

The solution of Laplacian problems over L-shaped domains with a singular function boundary integral method

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SUMMARY

The singular function boundary integral method is applied for the solution of a Laplace equation problem over an L-shaped domain. The solution is approximated by the leading terms of the local asymptotic solution expansion, while the Dirichlet boundary conditions are weakly enforced by means of Lagrange multipliers. Estimates of great accuracy are obtained for the leading singular coefficients, as well as for the Lagrange multipliers. Comparisons are made with recent numerical results in the literature. Copyright © 2002 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In many engineering problems (e.g. fracture mechanics applications [1]), governed by elliptic partial differential equations, boundary singularities arise when there is a sudden change in the boundary conditions (along a smooth boundary) or on the boundary itself. Singularities are known to affect adversely the accuracy and the convergence of standard numerical methods, such as finite element, boundary element, finite difference and spectral methods. Grid refinement is the most usual approach in these methods, aiming to improve the convergence rate and accuracy. However, most adaptive grid refinement schemes cause significant computational cost and their efficiency is not always satisfactory. To take into account the form of the singularity more effectively and achieve better accuracy and faster convergence, special methods are often required which incorporate the form of the singularity in the numerical scheme, which is, in general, more effective than mesh refinement (see References [2–4] and references therein). We should also add here the early works of Symm [5] Papamichael and Symm [6] and Xanthis *et al.* [7] who developed singular boundary integral methods. Two notable alternatives to singular methods are the *hp* version of the finite element method with

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geometrically graded meshes [8, 9], and a recently proposed multigrid finite element method on quasi-uniform meshes [10]. The former method has the potential of approximating singular solutions at an exponential rate of convergence, and the latter method is the only known way, to our knowledge, through which the $O(h^p)$ convergence rate can be retained using quasi-uniform meshes on problems with corner singularities.

For the two-dimensional Laplace equation, the asymptotic solution in polar co-ordinates (r, θ) centred at the singular point, is given by [11]

$$u(r, \theta) = \sum_{j=1}^{\infty} \alpha_j r^{\mu_j} f_j(\theta), \quad r, \theta \in V \quad (1)$$

where V is a simply connected domain, u is the dependent variable, α_j are the unknown singular coefficients, μ_j are the singularity powers arranged in ascending order, and the functions $f_j(\theta)$ represent the θ -dependence of the eigensolution. Of particular interest to engineering mechanics, especially in the field of elasticity, are the leading singular coefficients α_j of the asymptotic expansion [12]. In fracture mechanics, the first coefficient α_1 represents the so-called stress intensity factor, a measure of the stress at which fracture occurs. In the case of Laplacian problems, the singular coefficients are also called generalized stress intensity factors [12] or flux intensity factors [1].

Of special interest are numerical methods for the solution of problems with singularities in which the singular coefficients are calculated directly (see References [2–7, 13]). In the work of Babuška and Miller [14, 15], the singular coefficients were calculated by *post-processing* the finite element solution. This was done using both an influence function extraction technique and the well-known energy release principle of fracture mechanics. Szabó and Yosibash [16, 17], also used a finite element post-solution operation method, based on the complementary weak formulation, in order to calculate the singular coefficients in heat transfer and elasticity problems involving re-entrant corners and abrupt changes in material properties. Their method is applicable in cases where the singularities are characterized by complex eigenpairs. Brenner [18] used a multigrid finite element method for the computation of singular solutions and stress intensity factors with piecewise linear functions on quasi-uniform meshes. This method was shown to be efficient and convergent at the optimal $O(h)$ rate. A review of singular intensity factor evaluation and modelling of singularities in boundary integral methods is provided by Mukhopadhyay *et al.* [19].

