $c < 0$, thus the cases $(1A)$ and $(1B)$ are eliminated and $2a < -1$ thus (2Fi) is eliminated and in all the other cases we do have convergence.

iii) Fix $\beta, \gamma, \ell, r, h$, where by $(A4)$ $\beta < \gamma + 2\ell$. If we regularize in a strong enough norm, i.e. if α is large enough to have $c < 0$ and $2a < -1$, then we are in one of the cases $(2Aii), (2B), (2C), (2D), (2E), (2Fii), (2Gii), (2H)$ or $(2I)$, thus we always have convergence regardless of the regularity of the noise.

iv) Fix $\alpha, \beta, \gamma, h, r$, where by $(A4)$ $\beta < \gamma + 2\ell$. If the operator K is too smoothing, i.e. if ℓ is very large, we have that $2a < -1$, $2a - 2c > -1, 2b - 2c > -1$ and $c < 0$. Therefore we are in the case $(2Aii)$ and we have convergence with convergence rate $O(\delta^{\frac{2b+1}{2(b+c-a)}})$. Since $b + c - a = h - r - \ell$ and $2b + 1 < -1$, we have that for large enough values of ℓ, the exponent

$\frac{2b+1}{2(b+c-a)}$ becomes very small and in fact

$$\lim_{\ell\to\infty} \frac{2b+1}{2(b+c-a)} = 0.$$

This means that as the forward operator becomes more smoothing, the convergence rate becomes slower, which was expected since the smoother the forward operator the more difficult it is to invert. As it is highlighted in [5], "whenever a forward problem has smoothing properties one has to expect the appearance of oscillations coming from small data perturbations in the solution of the inverse problem".

v) We now check that our rates are consistent with the rates of Theorem 2.3.5 in Section 2.3. In Theorem 2.3.5 we have $\beta = \gamma = 0$ and fixed $\alpha \geq 0$. As stated in Remark 2.3.6(iii), Theorem 2.3.5 provides the best possible rate, $O(\delta^{\frac{4\ell+2\alpha}{6\ell+2\alpha}})$, for the *a-priori* information $u^\dagger \in \mathcal{H}^{4\ell+2\alpha}$. In the setup that we have here, we model the *a-priori* assumption $u^\dagger \in \mathcal{H}^{4\ell+2\alpha}$ by the assumption

$$h < -\frac{1}{2} - 4\ell - 2\alpha,$$

where h is assumed to be arbitrarily close to $-\frac{1}{2} - 4\ell - 2\alpha$. Observe that this assumption actually implies that $u^\dagger \in \mathcal{H}^{4\ell+2\alpha+\epsilon}$ for a small $\epsilon > 0$. Since we do not have any *a-priori* assumptions on the regularity of the noise, we set $r < -\frac{1}{2}$ with r arbitrarily close to $-\frac{1}{2}$. Then

$$2a - 2c = 4\ell + 2r > -1$$

and

$$2a = -4\ell + 2r - 4\alpha < -1,$$

since we assume that r is arbitrarily close to $-\frac{1}{2}$ and $\ell < 0, \alpha \geq 0$ are fixed. Furthermore,

$$2b - 2c = 8\ell + 4\alpha + 2h < -1$$

and

$$c = -4\ell - 2\alpha < 0,$$

thus we are in the case $(2Fii)$ and we have convergence with convergence rate

$$O(\delta^{\frac{2c}{4c-2a-1}}).$$

Observe that

$$\frac{2c}{4c-2a-1} = \frac{4\ell+2\alpha}{6\ell+2\alpha+r+\frac{1}{2}},$$

where $r+\frac{1}{2}$ is negative but arbitrarily close to 0. Consequently, we get a convergence rate, which is very close, but slightly faster than the convergence rate provided by Theorem 2.3.5. The fact that our rate is slightly faster, is attributed to the fact that by assuming that $r<-\frac{1}{2}$ with r arbitrarily close to $-\frac{1}{2}$, we are in fact assuming that $\eta \in \mathcal{H}^{\epsilon}$ for some small $\epsilon>0$, thus we have *a-priori* information on the regularity of the noise. Note that even if we allow h to get smaller, thus assuming even more regularity on $u^{\dagger}$, we do not get a faster convergence rate, since we stay in the same case and h does not appear in $\frac{2c}{4c-2a-1}$ which suggests that the Tikhonov Regularization in Hilbert Scales method for $\alpha \geq 0$, $\beta=\gamma=0$, saturates for $u^{\dagger} \in \mathcal{H}^{4\ell+2\alpha}$, again in agreement with Theorem 2.3.5.

vi) Fix $\alpha=\gamma=0,\ \ell=1, h=-10<-\frac{1}{2}$ and let

$$\beta=0<2\ell$$

and

$$-\frac{1}{2}-2\ell=-\frac{5}{2}<r=-2<-\frac{1}{2}.$$

Then $2a-2c=0>-1$, $2b-2c=-12<-1$ and $2a=-8<-1$, while $c=-4<0$, thus we are in case $(2Fii)$ and we have convergence with convergence rate $O(\delta^{\frac{2c}{2(b+c-a)}})=O(\delta^{\frac{2}{5}})$.
If we change β to $\beta=1$, then $r=-2$ is still admissible since $-2<-\frac{1}{2}-1$ and we have $2a-2c=0>-1$, $2b-2c=-16<-1$, $2a=-4<-1$ and $c=-2<0$. Hence, we are again in case $(2Fii)$ and we have convergence with convergence rate $O(\delta^{\frac{2c}{2(b+c-a)}})=O(\delta^{\frac{1}{5}})$, i.e. with a slower rate. Thus in this case if the norm where we measure the data error, becomes stronger, i.e. if the least squares norm becomes stronger, we have a slower convergence rate.

To understand why this makes sense, note that for larger β, the contribution of the least squares part to the Tikhonov Functional becomes more important related to the regularization term, thus the effect of regularization becomes smaller and continuity with respect to data is less enforced, hence we have slower convergence rates.

Furthermore, if we are in case $(2Aii)$, thus we have convergence with convergence rate

$$O(\delta^{\frac{2b+1}{2(b+c-a)}})$$

and we change β in a way that does not take as to another case, there is no change to the convergence rate, since β does not appear in $\frac{2b+1}{2(b+c-a)}$.

This can be interpreted by the fact that in order to move from $(2Fii)$ to $(2Aii)$, i.e. from $2b-2c<-1$ to $2b-2c>-1$, it means that either β got smaller, or α got larger if everything else is fixed. Thus this observation suggests, that for strong enough regularization norm, i.e. for large enough α, the regularizing effect of the penalty term is not affected from small changes in β.

vii) Suppose that we are in case $(2Aii)$, thus we have convergence with convergence rate

$$O(\delta^{\frac{2b+1}{2(b+c-a)}}).$$

Then, as noted in the proof we have $2b+1=h+\gamma+1<-1$ and $b+c-a=h-2\ell-r<2b+1$, thus if we allow γ to decrease by a small amount which does not take us to another case, we have that $2b+1<-1$ becomes smaller while $b+c-a<2b+1<-1$ remains unchanged, hence we have a faster convergence rate. This agrees with the intuition that convergence in weaker norms should give faster rates, even though sometimes this is not strictly true: suppose that we are in case $(2Fii)$, thus we have convergence with convergence rate

$$O(\delta^{\frac{2c}{2(b+c-a)}}).$$

Then, if we allow γ to change by a small amount which does not take us to another case, there is no change to the convergence rate since γ does not appear in $\frac{2c}{2(b+c-a)}$.

viii) Finally, note that in cases $(2Aii), (2Fii)$ and $(2Gii)$, if we set $a = -\frac{1}{2}$ the rate of convergence does not degenerate. This happens because the problem with a being equal to $-\frac{1}{2}$, is independent from λ (hence from δ), since it originates on the constant multiplying $\lambda^{2a-2c+1}$ in the expression of M_α.

Chapter 4

The Bayesian Approach

In the previous chapters we have presented deterministic regularization methods which approximate the solution u of the inverse problem

$$y = Ku, \quad u \in X, \ y \in Y, \tag{4.1}$$

where K is a linear, compact and self-adjoint operator and X, Y are Hilbert spaces. More precisely we have considered the case where our observation y is polluted by the presence of some additive noise η

$$y^{\delta} = Ku + \eta \tag{4.2}$$

and we have the knowledge that the norm of the noise is less than δ. We have shown convergence rate results for the convergence of the Tikhonov regularized approximate solution u_{δ}^{λ} to the true solution $u^{\dagger}$, as the noise disappears,

$$||u^{\dagger} - u_{\lambda}^{\delta}|| = O(f(\delta)),$$

which is an accepted quality criterion for a deterministic regularization method.

The deterministic theory of inverse problems can be criticized for the fact that the aforementioned convergence results depend on a norm bound of the noise which is a worst-case scenario [13]. It is often the case that we have more available information than a norm bound on the size of the noise, or the space where the solution lives in. In particular, we may have information on the statistical structure of the noise and of the solution.

Furthermore, the choice of the norms $||.||_s$, $s = \alpha, \beta, \gamma$, used in the previous chapter is somewhat arbitrary and not clearly linked to modelling assumptions [20].

In the first section of this chapter we give a brief presentation of the *Bayesian approach to inverse problems*, which addresses the issues mentioned above in a promising way. A more extensive introduction to the area is available in [20] and [15]. In Section 4.2 we prove posterior consistency results. As in the previous chapters, we adopt a Hilbert space setting. A short presentation of the tools from Probability and Measure Theory used in this chapter, is provided in the Appendix in Section 4.3.

4.1 A Change of Perspective

The main innovation in the Bayesian approach to inverse problems is that we express all the quantities in the model as random variables. We express our prior beliefs about the solution of the problem in the form of the *prior distribution*, μ_0, which is the distribution of u. Our knowledge of the statistical structure of the noise is expressed in terms of the noise distribution, P, which by equation (4.2) provides the distribution for the observations y given the solution u, P^u, called *data likelihood*. Note here that we will always assume that η and u are statistically mutually independent.

Together, the prior distribution on u and the distribution of $y|u$ (equivalently the distribution on η) specify a joint distribution for the pair (u, y). In the Bayesian approach the notion of solution is not a single approximation as it is in the deterministic theory. The solution is now a probability distribution on X, the *posterior distribution* μ^y, which is the distribution of $u|y$ containing information about the relative probability of the possible states of the solution u, given the data y.

To give some intuition lets consider the finite-dimensional case, [20]. Suppose we have the inverse problem

$$y = Ku \tag{4.3}$$

and consider the additive noise model

$$y = Ku + \eta, \tag{4.4}$$

where $u \in \mathbb{R}^n$, $y, \eta \in \mathbb{R}^q$ and K is a $q \times n$ matrix.

