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Special Boundary Approximation Methods for Laplace Equation Problems with Boundary Singularities—Applications to the Motz Problem

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Abstract—We investigate the convergence of special boundary approximation methods (BAMs) used for the solution of Laplace problems with a boundary singularity. In these methods, the solution is approximated in terms of the leading terms of the asymptotic solution around the singularity. Since the approximation of the solution satisfies identically the governing equation and the boundary conditions along the segments causing the singularity, only the boundary conditions along the rest of the boundary need to be enforced. Four methods of imposing the essential boundary conditions are considered: the penalty, hybrid, and penalty/hybrid BAMs and the BAM with Lagrange multipliers. *A priori* error analyses and numerical experiments are carried out for the case of the Motz problem, and comparisons between all methods are made.

Keywords—Elliptic equation, Boundary singularity, Singular coefficients, Error estimates, Convergence.

1. INTRODUCTION

As in Li [1], we employ the term *boundary approximation method* (BAM) for numerical methods used for the solution of boundary value problems when the solution is approximated over the entire domain as a linear combination of certain particular solutions of the governing equation. Since the governing equation is identically satisfied, only the enforcement of the boundary conditions

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is necessary in order to obtain the unknown coefficients of the above linear combination. BAMs include the boundary element method [2] and the method of fundamental solutions [3], in which the approximate solution is expressed in terms of fundamental solutions of the governing equation. The main advantage of the BAMs is that the dimension of the problem is reduced by one, which implies that the required computational cost is considerably reduced.

Special BAMs can be developed in the case of elliptic boundary value problems with a boundary singularity. If the local asymptotic solution around the singularity is known and converges over the entire solution domain, then the leading terms of the solution expansion can be used for the approximation of the solution. The additional advantages of such special BAMs are the following.

(a) Since the boundary conditions along the boundary parts causing the singularity are identically satisfied, application of the boundary conditions is necessary only along the remaining parts of the boundary.
(b) The singular coefficients, i.e., the leading coefficients of the asymptotic solution expansion, are calculated directly.
(c) The accuracy and the rate of convergence are considerably improved, compared to those of standard numerical methods which are seriously affected by the presence of singularities [1,4–6].

The approximation of the solution with the leading terms of the local asymptotic expansion may be employed only locally, i.e., in a subdomain Ω_1 containing the singularity. Such an approach is mandatory if the domain of convergence of the asymptotic solution is a subset of the domain Ω (which should be a superset of Ω_1). Then, one may use another set of particular solutions or employ standard numerical methods in order to approximate the solution and apply the boundary conditions in the remaining part Ω_2 of the domain ($\Omega = \Omega_1 \cup \Omega_2$). Obviously, in the latter case, the method is not a BAM. A difficulty associated with this approach comes from the need of imposing proper coupling conditions along the interface of Ω_1 and Ω_2 (see, e.g., [7]). Li [1] considered a benchmark Laplace equation problem with a boundary singularity, known as the Motz problem, and investigated different coupling techniques when finite elements, finite differences, and the finite-volume method are employed over Ω_2.

What distinguishes the various special BAMs used for solving elliptic boundary value problems with a boundary singularity is the way the essential boundary conditions are enforced. Li *et al.* [7] and Arad *et al.* [8] employed least-squares techniques, whereas Georgiou and co-workers [4–6] employed Lagrange multipliers. Li [1] also considered other techniques, such as the penalty method, the hybrid method and the penalty/hybrid method which can be viewed as a combination of the former two methods.

The objective of the present work is to carry out a priori error analyses for various special BAMs which will allow the optimal choice of the parameters involved, leading to exponential convergence rates. For demonstration purposes, we have chosen to study the Motz problem [9].

In Section 2, we consider a general Laplace equation problem with Dirichlet and mixed boundary conditions and formulate the corresponding Galerkin and minimization problems with the penalty, the hybrid and the penalty/hybrid BAMs. For comparison purposes, the BAM with Lagrange multipliers [4,6] is also considered. In Section 3, the application of the above four methods to the Motz problem is demonstrated, and in Sections 4–7, the corresponding error analyses are presented. Finally, in Section 8, we present some representative numerical experiments validating the error analyses, and make comparisons between all BAMs under study.