In References [20, 21], Georgiou and co-workers developed a singular function boundary integral method for the solution of plane Laplacian problems with boundary singularities. This is based on the approximation of the solution by the leading terms of the local solution expansion:

$$\bar{u} = \sum_{j=1}^{N_s} \alpha_j W^j \quad (2)$$

where N_s is the number of basis functions, and

$$W^j \equiv r^{\mu_j} f_j(\theta) \quad (3)$$

are the singular functions which exactly satisfy the governing equation and the boundary conditions along the boundary causing the singularity. Using Galerkin's principle, the Laplace equation is weighted by the singular functions. The volume integral is then reduced to a

boundary one using Green's second identity. The Dirichlet boundary conditions are weakly enforced by means of Lagrange multipliers. Since the final discretized equations are boundary integrals, the dimension of the problem is reduced by one, and thus the computational cost is significantly reduced. The method has been tested on standard Laplacian problems, such as the so-called Motz problem, yielding extremely accurate estimates of the leading singular coefficients. It also exhibits exponential convergence with respect to the number of singular functions [20, 21]. The method of Georgiou and co-workers [20, 21] is applicable only if expansion (2) is valid over the entire domain of solution.

The objective of the present paper is to calculate the singular coefficients in problems involving re-entrant corners (i.e. with L-shaped domains) and compare the results of the singular function boundary integral method [20, 21] with those of Arad *et al.* [1], and the multigrid finite element results of Brenner [18]. The method of Arad *et al.* [1] is similar to ours since it is based on the approximation of the solution by the leading terms of the local expansion. An essential difference is that, instead of using Lagrange multipliers for the enforcement of the boundary conditions away from the boundary causing the singularity, they minimize a discrete functional which sums the squares of the distances between the approximation and the boundary conditions at a given number of boundary points. (Note that the same idea was used earlier by Li *et al.* [22].) Arad *et al.* [1] used their method to solve the Motz problem and a Laplacian problem over an L-shaped domain. In the case of the Motz problem, their results for the leading singular coefficients are as accurate as those obtained with the singular boundary integral method. However, the convergence of their method with the number of singular functions appears to be much slower [1, 20]. Moreover, their method is computationally more costly due to the non-linearity introduced by the least-squares method.

Symm [5] solved two other Laplacian problems over L-shaped domains using a singular boundary integral method. The accuracy of his solution was restricted to four significant digits; he did not provide estimates for the leading singular coefficients. In Symm's method, the approximation of the solution is expanded around the singularity into a series of special solutions to the Laplace equation and is regularized by subtracting the four leading terms of the local expansion. The regularized solution is then calculated by the standard boundary element method, under the assumption that it vanishes together with its normal derivative at nodal points near the singular point. As noted by Igarashi and Honma [23], this assumption is valid only when those nodal points are located sufficiently near the singular points. Igarashi and Honma [23] modified Symm's method relaxing the above assumption, and showed that, in the case of the Motz problem, their method gives almost the same results as those of the Symm's method, when the boundary is subdivided into sufficiently fine elements. However, their results for the leading singular coefficients converge only up to five significant digits and are not as accurate as those calculated by the methods of Olson *et al.* [2], Li *et al.* [22], Georgiou *et al.* [20] and Arad *et al.* [1]. Igarashi and Honma [23] also applied their method to a Laplacian problem over an L-shaped domain and showed that with their method the convergence in the computation of the capacitance of the domain is improved compared with the method of Wigley [24] which involves a time-consuming iterative procedure. The accuracy of the calculated singular coefficients is restricted to five significant digits.

The outline of the present paper is as follows: in Section 2, we consider the Laplacian problem over an L-shaped domain solved by Arad *et al.* [1], and present the formulation of the boundary integral method. The numerical results are presented in Section 3, where the fast convergence of the method with respect to the number of singular functions is demonstrated,

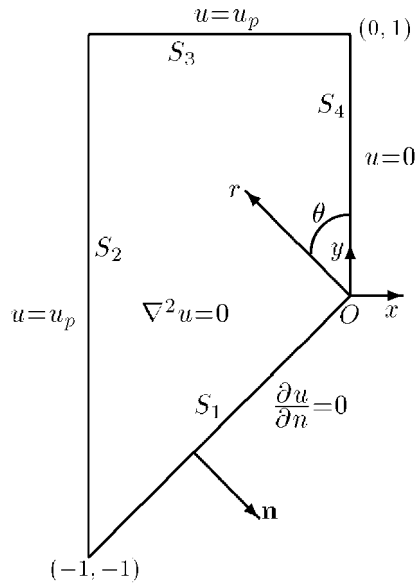


Figure 1. Geometry and boundary conditions.

and comparisons are made with the results provided by Arad *et al.* [1] and by Brenner [18]. The conclusions are summarized in Section 4.