Assume π_0 is the probability distribution function (p.d.f.) of the prior distribution μ_0 on u and that π^y is the p.d.f. of the posterior distribution μ^y. Let the noise η be a random variable with density ρ. Then the data likelihood has density

$$\rho(y|u) = \rho(y - Ku). \tag{4.5}$$

By *Bayes* formula (Theorem 4.3.11, Appendix), we have that

$$\pi^y(u) \propto \rho(y - Ku)\pi_0(u), \tag{4.6}$$

which means that the posterior distribution, μ^y, has a *Radon-Nikodym* derivative with respect to the prior distribution which is proportional to the data likelihood's density

$$\frac{d\mu^y}{d\mu_0}(u) \propto \rho(y - Ku). \tag{4.7}$$

The right hand side is non-negative, since it is a probability density, thus we may express it as the exponential of the negative of a function $\Phi(u; y)$, to get

$$\frac{d\mu^y}{d\mu_0}(u) \propto \exp\left(-\Phi(u; y)\right). \tag{4.8}$$

In general the calculation of the posterior distribution is complicated. We can simplify things by the use of a class of *conjugate priors* with respect to the data likelihood, [1]:

Definition 4.1.1. *Let $\mathcal{F}$ denote a family of data likelihood distributions indexed by u. A class $\mathcal{P}$ of prior distributions is said to be a conjugate family for $\mathcal{F}$, if the posterior distribution μ^y is in the class $\mathcal{P}$ for all $P^u \in \mathcal{F}$ and $\mu_0 \in \mathcal{P}$.*

For instance, since K is linear, the *Gaussian* family is a class of conjugate priors with respect to a *Gaussian* observational noise. Indeed, in this thesis we study only the case where both the prior and the observational noise are *Gaussian*. Since we are considering linear inverse problems in Hilbert spaces, this implies that the posterior distribution is also Gaussian and we can easily calculate its mean and covariance operator by Theorem 4.3.9 (Appendix).

In finite dimensions things are even simpler: suppose that $\eta \sim \mathcal{N}(0, B)$ and $\mu_0 = \mathcal{N}(m_0, \Sigma_0)$, where B and Σ_0 are $q \times q$ and $n \times n$ symmetric positive definite matrices, respectively. Then (4.6) becomes

$$\pi^y(u) \propto \exp\left(-\frac{1}{2}\left|B^{-\frac{1}{2}}(y - Ku)\right|^2 - \frac{1}{2}\left|\Sigma_0^{-\frac{1}{2}}(u - m_0)\right|^2\right) \tag{4.9}$$

and likewise (4.7) becomes

$$\frac{d\mu^y}{d\mu_0}(u) \propto \exp\left(-\frac{1}{2}\left|B^{-\frac{1}{2}}(y - Ku)\right|^2\right). \tag{4.10}$$

Note that since the logarithm of π^y is quadratic in u, the posterior distribution μ^y is also Gaussian $\mathcal{N}(m, \Sigma)$, where m and Σ can be calculated by completing the square:

$$m = m_0 + \Sigma_0 K^*(B + K\Sigma_0 K^*)^{-1}(y - Km_0), \tag{4.11}$$

$$\Sigma = \Sigma_0 - \Sigma_0 K^*(B + K\Sigma_0 K^*)^{-1} K\Sigma_0. \tag{4.12}$$

Observe that the mean of the posterior, m, is a random variable, since it depends on y which is a random variable. Hence the posterior measure, μ^y, is itself a random variable.

As mentioned earlier, in the Bayesian approach we no longer have a single solution u of (4.3), but instead we have a probability measure, the posterior measure, thus we need a way of extracting information from it. One way of obtaining information from the posterior distribution is to find the *Maximum a-posteriori estimator* (MAP estimator), which is a point u that maximizes the posterior p.d.f. π^y, [20], [15]. In the above case, the

MAP estimator is

$$\underset{u\in\mathbb{R}^n}{\operatorname{argmin}}\left(\frac{1}{2}\left|B^{-\frac{1}{2}}(y-Ku)\right|^2+\frac{1}{2}\left|\Sigma_0^{-\frac{1}{2}}(u-m_0)\right|^2\right) \tag{4.13}$$

which is the solution of a regularized minimization problem, similar to the ones we have examined in the previous chapters and which is recognized as the posterior mean given by (4.11).

From (4.9) and (4.13), it is already apparent that there is a close relation of the Bayesian approach in the Gaussian case, with Tikhonov Regularization in Hilbert Scales. The posterior mean, is the minimizer of a Tikhonov Functional with weighted norms and the p.d.f. of the posterior measure, π^y is the exponential of minus a Tikhonov Functional. In [6], this relation of Tikhonov Regularization with Bayesian analysis is explored in infinite dimensions.

In [20], it is shown that (4.8) generalizes naturally to infinite-dimensional cases. It is demonstrated, [20, Chapter 3], that many inverse problems can be formulated in the Bayesian approach with the posterior distribution taking this form.

Furthermore, it is proved that in many problems $\Phi(u; y)$ satisfies certain continuity and bound conditions [20, Assumption 2.7]. These conditions, as it is shown in [20, Chapter 4], secure the existence of the posterior distribution given by (4.8) together with its stability in the *Hellinger distance* with respect to small changes in the data. Similarly it is shown that the posterior distribution can be approximated by finite-dimensional approximations of Φ or K, again in the Hellinger distance.

Note that the conditions that we require $\Phi(u; y)$ to satisfy, are properties of the forward problem and they have nothing to do with inverse problems or probability. As it is demonstrated in [20, Chapter 3], probability comes into the picture through the choice of prior measure μ_0: the conditions on $\Phi(u; y)$ are satisfied on a Hilbert space X. For formula (4.8) or its generalization to hold, we need the prior measure μ_0 to charge the space X with full measure, $\mu_0(X)=1$.

This choice of the prior, which is not arbitrary since it depends on the forward problem, along with the statistical properties of the noise, determine the norms that appear in the relations (4.9), (4.10) and the MAP estimator (4.13) and also their generalization in infinite dimensions. Thus, the Bayesian approach provides a logical way of choosing the norms in the Tikhonov regularized approximation of the solution of the inverse problem (4.2), which can replace the somewhat arbitrary choice that we have in the deterministic setup.

4.2 Posterior Consistency - Small Noise Limit

Suppose we have the inverse problem

$$y = Ku, \ \ u \in X, \ y \in Y, \tag{4.14}$$

which is polluted by noise and consider the *data likelihood model*

$$y = Ku + \eta, \tag{4.15}$$

where $\eta \sim \mathcal{N}(0, C_1)$, $C_1 = \delta^2 C_1'$ and C_1' is a self-adjoint, positive definite, trace-class operator. In addition, assume that $u \sim \mathcal{N}(m_0, C_0)$ where $C_0 = \tau^2 C_0'$ is a self-adjoint, positive definite and trace-class operator. As we have already mentioned, using Theorem 4.3.9(Appendix), we can show that the posterior distribution is also Gaussian and we can calculate its mean and covariance operator m and C respectively, which depend on m_0, C_0, C_1, K and y.

In this section we examine the behaviour of the posterior measure as the size of the noise, which is modelled by its covariance operator, tends to 0. This small noise limit provides another link between the Bayesian and classical approaches and is a test of the consistency of the posterior distribution.

According to Diaconis and Freedman [4], Bayesian statisticians can be divided in two categories: the "classical Bayesians" like Laplace and Bayes and the "subjectivists" like de Finetti and Savage. Classical Bayesians' point of view is that there exists a true solution which we want to estimate from the noisy data and prior beliefs about the solution are modelled by the prior distribution. On the other hand, subjectivists do not accept the existence of a true solution and for them probabilities represent only degrees of belief.

Posterior consistency is clearly of interest to the classical Bayesians: as the observations get more and more accurate the posterior should converge to a Dirac on the true solution. We now provide a test which formalizes this notion of posterior consistency for the Gaussian case considered in this thesis.

Assume that we have measurements of problem (4.14) of the form

$$y^\delta = Ku^\dagger + \delta\xi, \tag{4.16}$$

where $\xi \sim \mathcal{N}(0, C_1')$; that is, suppose our measurements come from the particular data likelihood model (4.15) for $\eta = \delta\xi$. Note that the posterior distribution depends on δ in two different ways: through its dependence on y^δ, since it is the conditional distribution of u given y^δ and through the appearance of δ in the covariance operator of the data likelihood, C_1. Classical posterior consistency tests examine the behaviour of the posterior distribution $\mu^{y^\delta,\delta}$ as both the observations and the data likelihood model become more and more accurate:

Definition 4.2.1. *For a given $u^\dagger \in X$, the pair $(u^\dagger, \mu^{y^\delta,\delta})$ is consistent in the frequentist bayesian sense, if as $\delta \to 0$, $\mu^{y^\delta,\delta}$ converges weakly to a Dirac measure on $u^\dagger$, ξ-almost surely.*

We define another classical posterior consistency test, which as the observations and the data likelihood model become more and more accurate, allows the prior to concentrate on the prior mean. This test originates from (4.9) and the observation that in finite dimensions, for the *Gaussian* case, the posterior distribution is proportional to the exponential of a functional which resembles the Tikhonov Regularization functional and where the penalty term of the Tikhonov functional only relates to the prior distribution. Thus, in accordance with the classical regularization theory, where in order to get convergence as the noise fades away we allow the regularization to disappear in a carefully chosen way (cf. Theorem 2.2.15, Theorem 3.1.5), here too we allow the covariance operator of the prior to go to 0. We implement this by allowing the parameter τ to be δ-dependent, thus the posterior distribution now depends on δ in an additional third way: through its dependence on the covariance operator of the prior $C_0 = \tau(\delta)^2 C_0'$.

Definition 4.2.2. *For a given $u^\dagger \in X$, the pair $(u^\dagger, \mu^{y^\delta,\delta,\tau(\delta)})$ is consistent in the regularized frequentist bayesian sense, if as $\delta \to 0$, $\mu^{y^\delta,\delta,\tau(\delta)}$ converges weakly to a Dirac measure on $u^\dagger$, ξ-almost surely.*

Posterior consistency is also of interest for the subjectivists, but in a different sense: after specifying a prior distribution, generate imaginary data, compute the posterior and consider whether the posterior distribution is an adequate representation of the updated prior [4]. In [4], Diaconis and Friedman call this the "what if" method: what if the data came out that way? We express this notion of posterior consistency through the following test, [20]:

Fix $y \in Y$ and consider it as an observation of the inverse problem (4.14). This y does not depend on δ, it is just an element of Y. We do not assume that the observations come from our data likelihood model. Note that in this case, the posterior distribution depends on δ only through its appearance in the covariance operator of the data likelihood. Subjectivistic posterior consistency tests examine the behaviour of the posterior distribution, $\mu^{y,\delta}$, as the noise in the data likelihood model disappears, that is as the model becomes more and more accurate.

Various possibilities can be examined. Does the posterior distribution converge weakly anywhere? If it does converge, does it converge to a Dirac so that uncertainty disappears? If it does converge to a Dirac, is the Dirac centered on a generalized inverse of y? Does the prior play any role in the limit, or does the information from the data swamp the prior and make it irrelevant in the limit?