2. FORMULATIONS FOR THE LAPLACE EQUATION

For simplicity, we present the formulations of the BAMs for the special case of the Laplace equation. These formulations are easily extended to more general elliptic problems; see, e.g., [1].

We consider the Laplace equation in a plane, simply connected polygonal domain Ω,

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \text{in } \Omega, \tag{2.1}$$

with mixed boundary conditions,

$$u = g_1, \qquad \text{on } \Gamma_1, \tag{2.2}$$

$$\frac{\partial u}{\partial n} + qu = g_2, \qquad \text{on } \Gamma_2, \tag{2.3}$$

where $\Gamma_1 \cup \Gamma_2 = \partial\Omega$, $|\Gamma_1| > 0$, the functions g_1, g_2 and q are sufficiently smooth, $q|_{\Gamma_2} \geq 0$, and n is the outward normal direction to the boundary.

2.1. Weak Formulations and Lagrange Multipliers

Before proceeding to the various descriptions of the BAMs it is instructive to present first the standard weak formulations of the problem (2.1)–(2.3), i.e., the Galerkin weak form and its equivalent variational formulation, and discuss briefly the use of Lagrange multipliers for the enforcement of the essential boundary condition (2.2) on Γ_1. Let us employ the following notation for the Sobolev spaces of interest,

$$H^1(\Omega) = \left\{v : v, v_x, v_y \in L^2(\Omega)\right\}, \tag{2.4}$$

$$H_0^1(\Omega) = \left\{v : v, v_x, v_y \in L^2(\Omega),\ v|_{\Gamma_1} = 0\right\}. \tag{2.5}$$

We are also interested in the following subset of $H^1(\Omega)$,

$$H_*^1(\Omega) = \left\{v : v, v_x, v_y \in L^2(\Omega),\ v|_{\Gamma_1} = g_1\right\}. \tag{2.6}$$

In the Galerkin method, a solution $u \in H_*^1(\Omega)$ is sought, such that

$$\iint_\Omega \nabla u \cdot \nabla v \, ds + \int_{\Gamma_2} quv \, d\ell = \int_{\Gamma_2} g_2 v \, d\ell, \qquad \forall\, v \in H_0^1(\Omega), \tag{2.7}$$

or

$$B(u, v) = F(v), \qquad \forall\, v \in H_0^1(\Omega), \tag{2.8}$$

where

$$B(u, v) = \iint_\Omega \nabla u \cdot \nabla v \, ds + \int_{\Gamma_2} quv \, d\ell \tag{2.9}$$

and

$$F(v) = \int_{\Gamma_2} g_2 v \, d\ell. \tag{2.10}$$

The solution u of the Galerkin problem (2.7) minimizes the quadratic functional,

$$I(v) = \frac{1}{2} B(v, v) - F(v), \qquad v \in H_*^1(\Omega), \tag{2.11}$$

or

$$I(v) = \frac{1}{2} \iint_\Omega (\nabla v)^2 \, ds + \frac{1}{2} \int_{\Gamma_2} qv^2 \, d\ell - \int_{\Gamma_2} g_2 v \, d\ell, \qquad v \in H_*^1(\Omega). \tag{2.12}$$