2. BOUNDARY INTEGRAL METHOD

Consider the Laplace equation problem depicted in Figure 1. This is equivalent to a Poisson equation problem, $\nabla^2 v = -1$, over an L-shaped domain, with homogeneous Dirichlet boundary conditions along the whole boundary. Note that along boundary parts S_2 and S_3 essential boundary conditions are applied. Due to symmetry, only half of the domain is considered. The transformation $v = u + u_p$, where

$$u_p(r, \theta) = -\frac{r^2}{6\pi} \left[\frac{3\pi}{2} + 2 \ln r \sin 2\theta + \left(2\theta - \frac{3\pi}{2} \right) \cos 2\theta \right] \tag{4}$$

leads to the problem shown in Figure 1. A singularity arises at $x = y = 0$. The local solution is given by

$$u = \sum_{j=1}^{\infty} \alpha_j r^{2(2j-1)/3} \sin \left[\frac{2}{3}(2j-1)\theta \right] \tag{5}$$

This problem is very important in fracture mechanics and the ‘stress intensity factor’, defined by $2\alpha_1/3$, is of great significance [1].

In the singular boundary integral method [20], the solution is approximated as a linear combination of the leading singular functions of the local expansion (5),

$$\bar{u} = \sum_{j=1}^{N_x} \alpha_j W^j = \sum_{j=1}^{N_x} \alpha_j r^{2(2j-1)/3} \sin \left[\frac{2}{3}(2j-1)\theta \right] \quad (6)$$

where N_x is the number of singular functions, and α_j are the approximations of the singular coefficients to be calculated. Obviously, the method can be used only if the solution expansion is valid everywhere in the problem domain. Note also that the singular functions W^j satisfy the governing equation over the domain, and the boundary conditions along the parts of the boundary that cause the singularity.

Applying Galerkin's principle, the governing equation is weighted by the singular functions,

$$\int_V \nabla^2 u W^i dV = 0, \quad i = 1, 2, \dots, N_x \quad (7)$$

Given that the singular functions W^j satisfy the Laplace equation, application of Green's second identity reduces the volume integral into a boundary one:

$$\int_S \left(\frac{\partial \bar{u}}{\partial n} W^i - \bar{u} \frac{\partial W^i}{\partial n} \right) dS = 0, \quad i = 1, 2, \dots, N_x \quad (8)$$

where n denotes the direction normal to the boundary. The above integral is identically zero along boundary parts S_1 and S_4 since the boundary conditions are identically satisfied there (see Figure 1). This is an important feature of the singular boundary integral method since integration over the boundary parts causing the singularity is avoided.

To impose the Dirichlet conditions along the remaining parts, S_2 and S_3 , we employ Lagrange multipliers which replace the corresponding normal derivatives. The boundary is then partitioned into three-node elements, i.e. the Lagrange multipliers are expanded in terms of quadratic basis functions, M^j :

$$\lambda_A = \frac{\partial \bar{u}}{\partial x} = \sum_{j=1}^{N_{\lambda_A}} \lambda_A^j M^j \quad \text{and} \quad \lambda_B = \frac{\partial \bar{u}}{\partial y} = \sum_{j=1}^{N_{\lambda_B}} \lambda_B^j M^j \quad (9)$$