In the particular problem considered in this thesis, since K is invertible for the suitable choice of space Y, we would like to have that as $\delta \to 0$, the posterior distribution converges to a Dirac distribution centered on $K^{-1}y$.

In [20, Chapter 2], posterior consistency for the finite dimensional case is examined from a subjectivist's point of view. As in Section 4.1, assume that we have the finite-dimensional inverse problem

$$y = Ku$$

and consider the data likelihood model

$$y = Ku + \eta,$$

where $u \in \mathbb{R}^n$, $y, \eta \in \mathbb{R}^q$, K is an $q \times n$ matrix and $\eta \sim \mathcal{N}(0, B)$ where $B = \delta^2 B_0$ is a symmetric, positive definite matrix . Furthermore, suppose we have the prior $\mu_0 = \mathcal{N}(m_0, \Sigma_0)$, where Σ_0 is symmetric and positive definite. Then the posterior distribution is also Gaussian, $\mu^y = \mathcal{N}(m, \Sigma)$ where m and Σ are given by formulae (4.11) and (4.12) respectively.

Suppose $y \in \mathbb{R}^q$ is a fixed observation. We emphasize here that y is just an element of $\mathbb{R}^q$, we are not assuming that it comes from our data likelihood model. The following two theorems are proved:

Theorem 4.2.3. *[20, Theorem 2.4] Assume that $q \geq n$ and that $\mathcal{N}(K) = \{0\}$. Then in the limit $\delta^2 \to 0$*

$$\mu^y \Rightarrow \delta_{u^\dagger}$$

where $u^\dagger$ is the solution of the least squares problem

$$u^\dagger = \operatorname*{argmin}_{u \in \mathbb{R}^n} \left| B_0^{-\frac{1}{2}}(y - Ku) \right|^2$$

Note that since K is injective, $u^\dagger$ is unique and is the best-approximate solution as defined in Section 2.1, the only difference being the fact that a weighted norm is used for the least squares approximation. For more details on weighted best-approximate solutions refer to [5, Chapter 8, Section 1].

As the above theorem states, in the overdetermined case, i.e. in the case where the observations are of higher dimension than the unknown parameter and there is no loss of information because of K, since K is injective, as the noise disappears the posterior converges weakly to a Dirac measure determined by K and the relative weights of the observational noise. As it is highlighted in [20], "uncertainty disappears and the prior plays no role in this limit".

Assume now that $q < n$ and that $rank(K) = q$ so that we may write

$$K = (K_0 \ \ 0)Q^*$$

where $Q \in \mathbb{R}^{n\times n}$ is orthogonal, $Q^*Q = I$, $K_0 \in \mathbb{R}^{q\times q}$ is invertible and $0 \in \mathbb{R}^{q\times(n-q)}$ is the zero matrix. Let $L_0 = \Sigma_0^{-1}$ and write

$$Q^*L_0Q = \begin{bmatrix} L_{11} & L_{12} \\ L_{12}^* & L_{22} \end{bmatrix}$$

where $L_{11} \in \mathbb{R}^{q\times q}, L_{12} \in \mathbb{R}^{q\times(n-q)}$ and $L_{22} \in \mathbb{R}^{(n-q)\times(n-q)}$. Both L_{11} and L_{22} are positive definite symmetric, because Σ_0 is.
If we write

$$Q = (Q_1 \ \ Q_2)$$

with $Q_1 \in \mathbb{R}^{q\times q}$ and $Q_2 \in \mathbb{R}^{n\times(n-q)}$, then Q_1^* projects onto a q-dimensional subspace $\mathcal{O}$ and Q_2^* projects onto an $(n-q)$-dimensional subspace $\mathcal{O}^\perp$, the orthogonal complement of $\mathcal{O}$.

Define $z \in \mathbb{R}^q$ to be the unique solution of $K_0 z = y$. Note that z is not some kind of true solution. We are just inverting y in the directions that we have the information to do so.

Define $w \in \mathbb{R}^q$ and $w' \in \mathbb{R}^{n-q}$ via the equation

$$\Sigma_0^{-1}m_0 = Q \begin{bmatrix} w \\ w' \end{bmatrix}$$

and set

$$z' = -L_{22}^{-1}L_{12}^*z + L_{22}^{-1}w' \in \mathbb{R}^{n-q}.$$

Theorem 4.2.4. *[20, Theorem 2.5] In the small noise limit,* $\delta^2 \to 0$,

$$\mu^y \Rightarrow \delta_z \otimes \mathcal{N}(z', L_{22}^{-1}),$$

where $\delta_z \otimes \mathcal{N}(z', L_{22}^{-1})$ *is viewed as a measure on* $\mathcal{O} \oplus \mathcal{O}^{\perp}$.

Thus, in the underdetermined case where the observations are of lower dimension than the unknown parameter but K has full rank, as the noise disappears, the posterior converges weakly to a Dirac measure in the space $\mathcal{O}$ where we have information and to a proper Gaussian measure in the space $\mathcal{O}^{\perp}$ where we have no information. The Dirac is centered on the inverse of y in the directions where we have sufficient information to invert. As it is stressed in [20], "the prior plays a role in the posterior measure in the limit of zero observational noise" since the formulae for L_{22}^{-1} and z' contain Σ_0 and m_0.

We are now going to examine posterior consistency for the Bayesian version of the infinite dimensional problem that we have considered in the previous chapters.

Suppose we have the inverse problem

$$y = \mathcal{A}^{-\ell} u, \tag{4.17}$$

where $\mathcal{A}$ is "Laplacian-like" as defined in Section 1.2, $\ell > 0$. Consider the data likelihood model

$$y = \mathcal{A}^{-\ell} u + \eta, \tag{4.18a}$$

where $\eta \sim \mathcal{N}(0, C_1)$ and $C_1 = \delta^2 \mathcal{A}^{-\beta}$, $\beta > \frac{1}{2}$. This implies that $y|u \sim \mathcal{N}(\mathcal{A}^{-\ell} u, C_1)$.

In addition, assume we have the prior

$$\mu_0 = \mathcal{N}(0, C_0), \quad \text{where} \quad C_0 = \tau^2 \mathcal{A}^{-\alpha},\ \alpha > \frac{1}{2}. \tag{4.18b}$$

By Lemma 4.3.10(Appendix), we have that $\eta \in \mathcal{H}^s$, $s < \beta - \frac{1}{2}$ P-

almost surely and that $u \in \mathcal{H}^s,\ s < \alpha - \frac{1}{2}$ μ_0-almost surely, thus by Lemma 1.2.2 $\mathcal{A}^{-\ell}u \in \mathcal{H}^s,\ s < \alpha + 2\ell - \frac{1}{2}$, μ_0-almost surely.

These imply that $y \in \mathcal{H}^s,\ s < \iota$ for μ_0-almost all u and P^u-almost all η, where $\iota := \min\left\{\alpha + 2\ell - \frac{1}{2}, \beta - \frac{1}{2}\right\}$. By the assumptions on α, β and ℓ we have that $\iota > 0$, thus $y \in \mathcal{H}$ almost surely. Therefore, the pair (u, y) is jointly Gaussian in $\mathcal{H} \times \mathcal{H}$, with $\mathbb{E}u = 0$ and $\mathbb{E}y = 0$ and has covariance operator with components

$$C_{11} := \mathbb{E}uu^* = C_0,$$

$$C_{22} := \mathbb{E}yy^* = \tau^2\mathcal{A}^{-2\ell-\alpha} + \delta^2\mathcal{A}^{-\beta},$$

$$C_{12} := \mathbb{E}uy^* = \tau^2\mathcal{A}^{-\ell-\alpha} = \mathbb{E}yu^* =: C_{21},$$

where the notation used is defined in Definition 4.3.1 in the Appendix and the calculation is a direct application of the definition of the covariance operator (Definition 4.3.2, Appendix) since u and η are mutually independent.

Using Theorem 4.3.9(Appendix), we have that $u|y \sim \mu^y = \mathcal{N}(m, C)$ where:

$$\begin{aligned} m = C_{12}C_{22}^{-1}y &= \tau^2\mathcal{A}^{-\ell-\alpha}\left(\tau^2\mathcal{A}^{-2\ell-\alpha} + \delta^2\mathcal{A}^{-\beta}\right)^{-1} y \\ &= \mathcal{A}^{-\ell+\beta}\left(\mathcal{A}^{-2\ell+\beta} + \frac{\delta^2}{\tau^2}\mathcal{A}^{\alpha}\right)^{-1} y \end{aligned} \tag{4.19}$$

and

$$\begin{aligned} C &= C_{11} - C_{12}C_{22}^{-1}C_{21} \\ &= \tau^2\mathcal{A}^{-\alpha} - \tau^2\mathcal{A}^{-\ell-\alpha}\left(\tau^2\mathcal{A}^{-2\ell-\alpha} + \delta^2\mathcal{A}^{-\beta}\right)^{-1}\tau^2\mathcal{A}^{-\ell-\alpha} \\ &= \tau^2\mathcal{A}^{-\alpha} - \tau^2\mathcal{A}^{-\alpha}\left(I + \frac{\delta^2}{\tau^2}\mathcal{A}^{2\ell+\alpha-\beta}\right)^{-1}. \end{aligned} \tag{4.20}$$

Remark 4.2.5. Using the diagonalization of $\mathcal{A}$, we can rewrite the posterior mean as

$$m = \sum_{k=1}^{\infty} \frac{\mu_k^{-\ell+\beta}}{\mu_k^{-2\ell+\beta} + \frac{\delta^2}{\tau^2}\mu_k^{\alpha}} y_k\phi_k,$$

which looks identical to the Tikhonov regularized approximation in Section

3.1 (cf. Lemma 3.1.3)

$$u_\lambda^\delta = \arg\min_{u\in\mathcal{H}^\alpha} I_{\lambda,\alpha,\beta}(u).$$

However, since now we do not have that $y \in \mathcal{H}^\beta$, unlike the deterministic theory (cf. Remark 3.1.4), there is no guarantee that $m \in \mathcal{H}^\alpha, P$-almost surely without additional assumptions (cf. Remark 4.2.7(ii)). Nevertheless, we do have that $m \in \mathcal{H}, P$-almost surely:

$$||m||^2 = \sum_{k=1}^{\infty} \frac{\mu_k^{-2\ell+2\beta}}{\mu_k^{-4\ell+2\beta} + 2\frac{\delta^2}{\tau^2}\mu_k^{-2\ell+\beta+\alpha} + \frac{\delta^4}{\tau^4}\mu_k^{2\alpha}} y_k^2$$

i) If $\alpha - 2\ell \geq \beta$, then $y \in \mathcal{H}^s,\ s < \beta - \frac{1}{2}, P$-almost surely and

$$||m||^2 \leq c\sum_{k=1}^{\infty} \mu_k^{\beta-\alpha} y_k^2 < \infty,$$

since $\beta - \alpha < \beta - \frac{1}{2}$.

ii) If $\alpha - 2\ell < \beta$, then $y \in \mathcal{H}^s,\ s < \alpha + 2\ell - \frac{1}{2}, P$-almost surely and

$$||m||^2 \leq c\sum_{k=1}^{\infty} \mu_k^{2\ell} y_k^2 < \infty,$$

since $\alpha > \frac{1}{2}$, hence $2\ell < \alpha + 2\ell - \frac{1}{2}$.