Thus, the equivalent minimization problem is to find $u \in H_*^1(\Omega)$, such that

$$I(u) = \min_{v \in H_*^1(\Omega)} I(v). \tag{2.13}$$

If now, the essential boundary condition (2.2) on Γ_1 is enforced by means of Lagrange multipliers $\lambda = \frac{\partial u}{\partial n}|_{\Gamma_1}$, then the weak form of the problem (2.1)–(2.3) becomes [10]. Find $(u, \lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma_1)$, such that

$$B(u,v) + G(u,v;\lambda,\mu) = F(v), \qquad \forall\,(v,\mu) \in H^1(\Omega) \times H^{-1/2}(\Gamma_1), \tag{2.14}$$

where $B(\cdot,\cdot)$ and $F(\cdot)$ are given by (2.9) and (2.10), respectively, and

$$G(u,v;\lambda,\mu) = -\int_{\Gamma_1} (\lambda v + \mu(u - g_1))\, d\ell, \tag{2.15}$$

with $(v,\mu) \in H^1(\Omega) \times H^{-1/2}(\Gamma_1)$ arbitrary test functions. Here, $H^{-1/2}$ is the dual space of $H^{1/2}$, defined as follows. If

$$H^{1/2}(\partial\Omega) = \left\{u \in H^1(\Omega) : u|_{\partial\Omega} \in L^2(\Omega)\right\} \tag{2.16}$$

is the *trace space* of functions in $H^1(\Omega)$, T denotes the *trace operator*, and the norm of $H^{1/2}(\partial\Omega)$ is defined as

$$\|\psi\|_{1/2,\partial\Omega} = \inf_{u \in H^1(\Omega)} \left\{\|u\|_{1,\Omega} : Tu = \psi\right\}, \tag{2.17}$$

then $H^{-1/2}(\partial\Omega)$ is defined as the *closure* of $H^0(\partial\Omega) \equiv L^2(\partial\Omega)$ with respect to the norm,

$$\|\varphi\|_{-1/2,\partial\Omega} = \sup_{\psi \in H^{1/2}(\partial\Omega)} \frac{\int_{\partial\Omega} \varphi\psi}{\|\psi\|_{1/2,\partial\Omega}}. \tag{2.18}$$

The reader is referred to [10,11] for more details.

It is clear that the Galerkin problem (2.14) takes the form,

$$\iint_\Omega \nabla u \cdot \nabla v\, ds + \int_{\Gamma_2} quv\, d\ell - \int_{\Gamma_1} (\lambda v + \mu u)\, d\ell = \int_{\Gamma_2} g_2 v\, d\ell - \int_{\Gamma_1} g_1 \mu\, d\ell.$$

Its solution $(u,\lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma_1)$ creates a stationary point for the (not positive definite) functional,

$$\begin{aligned} I(v,\lambda) &= \frac{1}{2}\left[B(v,v) + G(v,v;\lambda,\lambda)\right] - F(v) \\ &= \frac{1}{2}\iint_\Omega (\nabla v)^2 ds + \frac{1}{2}\int_{\Gamma_2} qv^2 d\ell - \int_{\Gamma_2} g_2 v\, d\ell - \int_{\Gamma_1} \lambda(v - g_1)\, d\ell. \end{aligned} \tag{2.20}$$

Note that the Lagrange multiplier function $\lambda = \frac{\partial u}{\partial n}|_{\Gamma_1}$ is treated as an additional unknown variable.

2.2. Boundary Approximation Methods

The basic characteristic of the boundary approximation methods is that the solution of problem (2.1)–(2.3) is sought in a finite-dimensional subspace,

$$V_N = \text{span}\,\{\Phi_i\}_{i=1}^N, \tag{2.21}$$

where $\{\Phi_i\}_{i=1}^N$ is a finite set of analytic, linearly independent basis functions, satisfying

$$\Delta\Phi_i = 0, \qquad \text{in } \Omega, \quad i = 1, \ldots, N. \tag{2.22}$$

Thus, the approximate solution $u_N \in V_N$ is of the form,

$$u_N = \sum_{i=1}^{N} a_i^N \Phi_i, \tag{2.23}$$

where a_i^N, $i = 1, \ldots, N$, are unknown coefficients to be determined. The admissible (or test) functions v also belong to V_N and do not necessarily satisfy the essential boundary condition on Γ_1. Due to (2.22), any function $v \in V_N$ satisfies the Laplace equation. Therefore, the double integrals in the Galerkin problems (2.7) or (2.19) and in the functionals (2.12) or (2.20) are reduced to boundary integrals,

$$\iint_\Omega \nabla u \cdot \nabla v \, ds = \int_{\partial\Omega} u \frac{\partial v}{\partial n} \, d\ell \qquad \text{and} \qquad \iint_\Omega (\nabla v)^2 \, ds = \int_{\partial\Omega} v \frac{\partial v}{\partial n} \, d\ell.$$