where N_{λ_A} and N_{λ_B} are the numbers of quadratic nodes along boundaries S_2 and S_3 , respectively. The nodal values of λ_A and λ_B are additional unknowns of the problem. Finally, the Dirichlet boundary conditions are weighted by the quadratic basis functions. The following system of $N_x + N_{\lambda_A} + N_{\lambda_B}$ equations is thus obtained:

$$- \int_{S_2} \left(\lambda_A W^i - \bar{u} \frac{\partial W^i}{\partial x} \right) dy + \int_{S_3} \left(\lambda_B W^i - \bar{u} \frac{\partial W^i}{\partial y} \right) dx = 0, \quad i = 1, 2, \dots, N_x \quad (10)$$

$$- \int_{S_2} \bar{u} M^i dy = \int_{S_2} u_p M^i dy, \quad i = 1, 2, \dots, N_{\lambda_A} \quad (11)$$

$$\int_{S_3} \bar{u} M^i dx = - \int_{S_3} u_p M^i dx, \quad i = 1, 2, \dots, N_{\lambda_B} \quad (12)$$

The above system of equations can be written in the following block form:

$$\begin{bmatrix} K_1 & K_2 & K_3 \\ K_2 & 0 & 0 \\ K_3 & 0 & 0 \end{bmatrix} \begin{bmatrix} A \\ \Lambda_A \\ \Lambda_B \end{bmatrix} = \begin{bmatrix} 0 \\ \Gamma_2 \\ \Gamma_3 \end{bmatrix} \quad (13)$$

where submatrices K_1 , K_2 and K_3 contain the coefficients of the unknowns α_j , λ_A^j and λ_B^j , respectively, as they are found after expanding Equations (10)–(12) and expressing them in linear algebraic form. Also, A is the vector of the singular coefficients and Λ_A and Λ_B are the vectors of the Lagrange multipliers. Vectors Γ_2 and Γ_3 contain the values of the RHS of integrals (11) and (12), respectively. Note that the integrands in Equation (10) are non-singular and that all the integrals are calculated far from the boundaries causing the singularity. It is easily shown that the coefficient matrix is symmetric. Moreover, it is singular if $N_a < N_\lambda$, where $N_\lambda = N_{\lambda_A} + N_{\lambda_B}$.

3. NUMERICAL RESULTS

In all results presented in this section, boundaries S_2 and S_3 are subdivided uniformly into $2N$ and N elements, respectively. Therefore, the number of Lagrange multipliers is $N_\lambda = N_{\lambda_A} + N_{\lambda_B} = 6N + 2$. As in References [20, 21], the integrals in Equations (10)–(12) are calculated numerically by subdividing each quadratic element into 10 subintervals and using a 15-point Gauss–Legendre quadrature for the numerical integration over each subinterval. The symmetry of the coefficient matrix is taken into account during the calculation of its elements, which means that only the elements on and above the main diagonal are calculated.

As pointed out above, the number of the singular functions, N_α , should be much greater than the number of the Lagrange multipliers, N_λ , because otherwise the stiffness matrix is ill-conditioned or singular. On the other hand, large values of N_α should be avoided because the contributions of the high-order singular functions become either negligible (for $r < 1$) or very large (if $r > 1$) beyond the limits double precision can handle.

We performed several series of runs to find the optimal values of N_α and N_λ . We varied N_λ from 8 up to 68, and N_α from a value slightly above N_λ up to $3N_\lambda$. For every run, plots of λ_A vs y and λ_B vs x were obtained. In using the singular function boundary integral method, we observed that λ_A and λ_B were characterized by oscillations at all values of N_α when $N_\lambda \geq 44$. This is illustrated in Figure 2, where the graphs of λ_A and λ_B were presented for $N_\lambda = 38$ and 56 while keeping the number of singular functions equal to 80. These plots indicate that, indeed, as N_λ approaches the value of N_α , the coefficient matrix becomes ill-conditioned and oscillations appear.