We first give a necessary and sufficient condition on α and β which, using Feldman-Hayek Theorem (Theorem 4.3.7, Appendix), secures that the posterior is a well defined Gaussian measure on $\mathcal{H}$ which is equivalent to the prior.

Theorem 4.2.6. *Suppose $y \in \mathcal{H}^{2\beta-\alpha-2\ell}$. Then the posterior and the prior distributions, $\mathcal{N}(m, C)$ and $\mathcal{N}(0, C_0)$ respectively, are equivalent Gaussian measures on $\mathcal{H}$ if and only if $\alpha > \beta - 2\ell + \frac{1}{4}$.*

Proof. Suppose $\alpha > \beta - 2\ell + \frac{1}{4}$, or equivalently $\beta - \alpha - 2\ell < -\frac{1}{4}$. We verify that the three conditions of Feldman-Hayek hold:

i) We first need to show that the Cameron-Martin spaces $\mathcal{R}(C_0^{\frac{1}{2}})$ and $\mathcal{R}(C^{\frac{1}{2}})$ of the prior and the posterior respectively coincide. By Lemma

4.3.8(Appendix), it suffices to show that there exist $K_1, K_2 > 0$ such that

$$K_1 \langle x, C_0 x\rangle \leq \langle x, Cx\rangle \leq K_2 \langle x, C_0 x\rangle\,,\ \forall x \in \mathcal{H}.$$

Fix $x \in \mathcal{H}$. Then, using the diagonalization of $\mathcal{A}$,

$$\langle x, Cx\rangle = \left\langle \sum_{j=1}^{\infty} x_j\phi_j, \tau^2 \sum_{k=1}^{\infty} \mu_k^{-\alpha}\left(1 - \frac{1}{1+\frac{\delta^2}{\tau^2}\mu_k^{2\ell+\alpha-\beta}}\right) x_k\phi_k \right\rangle$$

$$= \tau^2 \sum_{k=1}^{\infty} \frac{\frac{\delta^2}{\tau^2}\mu_k^{2\ell+\alpha-\beta}}{1+\frac{\delta^2}{\tau^2}\mu_k^{2\ell+\alpha-\beta}} \mu_k^{-\alpha} x_k^2 = \tau^2 \sum_{k=1}^{\infty} \frac{\frac{\delta^2}{\tau^2}}{\mu_k^{\beta-\alpha-2\ell}+\frac{\delta^2}{\tau^2}} \mu_k^{-\alpha} x_k^2$$

and

$$\langle x, C_0 x\rangle = \left\langle \sum_{j=1}^{\infty} x_j\phi_j, \tau^2 \sum_{k=1}^{\infty} \mu_k^{-\alpha} x_k\phi_k \right\rangle = \tau^2 \sum_{k=1}^{\infty} \mu_k^{-\alpha} x_k^2.$$

Since $\beta - \alpha - 2\ell < -\frac{1}{4} < 0$, by Assumption 1.2.1(iii), we have that for some $K_2 > 0$

$$\frac{\frac{\delta^2}{\tau^2}}{\mu_k^{\beta-\alpha-2\ell}+\frac{\delta^2}{\tau^2}} \geq \frac{\frac{\delta^2}{\tau^2}}{(c_1^+)^{\beta-\alpha-2\ell}+\frac{\delta^2}{\tau^2}} \geq K_2, \forall k \in \mathbb{N},$$

thus $\langle x, Cx\rangle \geq K_2 \langle x, C_0 x\rangle$. Furthermore, since $\mu_k \geq 0$ we have that

$$\frac{\frac{\delta^2}{\tau^2}}{\mu_k^{\beta-\alpha-2\ell}+\frac{\delta^2}{\tau^2}} \leq 1,$$

thus for $K_1 = 1$, $\langle x, Cx\rangle \leq K_1 \langle x, C_0 x\rangle$.
Note that since $\mathcal{R}(C_0^{\frac{1}{2}}) = \mathcal{H}^\alpha$, we have that the Cameron-Martin space of both the prior and the posterior measure is $E = \mathcal{H}^\alpha$.

ii) We need to show that $m \in \mathcal{H}^\alpha$. Indeed,

$$||m||_\alpha^2 = \sum_{k=1}^{\infty} \mu_k^\alpha \frac{\mu_k^{-2\ell+2\beta}}{\left(\mu_k^{-2\ell+\beta}+\frac{\delta^2}{\tau^2}\mu_k^\alpha\right)^2} y_k^2 \leq \sum_{k=1}^{\infty} \frac{\mu_k^{\alpha-2\ell+2\beta}}{\frac{\delta^4}{\tau^4}\mu_k^{2\alpha}} y_k^2 = c\sum_{k=1}^{\infty} \mu_k^{2\beta-\alpha-2\ell} y_k^2$$

which is finite if and only if $y \in \mathcal{H}^{2\beta-\alpha-2\ell}$.

iii) Finally, we need to verify that the operator $T = (C_0^{-\frac{1}{2}}C^{\frac{1}{2}})(C_0^{-\frac{1}{2}}C^{\frac{1}{2}})^* -$

I is Hilbert-Schmidt. Since in our case C_0 and C commute and are both self-adjoint, we have that

$$T = C_0^{-1}C - I = \frac{1}{\tau^2}\mathcal{A}^{\alpha}\left(\tau^2\mathcal{A}^{-\alpha} - \tau^2\mathcal{A}^{-\alpha}\left(I + \frac{\delta^2}{\tau^2}\mathcal{A}^{2\ell+\alpha-\beta}\right)^{-1}\right) - I$$

$$= I - \left(I + \frac{\delta^2}{\tau^2}\mathcal{A}^{2\ell+\alpha-\beta}\right)^{-1} - I = -\left(I + \frac{\delta^2}{\tau^2}\mathcal{A}^{2\ell+\alpha-\beta}\right)^{-1}.$$

By the assumption we have $\beta - \alpha - 2\ell < -\frac{1}{4}$, therefore $4\beta - 4\alpha - 8\ell < -1$, hence T is Hilbert-Schmidt since, by Assumption 1.2.1(iii), its eigenvalues are square summable

$$\sum_{k=1}^{\infty}\frac{1}{(1+\frac{\delta^2}{\tau^2}\mu_k^{2\ell+\alpha-\beta})^2} \leq c\sum_{k=1}^{\infty}\mu_k^{2\beta-2\alpha-4\ell} \leq c\sum_{k=1}^{\infty}k^{4\beta-4\alpha-8\ell} < \infty.$$

Conversely, suppose $\beta - \alpha - 2\ell \geq -\frac{1}{4}$ or equivalently $2\ell + \alpha - \beta \leq \frac{1}{4}$. It suffices to show that one of the three conditions of the Feldman-Hayek Theorem fails. We show that the third condition fails, that is we show that T is not Hilbert-Schmidt. Indeed, the sum of the squares of its eigenvalues is

$$\sum_{k=1}^{\infty}\frac{1}{(1+\frac{\delta^2}{\tau^2}\mu_k^{2\ell+\alpha-\beta})^2} := S.$$

i) If $2\ell + \alpha - \beta \leq 0$ then by Assumption 1.2.1(iii), there exists $c > 0$ such that $\mu_k^{2\ell+\alpha-\beta} \leq c$, $\forall k \in \mathbb{N}$, thus

$$S \geq \sum_{k=1}^{\infty}\frac{1}{(1+\frac{\delta^2}{\tau^2}c)^2} = \infty.$$

ii) If $0 \leq 2\ell+\alpha-\beta \leq \frac{1}{4}$ then by Assumption 1.2.1(iii), there exists $c > 0$ such that $1 + \frac{\delta^2}{\tau^2}\mu_k^{2\ell+\alpha-\beta} \leq c\mu_k^{2\ell+\alpha-\beta}$, $\forall k \in \mathbb{N}$, thus

$$S \geq c\sum_{k=1}^{\infty}\mu_k^{2\beta-2\alpha-4\ell} \geq c\sum_{k=1}^{\infty}k^{4\beta-4\alpha-8\ell} = \infty$$

since by the assumption $4\beta - 4\alpha - 8\ell \geq -1$.

□

Remark 4.2.7. In the last theorem we prove that if $2\ell + \alpha - \beta < -\frac{1}{4}$, then the posterior and the prior are not equivalent. We now try to interpret this.

In the following, we say that the posterior is smoother than the prior (or vice-versa), if $s_1 > s_2$ where $\mathcal{H}^{s_1}$ is the best space in which we can secure that draws from the posterior live in with probability 1 and $\mathcal{H}^{s_2}$ is the best space in which we can secure that draws from the prior live in with probability 1.

i) Suppose $\lambda_k^{C_0}$ are the eigenvalues of C_0 and λ_k^C the eigenvalues of C. Then the third condition of the Feldman-Hayek Theorem requires that

$$\sum_{k=1}^{\infty} \left(\frac{\lambda_k^C}{\lambda_k^{C_0}} - 1 \right)^2 < \infty. \tag{4.21}$$

In our case

$$\lambda_k^C = \tau^2 \mu_k^{-\alpha} \left(1 - \frac{1}{1 + \frac{\delta^2}{\tau^2} \mu_k^{2\ell+\alpha-\beta}} \right)$$

and

$$\lambda_k^{C_0} = \tau^2 \mu_k^{-\alpha}.$$

Note that

$$0 < 1 - \frac{1}{1 + \frac{\delta^2}{\tau^2} \mu_k^{2\ell+\alpha-\beta}} \leq 1,$$

therefore

$$0 < \frac{\lambda_k^C}{\lambda_k^{C_0}} \leq 1.$$

This means that the prior is never smoother than the posterior and that condition (4.21) can only fail if one of the following is true:

a) $\frac{\lambda_k^C}{\lambda_k^{C_0}} \overset{k\to\infty}{\longrightarrow} 0$, in which case the posterior is smoother,

b) $O(\lambda^C) = O(\lambda^{C_0})$ but with different constants, so even though the posterior and the prior are equally smooth, they are not equivalent.

If $\alpha < \beta - 2\ell$, then

$$\frac{\lambda_k^C}{\lambda_k^{C_0}} = 1 - \frac{1}{1 + \frac{\delta^2}{\tau^2} \mu_k^{2\ell+\alpha-\beta}} = \frac{\frac{\delta^2}{\tau^2} \mu_k^{2\ell+\alpha-\beta}}{1 + \frac{\delta^2}{\tau^2} \mu_k^{2\ell+\alpha-\beta}} \leq \frac{\delta^2}{\tau^2} \mu_k^{2\ell+\alpha-\beta} \overset{k\to\infty}{\longrightarrow} 0,$$

thus the posterior is smoother than the prior.