The essential boundary condition on Γ_1 can be enforced using different techniques [1]. The variational formulations for the penalty, the hybrid and the penalty/hybrid BAMs are conveniently combined by introducing the parameters $w \geq 0$ and $\alpha \in [0, 1]$. An approximate solution $u_N \in V_N$ is sought, such that

$$I(u_N) = \min_{v \in V_N(\Omega)} I(v), \tag{2.24}$$

where

$$\begin{aligned} I(v) = \frac{1}{2} \int_{\partial\Omega} v \frac{\partial v}{\partial n} \, d\ell + \frac{1}{2} \int_{\Gamma_2} q v^2 \, d\ell - \int_{\Gamma_2} g_2 v \, d\ell \\ + w^2 \int_{\Gamma_1} (v - g_1)^2 \, d\ell - \alpha \int_{\Gamma_1} \frac{\partial v}{\partial n} (v - g_1) \, d\ell. \end{aligned} \tag{2.25}$$

In the penalty BAM, $w > 0$ and $\alpha = 0$; in the hybrid BAM, $w = 0$ and $\alpha = 1$; and in the penalty/hybrid BAM, $w \geq 0$ and $0 \leq \alpha \leq 1$ with $w^2 + \alpha^2 > 0$. The functional (2.25) involves only boundary integrals. This is, of course, also true for the equivalent Galerkin problem. Find $u \in H^1(\Omega)$ such that

$$\begin{aligned} \int_{\partial\Omega} u \frac{\partial v}{\partial n} \, d\ell + \int_{\Gamma_2} q u v \, d\ell + 2w^2 \int_{\Gamma_1} u v \, d\ell - \alpha \int_{\Gamma_1} \left(\frac{\partial u}{\partial n} v + u \frac{\partial v}{\partial n} \right) d\ell \\ = \int_{\Gamma_2} g_2 v \, d\ell + 2w^2 \int_{\Gamma_1} g_1 v \, d\ell - \alpha \int_{\Gamma_1} g_1 \frac{\partial v}{\partial n}, \, d\ell \qquad \forall\, v \in H^1(\Omega). \end{aligned} \tag{2.26}$$

The discrete problem is obtained by replacing u with $u_N \in V_N \subset H^1(\Omega)$ above and requiring that (2.26) holds for all $v \in V_N$.

In the BAM with Lagrange multipliers, the functional,

$$I(v, \lambda) = \frac{1}{2} \int_{\partial\Omega} v \frac{\partial v}{\partial n} \, d\ell + \frac{1}{2} \int_{\Gamma_2} q v^2 \, d\ell - \int_{\Gamma_2} g_2 v \, d\ell - \int_{\Gamma_1} \lambda \colon (v - g_1) \, d\ell$$

is minimized over all $(v, \lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma_1)$.

The similarity of the BAM with Lagrange multipliers with the hybrid BAM is obvious; the main difference is that the normal derivative $\frac{\partial v}{\partial n}|_{\Gamma_1} = \lambda$ is treated as an additional unknown variable. This is usually approximated locally in terms of polynomial basis functions. For completeness, we state the associated Galerkin problem, which reads. Find $(u, \lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma_1)$, such that

$$\int_{\partial\Omega} u \frac{\partial v}{\partial n} \, ds + \int_{\Gamma_2} q u v \, d\ell - \int_{\Gamma_1} [\lambda v + \mu \, (u - g_1)] \, d\ell = \int_{\Gamma_2} g_2 v \, d\ell,$$

for all $(v, \mu) \in H^1(\Omega) \times H^{-1/2}(\Gamma_1)$. As before, the discrete problem is obtained by replacing (u, λ) above with $(u_N, \lambda_h) \in [V_N \times \Lambda_h] \subset [H^1(\Omega) \times H^{-1/2}(\Gamma_1)]$ and requiring that (2.28) holds for all $(v, \mu) \in (V_N \times \Lambda_h)$. The precise definition of the finite-dimensional subspace $\Lambda_h \subset H^{-1/2}(\Gamma_1)$ is given in Section 4.4 ahead.