The smoothness of the calculated Lagrange multipliers (checked by plotting λ_A vs y and λ_B vs x) provides a good measure of the quality of the solution. Our calculations with different values of N_α and N_λ show that the optimal value for N_λ is 38 (see Figure 2). For smaller values of N_λ , λ_A and λ_B are still smooth but their approximations are, of course, less satisfactory because the boundary is less refined. In all runs, we observed that for $N_\lambda \leq 38$ the approximation of the solution was the best possible when the value of N_α was about $2N_\lambda$. For $N_\alpha > 2N_\lambda$ the solution deteriorated. This means that there is an upper bound on the number of singular coefficients one should use.

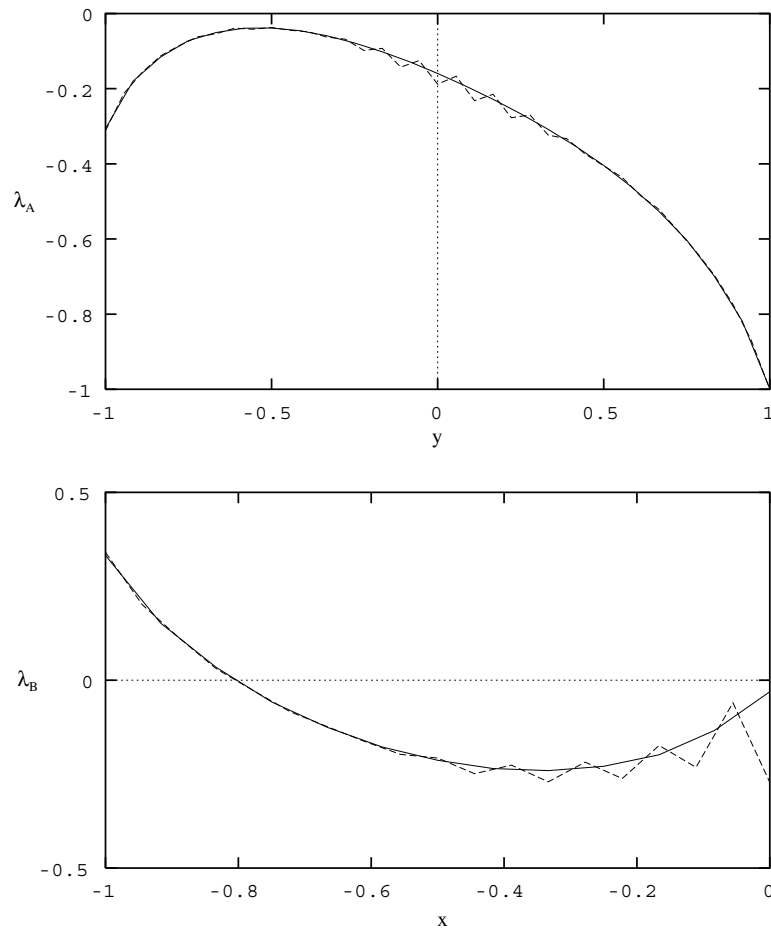


Figure 2. Calculated Lagrange multipliers with $N_x = 80$ and $N_x = 38$ (solid line) and 56 (dashed line).

Table I. Convergence of the solution with N_x when $N_x = 80$.

N_x	α_1	α_2	α_3	α_5	α_{10}	α_{15}
8	0.40187481	0.09363398	-0.0094030	-0.0083099	-0.0002086	-0.0002264
14	0.40192971	0.09364730	-0.0093818	-0.0083517	-0.0005154	-0.0001607
26	0.40193100	0.09364829	-0.0093831	-0.0083588	-0.0005649	-0.0001351
38	0.40193103	0.09364829	-0.0093830	-0.0083588	-0.0005653	-0.0001378
50	0.40193103	0.09364828	-0.0093830	-0.0083589	-0.0005652	-0.0001376
56	0.40193103	0.09364827	-0.0093831	-0.0083588	-0.0005653	-0.0001376
62	0.40193104	0.09364828	-0.0093830	-0.0083589	-0.0005653	-0.0001375

In Table I, we show the effect of N_x on the calculated values of α_1 , α_2 , α_3 , α_5 , α_{10} and α_{15} , obtained with $N_x = 80$. The results indicate that the values of singular coefficients converge rapidly with N_x and that very accurate estimates are obtained at least for the 15 leading

Table II. Convergence of the solution with N_x when $N_\lambda = 38$.