If $\alpha = \beta - 2\ell$, then

$$\frac{\lambda_k^C}{\lambda_k^{C_0}} = \frac{\frac{\delta^2}{\tau^2}}{1 + \frac{\delta^2}{\tau^2}},$$

thus the posterior and the prior are equally smooth, but not equivalent.

If $\beta - 2\ell < \alpha \leq \beta - 2\ell + \frac{1}{4}$, then

$$\frac{\lambda_k^C}{\lambda_k^{C_0}} = 1 - \frac{1}{1 + \frac{\delta^2}{\tau^2}\mu_k^{2\ell+\alpha-\beta}} \overset{k\to\infty}{\longrightarrow} 1.$$

In this case, even though the terms of the sum in (4.21) do converge to 0, they do not converge fast enough to secure that the sum is finite, thus the posterior and the prior are again equally smooth, but not equivalent.

ii) In Theorem 4.2.6 we assume that $y \in \mathcal{H}^{2\beta-\alpha-2\ell}$ which secures that the posterior mean lives in the Cameron-Martin space $E = \mathcal{R}(C_0^{\frac{1}{2}})$, i.e. $m \in \mathcal{H}^\alpha$. It is interesting to check what happens to the posterior mean, m, if we allow random y from the model (4.18a).

By Remark 4.2.5, we already now that $m \in \mathcal{H}$, always. If $\beta - \alpha - 2\ell < -\frac{1}{2}$, i.e. if $\alpha > \beta - 2\ell + \frac{1}{2}$, then $m \in \mathcal{H}^\alpha$ almost surely. Indeed,

$$||m||_\alpha^2 = \sum_{k=1}^{\infty} \frac{\mu_k^{-2\ell+2\beta+\alpha}}{(\mu_k^{-2\ell+\beta} + \frac{\delta^2}{\tau^2}\mu_k^\alpha)^2} y_k^2 \leq c \sum_{k=1}^{\infty} \mu_k^{-2\ell+2\beta-\alpha} y_k^2.$$

Since $y \in \mathcal{H}^s$, almost surely for $s < \iota = \min\left\{\alpha + 2\ell - \frac{1}{2}, \beta - \frac{1}{2}\right\}$, in order to show that $m \in \mathcal{H}^\alpha$ we need $\iota > -2\ell + 2\beta - \alpha$, i.e.

$$\alpha + 2\ell - \frac{1}{2} > -2\ell + 2\beta - \alpha$$

and

$$\beta - \frac{1}{2} > -2\ell + 2\beta - \alpha$$

which are equivalent to

$$\beta - \alpha - 2\ell < -\frac{1}{4}$$

and

$$\beta - \alpha - 2\ell < -\frac{1}{2}$$

respectively. If $\alpha > \beta - 2\ell + \frac{1}{2}$ then both of the requirements are met and we have that $m \in \mathcal{H}^\alpha$ almost surely. So, in order to secure for random data from the model (4.18a), that the prior and the posterior are equivalent almost surely, we in fact need a stronger condition on α than the one provided by the last theorem, i.e. we need

$$\alpha > \beta - 2\ell + \frac{1}{2},$$

instead of $\alpha > \beta - 2\ell + \frac{1}{4}$.

It is also interesting, that we can only secure that $m \notin \mathcal{H}^\alpha$ almost surely, in only one special case, i.e. when $\alpha = \beta - 2\ell$. Indeed, since $\eta \sim \mathcal{N}(0, \delta^2 \mathcal{A}^{-\beta})$ we have that almost surely $\eta \notin \mathcal{H}^\beta$, since draws from a Gaussian measure in infinite dimensions do not lie in the corresponding Cameron-Marin space. Therefore, almost surely $y \notin \mathcal{H}^\beta$ and we have that:

a) if $\alpha \leq \beta - 2\ell$, then

$$||m||_\alpha^2 \geq \sum_{k=1}^{\infty} \frac{\mu_k^{-2\ell+2\beta+\alpha}}{c\mu_k^{-4\ell+2\beta}} y_k^2 = c \sum_{k=1}^{\infty} \mu_k^{2\ell+\alpha} y_k^2$$

which we can secure that is infinite for $2\ell + \alpha \geq \beta$.

b) if $\alpha \geq \beta - 2\ell$, then

$$||m||_\alpha^2 \geq \sum_{k=1}^{\infty} \frac{\mu_k^{-2\ell+2\beta+\alpha}}{c\mu_k^{2\alpha}} y_k^2 = c \sum_{k=1}^{\infty} \mu_k^{-2\ell+2\beta-\alpha} y_k^2$$

which we can secure that is infinite for $-2\ell + 2\beta - \alpha \geq \beta$ since almost surely $y \notin \mathcal{H}^\beta$.

Thus we can only show that the posterior mean almost surely does not lie in the Cameron-Martin space for $\alpha = \beta - 2\ell$. One might have expected that for small values of α the posterior mean explodes, since for small values of α, as we have seen in (i) of this remark, the prior covariance does not dominate in the expression for the posterior covariance, which means that the regularization is not helping. However, since by allowing α to get smaller, we are also weakening the requirement that $m \in \mathcal{H}^\alpha$, this does not happen except for one critical value $\alpha = \beta - 2\ell$.

The fact that for large values of α we have that the posterior and the prior are equivalent, even for random data from the model (4.18a), can be explained by the fact that, as we have seen in (i) of this remark, for large values of α the prior covariance dominates in the expression for the posterior covariance.

Remark 4.2.8. When studying classical posterior consistency it is of interest to know when the condition on y required for Theorem 4.2.6 holds. A simple calculation shows that if $\beta - \alpha - 2\ell < -\frac{1}{2}$ and $2\beta - \alpha - 4\ell < 0$, then the assumed measurements of the form

$$y^\delta = \mathcal{A}^{-\ell} u^\dagger + \delta\xi,$$

where $u^\dagger \in \mathcal{H}$ is fixed and $\xi \sim \mathcal{N}(0, \mathcal{A}^{-\beta})$, belong to the space $\mathcal{H}^{2\beta-\alpha-2\ell}$ ξ-almost surely.

Indeed, by Lemma 4.3.10(Appendix), since $\beta - \alpha - 2\ell < -\frac{1}{2}$ we have that $2\beta - \alpha - 2\ell < \beta - \frac{1}{2}$, thus $\xi \in \mathcal{H}^{2\beta-\alpha-2\ell}$ ξ-almost surely.

Furthermore, $\mathcal{A}^{-\ell} u^\dagger \in \mathcal{H}^{2\ell}$ by Lemma 1.2.2 and since $2\beta - \alpha - 4\ell < 0$ we have that $2\ell > 2\beta - \alpha - 2\ell$, thus $\mathcal{A}^{-\ell} u^\dagger \in \mathcal{H}^{2\beta-\alpha-2\ell}$. Hence, $y \in \mathcal{H}^{2\beta-\alpha-2\ell}$ ξ-almost surely.

Note that the above two conditions are not mutually exclusive.

We now examine classical posterior consistency for the inverse problem (4.17) and the data likelihood model (4.18).

Theorem 4.2.9. *Suppose we have the inverse problem* (4.17) *with the data likelihood model* (4.18a) *and prior* (4.18b) *and assume* $\beta - 2\ell > \frac{1}{2}$. *Then, the posterior measure* $\mu^y = \mathcal{N}(m, C)$, *where* m *and* C *are given by* (4.19) *and* (4.20) *respectively, is consistent in the classical sense.*

Proof. Assume we have measurements of the problem (4.17) of the form

$$y^\delta = \mathcal{A}^{-\ell}u^\dagger + \delta\xi,$$

where $\xi \sim \mathcal{N}(0, \mathcal{A}^{-\beta})$ and $u^\dagger \in \mathcal{H}$ is fixed. By Definition 4.2.1, we need to show that ξ-almost surely $\mu^{y^\delta,\delta} \Rightarrow \delta_{u^\dagger}$, as $\delta \to 0$. By Lemma 4.3.4(Appendix), it suffices to show that ξ-almost surely $\langle m, x\rangle \to \langle u^\dagger, x\rangle$ and $\langle Cx, x\rangle \to 0$, $\forall x \in \mathcal{H}$ as $\delta \to 0$.

Indeed, fix $x \in \mathcal{H}$.

Using the diagonalization of $\mathcal{A}$, by (4.19) we have that

$$\langle m, x\rangle = \left\langle \mathcal{A}^{-\ell+\beta}\left(\mathcal{A}^{-2\ell+\beta} + \frac{\delta^2}{\tau^2}\mathcal{A}^{\alpha}\right)^{-1} y^\delta, x\right\rangle$$

$$= \sum_{k=1}^{\infty} \frac{\mu_k^{-\ell+\beta}}{\mu_k^{-2\ell+\beta} + \frac{\delta^2}{\tau^2}\mu_k^{\alpha}} y_k^\delta x_k.$$

Define $q_k = \langle u^\dagger, \phi_k\rangle$. Then $y_k^\delta = \mu_k^{-\ell} q_k + \delta\xi_k$ and

$$\langle m, x\rangle = \sum_{k=1}^{\infty} \frac{\mu_k^{-2\ell+\beta} q_k x_k}{\mu_k^{-2\ell+\beta} + \frac{\delta^2}{\tau^2}\mu_k^{\alpha}} + \sum_{k=1}^{\infty} \frac{\delta\mu_k^{-\ell+\beta}\xi_k x_k}{\mu_k^{-2\ell+\beta} + \frac{\delta^2}{\tau^2}\mu_k^{\alpha}}. \qquad (i)$$

Observe that $\forall k \in \mathbb{N}$

$$\frac{\mu_k^{-2\ell+\beta} q_k x_k}{\mu_k^{-2\ell+\beta} + \frac{\delta^2}{\tau^2}\mu_k^{\alpha}} \le q_k x_k \quad \text{and} \quad \lim_{\delta\to 0} \frac{\mu_k^{-2\ell+\beta} q_k x_k}{\mu_k^{-2\ell+\beta} + \frac{\delta^2}{\tau^2}\mu_k^{\alpha}} = q_k x_k,$$

since τ is fixed, thus $\frac{\delta^2}{\tau^2} \to 0$, as $\delta \to 0$, where

$$\sum_{k=1}^{\infty} q_k x_k = \langle u^\dagger, x\rangle < \infty$$

since $u^\dagger, x \in \mathcal{H}$. Hence, the first term in (i) converges to $\langle u^\dagger, x\rangle$, as $\delta \to 0$.