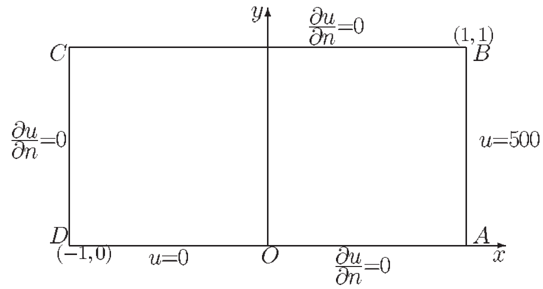


Figure 1. Geometry and boundary conditions of the Motz problem.

3. APPLICATION OF THE BAMS TO THE MOTZ PROBLEM

The Motz problem [9] is a benchmark Laplace equation problem that is very often used for testing various special numerical methods proposed in the literature for the solution of elliptic boundary value problems with boundary singularities. Figure 1 shows the geometry and the boundary conditions as modified by Wait and Mitchell [12]. The boundary value problem is stated as follows,

$$\Delta u = 0, \qquad \text{in } \Omega = \{(x,y) \mid -1 \le x \le 1,\ 0 \le y \le 1\}, \tag{3.1}$$

$$u|_{\overline{OD}} = 0, \tag{3.2}$$

$$u|_{\overline{AB}} = 500, \tag{3.3}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\overline{OA}\cup\overline{BC}\cup\overline{CD}} = 0. \tag{3.4}$$

A singularity arises at $x = y = 0$, where the boundary condition suddenly changes from $u = 0$ to $\frac{\partial u}{\partial y} = 0$. The local solution is given by

$$u = \sum_{i=1}^{\infty} a_i r^{(2i-1)/2} \cos\left[\left(\frac{2i-1}{2}\right)\theta\right], \tag{3.5}$$

where (r,θ) are the polar coordinates centered at the origin. The above expansion is valid in the entire solution domain [13], with a radius of convergence at least as large as two [14]. The values of the coefficients a_i, known as *singular coefficients* or *generalized stress intensity factors* are of interest. Rosser and Papamichael obtained the exact solution of the Motz problem using a conformal mapping technique and computed accurate approximations to the first 20 coefficients expressing them in terms of the coefficients in the series expansions of various elliptic functions and integrals involved in their conformal maps [14,15].

Many special numerical schemes have been proposed for the solution of the Motz problem, including finite-difference, global-element, boundary-element, and finite-element methods. Early works include those of Symm [16] and Papamichael and Symm [17] who developed singular boundary integral methods. Recent methods include those of Georgiou *et al.* [6] and Li and Lu [18]. The reader is referred to these papers for discussions about other numerical methods used for the solution of the Motz problem and the calculation of the singular coefficients, and for additional references.

Let us now consider the following approximation of the solution,

$$u_N = \sum_{i=1}^{N} a_i^N r^{(2i-1)/2} \cos\left[\left(\frac{2i-1}{2}\right)\theta\right] \tag{3.6}$$

or

$$u_N = \sum_{i=1}^{N} a_i^N \Phi_i, \tag{3.7}$$

where the basis functions,

$$\Phi_i = r^{(2i-1)/2} \cos\left[\left(\frac{2i-1}{2}\right)\theta\right], \tag{3.8}$$

are the singular functions appearing in the local solution expansion (3.5) and a_i^N are the approximations of the singular coefficients a_i. Since the singular functions Φ_i are solutions of the Laplace equation, the theory of the previous section applies with

$$\Gamma_1 = \overline{OD} \cup \overline{AB}, \qquad \Gamma_2 = \overline{OA} \cup \overline{BC} \cup \overline{CD}, \tag{3.9}$$
$$g_1|_{\overline{OD}} = 0, \qquad g_1|_{\overline{AB}} = 500, \qquad q|_{\Gamma_2} = 0, \qquad \text{and} \qquad g_2|_{\Gamma_2} = 0.$$