N_x	α_1	α_2	α_5	α_{10}	α_{15}
50	0.401931033	0.093648287	-0.00835895	-0.00056546	-0.0001374
60	0.401931032	0.093648288	-0.00835882	-0.00056548	-0.0001377
70	0.401931032	0.093648287	-0.00835882	-0.00056532	-0.0001379
80	0.401931032	0.093648286	-0.00835881	-0.00056534	-0.0001378
90	0.401931033	0.093648287	-0.00835882	-0.00056536	-0.0001378
100	0.401931033	0.093648288	-0.00835882	-0.00056561	-0.0001375
110	0.401931033	0.093648289	-0.00835880	-0.00056535	-0.0001380
120	0.401931033	0.093648285	-0.00835883	-0.00056522	-0.0001381
130	0.401931032	0.093648283	-0.00835885	-0.00056513	-0.0001382

Table III. Converged values of the leading singular coefficients with $N_\lambda = 38$ and $N_x = 90$.

i	α_i
1	0.40193103
2	0.09364829
3	-0.0093830
4	-0.0298851
5	-0.0083588
6	-0.0047302
7	-0.0015451
8	-0.001098
9	-0.000719
10	-0.000565
11	-0.000395
12	-0.000296
13	-0.000219
14	-0.000173
15	-0.000138

coefficients. Also, one notices that when the difference between N_λ and N_x is large then the solution is not accurate, as emphasized above. In Table II, we show the effect of N_x on the values of some singular coefficients calculated with $N_\lambda = 38$. The method exhibits exponential convergence with respect to N_x . However, for values of N_x greater than 90, the accuracy of the solution appears to start deteriorating due to the fact that the system becomes ill-conditioned. In Table III, we tabulate the converged values of the singular coefficients calculated with optimal choices of $N_x = 90$ and $N_\lambda = 38$. Evidently, the contributions of the higher-order terms are progressively vanishing. The CPU time required for the above run is 2.14 s on an IBM RS6000 (Processor type: Power PC 604e/375 MHz).

Note that the value 0.40193103 of the leading singular coefficient is accurate to the eighth significant digit (this accuracy is obtained even for $N_x = 50$ and $N_\lambda = 38$), while Arad *et al.* [1] provide the value 0.401920085 which is accurate only to the fourth significant digit. (Arad *et al.* [1] have not studied the convergence of their method in the case of the present problem.) Brenner [18] reported for the first singular coefficient the values 0.40193219,

0.40193032 and 0.40193057, which are accurate to the fifth decimal digit. These values have been obtained by the standard full multigrid algorithm and two modified full multigrid algorithms, respectively (see Reference [18] for more details). Neither Arad *et al.* [1] nor Brenner [18] provide estimates for the higher-order singular coefficients.

As a conclusion, using Lagrange multipliers for the enforcement of the essential boundary conditions is more effective than the minimization of a least-squares functional suggested by Arad *et al.* [1]. It is also computationally less costly, since it preserves the linearity of the problem. Finally, when compared to the multigrid methods from Reference [18], the singular function boundary integral method yields more accurate estimates not only for the first but also for the other leading singular coefficients.

4. CONCLUSIONS

A Laplace equation problem over an L-shaped domain has been solved using the singular function boundary integral method. The calculated values of the singular coefficients are of great accuracy. Comparisons between the results of the present method and those of the method of Arad *et al.* [1] indicate that the singular boundary integral method converges faster with respect to the number of singular functions and yields more accurate estimates for the leading singular coefficients.

The method can be applied for other singular problems with a different shape of the domain and with different singularities, provided that the local solution expansion is known and holds over the entire domain. Extensions to other singular elliptic problems are currently under investigation.

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