By Lemma 4.3.10(Appendix), we have that $\xi \in \mathcal{H}^s$ ξ-almost surely, $\forall s < \beta - \frac{1}{2}$, hence $\xi \in \mathcal{H}^{2\ell}$ ξ-almost surely, since $\beta - \frac{1}{2} > 2\ell$. The second term can be estimated as follows:

$$\sum_{k=1}^{\infty} \frac{\delta \mu_k^{-\ell+\beta} \xi_k x_k}{\mu_k^{-2\ell+\beta} + \frac{\delta^2}{\tau^2} \mu_k^{\alpha}} \leq \delta \sum_{k=1}^{\infty} \mu_k^{\ell} \xi_k x_k$$

where the sum is finite by Cauchy-Schwartz inequality, since $\xi \in \mathcal{H}^{2\ell}$ ξ-almost surely and $x \in \mathcal{H}$. Thus, as $\delta \to 0$ the second term vanishes ξ-almost surely.

For the convergence of the covariance operator, using (4.20) and the diagonalization of $\mathcal{A}$, we have

$$\begin{aligned}
\langle Cx, x \rangle &= \left\langle \left(\tau^2 \mathcal{A}^{-\alpha} - \tau^2 \mathcal{A}^{-\alpha} \left(I + \frac{\delta^2}{\tau^2} \mathcal{A}^{2\ell+\alpha-\beta} \right)^{-1} \right) x, x \right\rangle \\
&= \sum_{k=1}^{\infty} \left(\tau^2 \mu_k^{-\alpha} - \tau^2 \mu_k^{-\alpha} \left(1 + \frac{\delta^2}{\tau^2} \mu_k^{2\ell+\alpha-\beta} \right)^{-1} \right) x_k^2 \\
&= \tau^2 \sum_{k=1}^{\infty} \mu_k^{-\alpha} \left(1 - \frac{1}{1 + \frac{\delta^2}{\tau^2} \mu_k^{2\ell+\alpha-\beta}} \right) x_k^2.
\end{aligned}$$

By the Dominated Convergence Theorem the last sum vanishes as $\delta \to 0$, since the parenthesis on the one hand is uniformly bounded by 1 and on the other hand $\forall k \in \mathbb{N}$ tends to 0 and since $x \in \mathcal{H}$, where $\alpha > \frac{1}{2} > 0$, thus $x \in \mathcal{H}^{-\alpha}$. □

Remark 4.2.10. As it is mentioned in the proof, the condition $\beta - 2\ell > \frac{1}{2}$, implies that $\xi \in \mathcal{H}^{2\ell}, \xi-$almost surely. Since by the assumption $u^\dagger \in \mathcal{H}$ thus $\mathcal{A}^{-\ell} u^\dagger \in \mathcal{H}^{2\ell}$, we have that $y^\delta \in \mathcal{H}^{2\ell}$. By Lemma 1.2.3 we know that $K^{-1} = \mathcal{A}^{\ell}$ is well defined and bounded in $\mathcal{H}^{2\ell}$, which means that the problem is well posed and no regularization was necessary (cf. Remark 3.1.2).

Theorem 4.2.11. *Suppose we have the inverse problem* (4.17) *with the data likelihood model* (4.18*a*) *and the prior* (4.18*b*) *and assume that* $\beta - 2\ell \leq \frac{1}{2}$*, i.e. the inverse problem* (4.17) *is ill-posed. Then, the posterior measure* $\mu^y = \mathcal{N}(m, C)$*, where* m *and* C *are given by (4.19) and (4.20) respectively,* *is consistent in the regularized classical sense, provided*

$$\frac{\delta}{\tau(\delta)} \to 0 \quad \textit{and} \quad \tau(\delta) \to 0, \quad \textit{as} \quad \delta \to 0. \tag{4.22}$$

Proof. Assume we have measurements of the problem (4.17) of the form

$$y^\delta = \mathcal{A}^{-\ell} u^\dagger + \delta\xi,$$

where $\xi \sim \mathcal{N}(0, \mathcal{A}^{-\beta})$ and $u^\dagger \in \mathcal{H}$ is fixed. By Definition 4.2.2, we need to show that ξ-almost surely $\mu^{y^\delta,\delta,\tau(\delta)} \Rightarrow \delta_{u^\dagger}$, as $\delta \to 0$. By Lemma 4.3.4(Appendix), it suffices to show that ξ-almost surely $\langle m, x\rangle \to \langle u^\dagger, x\rangle$ and $\langle Cx, x\rangle \to 0$, $\forall x \in \mathcal{H}$ as $\delta \to 0$.

Indeed, fix $x \in \mathcal{H}$.

For the proof that ξ-almost surely $\langle m, x\rangle \to \langle u^\dagger, x\rangle$, as $\delta \to 0$ it suffices to show that ξ-almost surely $||m - u^\dagger|| \to 0$, as $\delta \to 0$.

Define $q_k = \langle u^\dagger, \phi_k\rangle$. Then $y_k^\delta = \mu_k^{-\ell} q_k + \delta\xi_k$ and like in the proof of Theorem 3.1.5 (for $\gamma = 0$), by Remark 4.2.5 we have

$$||m - u^\dagger||^2 = \sum_{k=1}^{\infty} \left(\frac{\mu_k^{-2\ell+\beta} q_k + \delta\mu_k^{-\ell+\beta}\xi_k}{\mu_k^{-2\ell+\beta} + \frac{\delta^2}{\tau^2}\mu_k^{\alpha}} - q_k \right)^2 \leq$$

$$\sum_{k=1}^{\infty} \frac{2\delta^2 \mu_k^{-2\ell+2\beta}\xi_k^2}{\mu_k^{-4\ell+2\beta} + \frac{\delta^4}{\tau^4}\mu_k^{2\alpha} + 2\frac{\delta^2}{\tau^2}\mu_k^{-2\ell+\beta+\alpha}} + \sum_{k=1}^{\infty} \frac{2\frac{\delta^4}{\tau^4}\mu_k^{2\alpha} q_k^2}{\mu_k^{-4\ell+2\beta} + \frac{\delta^4}{\tau^4}\mu_k^{2\alpha} + 2\frac{\delta^2}{\tau^2}\mu_k^{-2\ell+\beta+\alpha}}.$$

The first term can be bounded by

$$\sum_{k=1}^{\infty} \frac{2\delta^2 \mu_k^{-2\ell+2\beta}\xi_k^2}{2\frac{\delta^2}{\tau^2}\mu_k^{-2\ell+\beta+\alpha}} = \tau^2 \sum_{k=1}^{\infty} \mu_k^{\beta-\alpha}\xi_k^2.$$

The sum is finite ξ-almost surely, since by Lemma 4.3.10(Appendix) we have that $\xi \in \mathcal{H}^s$, $s < \beta - \frac{1}{2}$ ξ-almost surely, where $\beta - \alpha < \beta - \frac{1}{2}$, since $\alpha > \frac{1}{2}$. Hence, as $\delta \to 0$, the first term vanishes since by the assumption we have that $\tau \to 0$ as $\delta \to 0$.

The second term, for $\frac{\delta}{\tau} \to 0$, vanishes by the Dominated Convergence Theorem, since for each $k \in \mathbb{N}$

$$\frac{2\frac{\delta^4}{\tau^4}\mu_k^{2\alpha}q_k^2}{\mu_k^{-4\ell+2\beta} + \frac{\delta^4}{\tau^4}\mu_k^{2\alpha} + 2\frac{\delta^2}{\tau^2}\mu_k^{-2\ell+\beta+\alpha}} \to 0$$

as $\frac{\delta}{\tau} \to 0$ and for fixed $\frac{\delta}{\tau}$, since $\mu_k > 0$, $\forall k \in \mathbb{N}$ we have

$$\frac{2\frac{\delta^4}{\tau^4}\mu_k^{2\alpha}q_k^2}{\mu_k^{-4\ell+2\beta} + \frac{\delta^4}{\tau^4}\mu_k^{2\alpha} + 2\frac{\delta^2}{\tau^2}\mu_k^{-2\ell+\beta+\alpha}} \leq \frac{2\frac{\delta^4}{\tau^4}\mu_k^{2\alpha}q_k^2}{\frac{\delta^4}{\tau^4}\mu_k^{2\alpha}} = 2q_k^2$$

which is summable because $u^\dagger \in \mathcal{H}$. Thus, since by the assumption $\frac{\delta}{\tau} \to 0$, as $\delta \to 0$, we have that the second term vanishes as $\delta \to 0$.

Hence, we have shown that ξ-almost surely $\left|\left|m - u^\dagger\right|\right|_\gamma \to 0$, as $\delta \to 0$.

For the convergence of the covariance operator, like in the proof of Theorem 4.2.9 we have

$$\langle Cx, x\rangle = \tau^2 \sum_{k=1}^{\infty} \mu_k^{-\alpha}\left(1 - \frac{1}{1 + \frac{\delta^2}{\tau^2}\mu_k^{2\ell+\alpha-\beta}}\right)x_k^2.$$

By the assumption we have that $\tau \to 0$ as $\delta \to 0$. Furthermore by the Dominated Convergence Theorem the last sum vanishes as $\frac{\delta^2}{\tau^2} \to 0$, since the parenthesis on the one hand is uniformly bounded by 1 and on the other hand $\forall k \in \mathbb{N}$ tends to 0 and since $x \in \mathcal{H}$, where $\alpha > \frac{1}{2} > 0$, thus $x \in \mathcal{H}^{-\alpha}$. Hence, since by the assumption $\frac{\delta^2}{\tau^2} \to 0$, as $\delta \to 0$, we have that $\langle Cx, x\rangle \to 0$ as $\delta \to 0$.

□

Remark 4.2.12.

i) The condition $\beta - 2\ell \leq \frac{1}{2}$ does not secure that $\xi \in \mathcal{H}^{2\ell}$, thus y^δ does not necessarily live in $\mathcal{H}^{2\ell}$ where K^{-1} is well defined and continuous, hence we do need regularization. By the assumption $\beta > \frac{1}{2}$, we do have that $\xi \in \mathcal{H}$, thus at least $y \in \mathcal{H}$.

In the proof, we substantially use the assumption that $\alpha > \frac{1}{2}$, which secures that the prior has full measure on $\mathcal{H}^s$, for $s < \alpha - \frac{1}{2}$ and which we can interpret as the requirement that the prior is indeed regularizing, i.e. with probability 1, samples from it live in a space which is at least as smooth as $\mathcal{H}$. In the classical Tikhonov regularization theory, we had the condition $\alpha \geq 0$, (cf. Theorem 3.1.5), which again was interpreted as the requirement that the penalty term in the Tikhonov functional is indeed regularizing, so the connection is apparent.

ii) Note that condition (4.22) of the last theorem says that $\tau \to 0$ in order to obtain the convergence of the posterior to $\delta_{u^\dagger}$. This condition on τ seems to be in conflict with the common understanding of the Bayesian approach, where zero covariance in the prior means that the mean of the prior should be taken as the true solution. To resolve this, note that compared to the variance of the noise, the variance of the prior distribution does tend to infinity, $\frac{\tau}{\delta} \to \infty$, since $\frac{\delta}{\tau} \to 0$, therefore $\tau >> O(\delta)$, i.e. the prior distribution becomes non-informative since its uncertainty becomes larger than the uncertainty in the noise [13].

iii) The link between the Bayesian approach and the Classical approach is apparent in the last proof: the proof of the convergence of the posterior mean, m, to the true solution, $u^\dagger$, is almost identical with the proof of the convergence of the Tikhonov regularized approximation, u_λ^δ, to $u^\dagger$ in Theorem 3.1.5, for $\lambda = \frac{\delta^2}{\tau^2}$ and $\eta_k = \delta\xi_k$. The only difference is the justification of the steps, since we no longer have that $\xi \in \mathcal{H}^\beta$ like in the Classical case.