Moreover, the essential boundary condition on $\overline{OD}$ and the natural boundary condition on $\overline{OA}$ are identically satisfied by all basis functions Φ_i. As a result, for all $v \in V_N$,

$$\int_{\overline{OD}} v\frac{\partial v}{\partial n}\, d\ell = \int_{\overline{OA}} v\frac{\partial v}{\partial n}\, d\ell = 0.$$

Therefore, the functional (2.25) becomes

$$I(v) = \frac{1}{2}\int_{\Gamma^*} v\frac{\partial v}{\partial n}\, d\ell + w^2 \int_{\overline{AB}} (v-500)^2\, d\ell - \alpha \int_{\overline{AB}} \frac{\partial v}{\partial n}(v-500)\, d\ell, \tag{3.10}$$

where

$$\Gamma^* = \overline{AB} \cup \overline{BC} \cup \overline{CD}, \tag{3.11}$$

and the solution is sought in the space,

$$H^1_M(\Omega) \doteq \left\{ v \in H^1(\Omega) : v\Big|_{\overline{OD}} = \frac{\partial v}{\partial n}\Big|_{\overline{AB}} = 0 \right\}. \tag{3.12}$$

For convenience, the minimization and Galerkin problems reached with the four BAMs studied in this work are summarized below.

Penalty BAM

Minimization Problem: Minimize

$$I_P(v) \doteq \frac{1}{2}\int_{\Gamma^*} v\frac{\partial v}{\partial n}\, d\ell + w^2 \int_{\overline{AB}} : (v-500)^2\, d\ell, \qquad v \in V_N \subset H^1_M(\Omega). \tag{3.13}$$

Galerkin Problem: Find $u_N \in V_N \subset H^1_M(\Omega)$, such that $\forall\, v \in V_N \subset H^1_M(\Omega)$,

$$\int_{\Gamma^*} u_N\frac{\partial v}{\partial n}\, d\ell + 2w^2 \int_{\overline{AB}} u_N v\, d\ell = 2w^2 \int_{\overline{AB}} 500v\, d\ell. \tag{3.14}$$

Hybrid BAM

Minimization Problem: Minimize

$$I_H(v) \doteq \frac{1}{2}\int_{\Gamma^*} v\frac{\partial v}{\partial n}\, d\ell - \int_{\overline{AB}} : \frac{\partial v}{\partial n} : (v-500)\, d\ell, \qquad v \in V_N \subset H^1_M(\Omega). \tag{3.15}$$

Galerkin Problem: Find $u_N \in V_N \subset H^1_M(\Omega)$, such that, $\forall v \in V_N \subset H^1_M(\Omega)$,

$$\int_{\Gamma^*} u_N\frac{\partial v}{\partial n}\, d\ell - \int_{\overline{AB}} \left(\frac{\partial u_N}{n} v + u_N\frac{\partial v}{\partial n}\right) d\ell = -\int_{\overline{AB}} 500\frac{\partial v}{\partial n}\, d\ell. \tag{3.16}$$

Penalty/Hybrid BAM

Minimization Problem: Minimize

$$I_{PH}(v) \doteq \frac{1}{2}\int_{\Gamma^*} v\frac{\partial v}{\partial n}\,d\ell + w^2\int_{\overline{AB}}(v-500)^2\,d\ell - \alpha\int_{\overline{AB}}\frac{\partial v}{\partial n}(v-500)\,d\ell,$$
$$v \in V_N \subset H_M^1(\Omega). \tag{3.17}$$