Remark 4.2.13. In both Theorem 4.2.9 and Theorem 4.2.11, it would make sense to ask when the posterior, μ^{y^δ}, and the prior, μ_0, are equivalent. By Theorem 4.2.6 and Remark 4.2.8, the conditions $\beta - \alpha - 2\ell < -\frac{1}{2}$ and $2\beta - \alpha - 4\ell < 0$ secure that μ^{y^δ} and μ_0 are equivalent ξ-almost surely.

We now examine subjectivistic posterior consistency.

Theorem 4.2.14. *Fix* $y \in \mathcal{H}^{2\ell}$. *Then*

$$\mu^y \Rightarrow \delta_{\mathcal{A}^\ell y}$$

as $\delta \to 0$.

Proof. By Lemma 4.3.4(Appendix), it suffices to show that $\langle m, x\rangle \to \langle \mathcal{A}^\ell y, x\rangle$ and $\langle Cx, x\rangle \to 0,\ \forall x \in \mathcal{H}$ as $\delta \to 0$.

Indeed, fix $x \in \mathcal{H}$.

Using the diagonalization of $\mathcal{A}$, by (4.19) we have that

$$\langle m, x\rangle = \left\langle \mathcal{A}^{-\ell+\beta}\left(\mathcal{A}^{-2\ell+\beta} + \frac{\delta^2}{\tau^2}\mathcal{A}^\alpha\right)^{-1} y, x\right\rangle$$

$$= \sum_{k=1}^\infty \frac{\mu_k^{-\ell+\beta}}{\mu_k^{-2\ell+\beta} + \frac{\delta^2}{\tau^2}\mu_k^\alpha} y_k x_k, \quad \forall x \in \mathcal{H}.$$

Since $\forall k \in \mathbb{N}$

$$\frac{\mu_k^{-\ell+\beta} y_k x_k}{\mu_k^{-2\ell+\beta} + \frac{\delta^2}{\tau^2}\mu_k^\alpha} \le \mu_k^\ell y_k x_k, \quad \lim_{\delta\to 0} \frac{\mu_k^{-\ell+\beta} y_k x_k}{\mu_k^{-2\ell+\beta} + \frac{\delta^2}{\tau^2}\mu_k^\alpha} = \mu_k^\ell y_k x_k$$

and since

$$\sum_{k=1}^\infty \mu_k^\ell y_k x_k = \langle \mathcal{A}^\ell y, x\rangle < \infty$$

by Cauchy-Schwartz since $y \in \mathcal{H}^{2\ell}$ and $x \in \mathcal{H}$, we have that $\langle m, x\rangle \to \langle \mathcal{A}^\ell y, x\rangle$ as $\delta \to 0$, by the Dominated Convergence Theorem.

For the convergence of the covariance operator, using (4.20) we have

$$\langle Cx, x\rangle = \left\langle \left(\tau^2\mathcal{A}^{-\alpha} - \tau^2\mathcal{A}^{-\alpha}\left(I + \frac{\delta^2}{\tau^2}\mathcal{A}^{2\ell+\alpha-\beta}\right)^{-1}\right) x, x\right\rangle$$

$$= \sum_{k=1}^\infty \left(\tau^2\mu_k^{-\alpha} - \tau^2\mu_k^{-\alpha}\left(1 + \frac{\delta^2}{\tau^2}\mu_k^{2\ell+\alpha-\beta}\right)^{-1}\right) x_k^2$$

$$= \tau^2 \sum_{k=1}^\infty \mu_k^{-\alpha}\left(1 - \frac{1}{1 + \frac{\delta^2}{\tau^2}\mu_k^{2\ell+\alpha-\beta}}\right) x_k^2$$

which vanishes as $\delta \to 0$, since the parenthesis goes to 0 and by the Dominated Convergence Theorem since $x \in \mathcal{H}$ and $\alpha > \frac{1}{2} > 0$.

□

In conclusion, we have shown that for a fixed $y \in \mathcal{H}^{2\ell}$, the posterior does converge to a Dirac as the noise fades away, centered on the inverse image of y. Uncertainty disappears and the prior does not play any role in the small noise limit.

Remark 4.2.15. In the subjectivistic posterior consistency, we assume that the data do not come from our data likelihood model. Nevertheless, it also makes sense to assume that y comes from a space which secures the equivalence of the posterior and the prior. In that case, in the last theorem, by Theorem 4.2.6, we would need $\beta-\alpha-2\ell < -\frac{1}{4}$ and a modified condition on y, in particular $y \in \mathcal{H}^c$ where $c = \max\{2\ell, 2\beta - \alpha - 2\ell\}$.

4.3 Appendix

A concise, yet self-sustained, presentation of the tools from Probability and Measure Theory used for the development of the theory of the Bayesian Approach to Inverse Problems can be found in [20, Chapter 6]. We hereby provide the particular tools used in this project, as can be found in [20, Chapter 6].

In the following $\mathcal{Z}$ is a Hilbert space.

Definition 4.3.1. *For any $z_1, z_2 \in \mathcal{Z}$ we define the operator $z_1 \otimes z_2$ by the identity*

$$(z_1 \otimes z_2)z = \langle z_2, z\rangle z_1, \ \forall z \in \mathcal{Z}.$$

We use $*$ to denote the adjoint of a linear operator between two Hilbert spaces. In particular we may view $z_1, z_2 \in \mathcal{Z}$ as linear operators from $\mathbb{R}$ to $\mathcal{Z}$ and then

$$z_1 \otimes z_2 = z_1 z_2^*.$$

Definition 4.3.2. *A measure μ on the Hilbert space $\mathcal{Z}$ has a mean, $m \in \mathcal{Z}$, and covariance operator, $C : \mathcal{Z} \to \mathcal{Z}$, given by*

$$m = \int_{\mathcal{Z}} z\mu(dz)$$

and

$$\langle z_1, Cz_2\rangle = \int_{\mathcal{Z}} \langle z_1, z-m\rangle\langle z-m, z_2\rangle \mu(dz),$$

respectively. For a random variable $x \sim \mu$*,*

$$m = \mathbb{E}(x)$$

and

$$C = \mathbb{E}(x-m)\otimes(x-m).$$

Theorem 4.3.3. *[20, Theorem 6.4] A Gaussian measure on* $(\mathcal{Z}, \mathcal{B}(\mathcal{Z}))$ *has a mean* m *and covariance operator* C*. Further, the characteristic function of the measure is*

$$\varphi(z) = \exp\left(i\langle z, m\rangle - \frac{1}{2}\langle z, Cz\rangle\right), \ \forall z \in \mathcal{Z}.$$

Lemma 4.3.4. *[20, Lemma 6.5] Consider a family of probability measures* $\mu^{(n)}$*. Assume that,* $\forall z \in \mathcal{Z}$

$$\varphi_{\mu^{(n)}}(z) \to \exp\left(i\langle z, m^+\rangle - \frac{1}{2}\langle z, C^+ z\rangle\right).$$

Then $\mu^{(n)} \Rightarrow \mathcal{N}(m^+, C^+)$*.*

Definition 4.3.5. *We define the Cameron-Martin space* E *associated with a Gaussian measure* $\mu = \mathcal{N}(0, C)$ *on* $\mathcal{Z}$ *to be the intersection of all linear spaces of full measure under* μ*.*

Lemma 4.3.6. *[20, Lemma 6.10] The Cameron-Martin space associated to a Gaussian measure* $\mathcal{N}(0, C)$ *on the Hilbert space* $(\mathcal{Z}, \langle ., .\rangle)$*, is the Hilbert space* $E := \mathcal{R}(C^{\frac{1}{2}})$ *with the inner product*

$$\langle ., .\rangle_C = \langle C^{-\frac{1}{2}}., C^{-\frac{1}{2}}.\rangle.$$

Theorem 4.3.7. *[20, Theorem 6.13]-Feldman Hayek Theorem Two Gaussian measures* $\mu_i = \mathcal{N}(m_i, C_i)$*,* $i = 1, 2$ *on a Hilbert space* $\mathcal{Z}$ *are either singular or equivalent. They are equivalent if and only if the following three conditions hold:*

(i) $\mathcal{R}(C_1^{\frac{1}{2}}) = \mathcal{R}(C_2^{\frac{1}{2}}) := E$*,*

(ii) $m_1 - m_2 \in E$,

(iii) *the operator* $T := (C_1^{-\frac{1}{2}} C_2^{\frac{1}{2}})(C_1^{-\frac{1}{2}} C_2^{\frac{1}{2}})^* - I$ *is Hilbert-Schmidt in* $\overline{E}$.

Lemma 4.3.8. *[20, Lemma 6.15] For any two positive definite, self-adjoint, bounded linear operators* C_i *on a Hilbert space* $\mathcal{Z}$, $i = 1, 2$, *the condition* $\mathcal{R}(C_1^{\frac{1}{2}}) \subset \mathcal{R}(C_2^{\frac{1}{2}})$ *holds if and only if there exists a constant* $\kappa > 0$ *such that*

$$\langle z, C_1 z \rangle \leq \kappa \langle z, C_2 z \rangle, \ \forall z \in \mathcal{Z}.$$

Theorem 4.3.9. *[20, Theorem 6.20] Let* $\mathcal{Z} = \mathcal{Z}_1 \oplus \mathcal{Z}_2$ *be a separable Hilbert space with projectors* $\Pi_i : \mathcal{Z} \to \mathcal{Z}_i$, $i = 1, 2$. *Let* $(z_1, z_2) \in \mathcal{Z}_1 \oplus \mathcal{Z}_2$ *be a* $\mathcal{Z}$*-valued Gaussian random variable with mean* $m = (m_1, m_2)$ *and positive definite covariance operator* C. *Define*

$$C_{ij} = \mathbb{E}(z_i - m_i) \otimes (z_j - m_j).$$

Then the conditional distribution of z_1 *given* z_2 *is Gaussian with mean*

$$m' = m_1 + C_{12} C_{22}^{-1} (z_2 - m_2)$$

and covariance operator

$$C' = C_{11} - C_{12} C_{22}^{-1} C_{21}.$$

Lemma 4.3.10. *[20, Lemma 6.27] Consider a Gaussian measure* $\mu = \mathcal{N}(0, \mathcal{A}^{-\alpha})$, *where* $\mathcal{A}$ *satisfies the Assumptions 1.2.1* $(i) - (iii)$ *and* $\alpha > \frac{1}{2}$. *Then* $u \sim \mu$ *is in* $\mathcal{H}^s$ *almost surely for any* $s < \alpha - \frac{1}{2}$.

Theorem 4.3.11 (Bayes Formula). *Let* $(u, y) \in \mathbb{R}^n \times \mathbb{R}^q$ *be a jointly distributed pair of random variables with Lebesgue density* $\rho(u, y)$. *Then the infinitesimal version of Bayes Formula is*

$$\rho(u|y) \propto \rho(y|u)\rho(u).$$

Chapter 5

Conclusion

The purpose of this thesis has been to explore the Classical and the Bayesian approach to Inverse Problems and to give some intuition on the connection of the Tikhonov Regularization method to the Bayesian approach to inverse problems. We have tried to achieve the above task, using as our guide the "Laplacian-like" inverse problem, which enabled the simplification of the calculations. We have proved convergence results in the classical approach, provided convergence rates and then went on to prove posterior consistency results in the Bayesian approach. The link has been apparent in many instances, for example note the similarity of the proof of Theorem 3.1.5 with the proof of Theorem 4.2.11, or the fact that the posterior mean m in the Bayesian approach to the "Laplacian-like" problem (cf. Remark 4.2.5), is identical to the Tikhonov regularized approximation u_λ^δ for $\gamma = 0$ (cf. Lemma 3.1.3) .

We hope that this work can serve as a foundation for our future exploration of the field and that we will be able to get results in more general cases by gradually erasing the assumptions which simplified the calculations here. The calculations in Chapter 3, which we believe are sharp, will also provide useful guidance in these more general situations.

In this work, things were greatly simplified by the assumption that the forward operator, K, is diagonalizable in the same eigenbasis as the operator $\mathcal{A}$ which induces the Hilbert Scale used and furthermore the fact that K is a power of $\mathcal{A}$. Moreover, the use of the same Hilbert Scale in both the least squares term and the penalty term, also simplified things. Note that in the Bayesian approach this originated from the fact that we used Gaussian prior, $\mathcal{N}(0, C_0)$ and Gaussian observational noise $\mathcal{N}(0, C_1)$, with covariance operators which can be related by scaling, since $C_0 = \tau^2 \mathcal{A}^{-\alpha}$ and $C_1 = \delta^2 \mathcal{A}^{-\beta}$.

A first attempt to erase the above assumptions has been made by Andrew Stuart and Stig Larson (unpublished notes), who assume a weaker connection between all the norms and the forward operator. In particular, instead of the scaling connection between the different norms as well as between the operator inducing the norms with the forward operator, a norm-equivalence relation between different weighted norms is assumed, which also suggests a weaker relation between the forward operator and the operators inducing the different weighted norms.

In Theorem 4.2.11 we have proved regularized frequentist bayesian posterior consistency, i.e. we have proved that under sufficient conditions, if we have data of the form

$$y^\delta = Ku^\dagger + \delta\xi,$$

where $\xi \sim \mathcal{N}(0, \mathcal{A}^{-\beta})$ and $u^\dagger \in \mathcal{H}$ is the true solution, then as $\delta \to 0$ the posterior distribution $\mathcal{N}(m, C)$ weakly converges to a Dirac centered on the true solution $u^\dagger$, ξ-almost surely. As it is well known, the weak convergence of measures is metrized by the Prokhorov metric, [2]. We have devoted a great effort for producing the convergence rates, in the classical case, provided in Section 3.2 of this thesis, thus we hope that in the future we will be able to use these rates to obtain convergence rates in the Prokhorov metric, for the convergence proved in Theorem 4.2.11.

The Ky-Fan metric, [12], is a metric which measures distances between random variables from a probability space $(\Omega, \mathcal{F}, P)$ to a metric space (X, d_X), which quantifies the convergence in probability. In [14], [19] and [13], the authors consider as (X, d_X) the metric space of all the random measures with the Prokhorov metric and calculate convergence rates in the Ky-Fan metric for the convergence of the random variable μ^y

to the constant random variable $\delta_{u^\dagger}$.

In Theorem 4.2.6, in order to secure the well definiteness of the posterior, we had to restrict the space were y lives. In [8], Florens and Simoni use similar techniques as those we employ here, to prove frequentist posterior consistency results for bounded linear forward operators. In [7], Florens and Simoni argue against the restriction of the space where the data are assumed to live, since the actual data may not satisfy this assumption and they proceed to suggest a method of regularizing the posterior covariance operator, in order to secure the well definiteness of the posterior.

Finally, in the results presented in this thesis, we make a substantial use of Gaussianity. Particularly, in Theorem 4.2.6 where we give necessary and sufficient conditions for the equivalence of the prior and the posterior measure, the use of Theorem 4.3.7 relies on the Gaussianity of both the prior and the noise. If we try to use the abstract theory, presented in [20], to obtain the same result, formally we would get that

$$\frac{d\mu^y}{d\mu_0}(u) \propto \exp\left(-\frac{1}{2\delta^2}\,||Ku||_\beta^2 + \frac{1}{2\delta^2}\langle Ku, y\rangle_\beta\right)$$
$$:= \exp(-\Phi(u,y)).$$

Then, by [20, Theorem 4.1], in order to have that the posterior is well defined, we would need to show that Φ is Lipschitz-continuous with respect to u, on some space X and then choose a prior μ_0, such that $\mu_0(X) = 1$. Since $\Phi(u, y)$ is finite if and only if $u \in \mathcal{H}^{\beta-2\ell}$, we would need the prior $\mu_0 = \mathcal{N}(0, \tau^2\mathcal{A}^{-\alpha})$ to charge $\mathcal{H}^{\beta-2\ell}$ with full measure. By Lemma 4.3.10, this is secured for $\beta - 2\ell < \alpha - \frac{1}{2}$ thus we have the condition

$$\alpha > \beta - 2\ell + \frac{1}{2},$$

which is stronger than the condition

$$\alpha > \beta - 2\ell + \frac{1}{4}$$

obtained in Theorem 4.2.6.

Note that in Remark 4.2.7(ii), we show that if we assume y to come from the data likelihood model (4.18a), then we need a stronger condition $\alpha > \beta - 2\ell + \frac{1}{2}$ which seems to agree with the abstract theory. However, since in the abstract theory y is assumed to live in some fixed Hilbert space Y, we still have a difference in the requirements of the two theories.

This observation, might suggest that the approach taken in the non-Gaussian case, [20, Chapter 4], based on the continuity of Φ, is too strong.

Bibliography

[1] James O. Berger, *Statistical decision theory and Bayesian analysis*, Springer Series in Statistics, Springer-Verlag, New York, 1993, Corrected reprint of the second (1985) edition. MR 1234489 (94j:62002)

[2] Patrick Billingsley, *Convergence of probability measures*, second ed., Wiley Series in Probability and Statistics: Probability and Statistics, John Wiley & Sons Inc., New York, 1999, A Wiley-Interscience Publication. MR 1700749 (2000e:60008)

[3] Robert Dautray and Jacques-Louis Lions, *Mathematical analysis and numerical methods for science and technology. Vol. 3*, Springer-Verlag, Berlin, 1990, Spectral theory and applications, With the collaboration of Michel Artola and Michel Cessenat, Translated from the French by John C. Amson. MR 1064315 (91h:00004a)

[4] Persi Diaconis and David Freedman, *On the consistency of Bayes estimates*, Ann. Statist. **14** (1986), no. 1, 1–67, With a discussion and a rejoinder by the authors. MR 829555 (88e:62016a)

[5] Heinz W. Engl, Martin Hanke, and Andreas Neubauer, *Regularization of inverse problems*, Mathematics and its Applications, vol. 375, Kluwer Academic Publishers Group, Dordrecht, 1996. MR 1408680 (97k:65145)

[6] Ben G. Fitzpatrick, *Bayesian analysis in inverse problems*, Inverse Problems **7** (1991), no. 5, 675–702. MR 1128635 (92h:62049)

[7] Jean-Pierre Florens and Anna Simoni, *Regularized posteriors in linear ill-posed inverse problems*, Preprint (2008).

[8] ———, *Regularizing priors for linear inverse problems*, Preprint (2010).

[9] Markus Hegland, *An optimal order regularization method which does not use additional smoothness assumptions*, SIAM J. Numer. Anal. **29** (1992), no. 5, 1446–1461. MR 1182739 (93j:65090)

[10] Erhard Heinz, *Beiträge zur Störungstheorie der Spektralzerlegung*, Math. Ann. **123** (1951), 415–438. MR 0044747 (13,471f)

[11] Edwin Hewitt and Karl Stromberg, *Real and abstract analysis. A modern treatment of the theory of functions of a real variable*, Springer-Verlag, New York, 1965. MR 0188387 (32 #5826)

[12] Andreas Hofinger, *The metrics of Prokhorov and Ky Fan for assessing uncertainty in inverse problems*, Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II **215** (2006), 107–125 (2007). MR 2373930 (2008k:60012)

[13] Andreas Hofinger and Hanna K. Pikkarainen, *Convergence rate for the Bayesian approach to linear inverse problems*, Inverse Problems **23** (2007), no. 6, 2469–2484. MR 2441014 (2010d:65016)

[14] ______, *Convergence rates for linear inverse problems in the presence of an additive normal noise*, Stoch. Anal. Appl. **27** (2009), no. 2, 240–257. MR 2503292

[15] Jari Kaipio and Erkki Somersalo, *Statistical and computational inverse problems*, Applied Mathematical Sciences, vol. 160, Springer-Verlag, New York, 2005. MR 2102218 (2005g:65001)

[16] Andreas Kirsch, *An introduction to the mathematical theory of inverse problems*, Applied Mathematical Sciences, vol. 120, Springer-Verlag, New York, 1996. MR 1479408 (99c:34023)

[17] M. Thamban Nair, Markus Hegland, and Robert S. Anderssen, *The trade-off between regularity and stability in Tikhonov regularization*, Math. Comp. **66** (1997), no. 217, 193–206. MR 1388891 (98a:65081)

[18] Frank Natterer, *Error bounds for Tikhonov regularization in Hilbert scales*, Applicable Anal. **18** (1984), no. 1-2, 29–37. MR 762862 (86e:65081)

[19] A. Neubauer and H. K. Pikkarainen, *Convergence results for the Bayesian inversion theory*, J. Inverse Ill-Posed Probl. **16** (2008), no. 6, 601–613. MR 2458288 (2010c:60011)

[20] A. M. Stuart, *Inverse problems: a Bayesian perspective*, Acta Numer. **19** (2010), 451–559. MR 2652785

[21] A. N. Tikhonov, *On the solution of incorrectly put problems and the regularisation method*, Outlines Joint Sympos. Partial Differential Equations (Novosibirsk, 1963), Acad. Sci. USSR Siberian Branch, Moscow, 1963, pp. 261–265. MR 0211218 (35 #2100)

[22] Curtis R. Vogel, *Computational methods for inverse problems*, Frontiers in Applied Mathematics, vol. 23, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2002, With a foreword by H. T. Banks. MR 1928831 (2003i:65004)