Galerkin Problem: Find $u_N \in V_N \subset H_M^1(\Omega)$, such that, $\forall\, v \in V_N \subset H_M^1(\Omega)$,

$$\int_{\Gamma^*} u_N\frac{\partial v}{\partial n}\,d\ell + 2w^2\int_{\overline{AB}} u_N v\,d\ell - \alpha\int_{\overline{AB}}\left(\frac{\partial u_N}{n}v + u_N\frac{\partial v}{\partial n}\right)\,d\ell$$
$$= 2w^2\int_{\overline{AB}} 500v\,d\ell - \alpha\int_{\overline{AB}} 500\frac{\partial v}{\partial n}\,d\ell. \tag{3.18}$$

BAM with Lagrange Multipliers

Minimization problem: Minimize

$$I_L(v,\lambda) \doteq \frac{1}{2}\int_{\Gamma^*} v\frac{\partial v}{\partial n}:\,d\ell - \int_{\overline{AB}}\lambda(v-500)\,d\ell,$$
$$(v,\lambda) \in [V_N \times \Lambda_h] \subset \left[H_M^1(\Omega) \times H^{-1/2}(\overline{AB})\right]. \tag{3.19}$$

Galerkin problem: Find $(u_N, \lambda_h) \in [V_N \times \Lambda_h] \subset \left[H_M^1(\Omega) \times H^{-1/2}(\overline{AB})\right]$, such that

$$\int_{\Gamma^*} u_N\frac{\partial v}{\partial n}\,d\ell - \int_{\overline{AB}}:\ [\lambda_h v + \mu\,(u_N - 500)]\,d\ell = 0,$$
$$\forall\,(v,\mu) \in [V_N \times \Lambda_h] \subset \left[H_M^1(\Omega) \times H^{-1/2}(\overline{AB})\right]. \tag{3.20}$$

4. ERROR ANALYSES

Before proceeding to the error analyses for the four BAMs, we first provide some useful results, which, for the sake of simplicity, are presented specifically for the Motz problem. We will often use the notation $\beta \approx \gamma$ to mean that there exist constants C_1 and C_2, such that

$$C_1\beta \le \gamma \le C_2\beta.$$

Also, throughout this section, the letters c and C denote generic positive constants which are generally different in each occurrence. Finally, we note that the error analyses that follow will give bounds on the error in approximating u by u_N; error bounds for the singular coefficients can be obtained from these and the fact that [1],

$$\left|a_i - a_i^N\right| \le C\,\|u - u_N\|_{L^2(\Omega)}, \tag{4.1}$$

with C a positive constant independent of N.

LEMMA 4.1. *Let v satisfy (3.1)–(3.4) and let Γ_1 be given by (3.9). Then,*

$$|v|_{1,\Omega}^2 + \|v\|_{0,\Gamma_1}^2 \approx \|v\|_{1,\Omega}^2. \tag{4.2}$$

PROOF. We first have

$$|v|_{1,\Omega}^2 + \|v\|_{0,\Gamma_1}^2 \le |v|_{1,\Omega}^2 + \|v\|_{0,\Omega}^2 = \|v\|_{1,\Omega}^2. \tag{4.3}$$

Next, we use Poincaré's inequality,

$$|v|_{1,\Omega}^2 \ge C\,\|v\|_{1,\Omega}^2,$$

to obtain

$$C\,\|v\|_{1,\Omega}^2 \le |v|_{1,\Omega}^2 + \|v\|_{0,\Gamma_1}^2. \tag{4.4}$$

Combining (4.3) and (4.4), we get (4.2).

In what follows, we will be using the norm,

$$\|v\|_H = \left\{\int_{\Gamma^*} v\frac{\partial v}{\partial n} + w^2\int_{\overline{AB}} v^2\right\}^{1/2}, \tag{4.5}$$

for $w \ge 1$, with Γ^* given by (3.11).

Lemma 4.2. *If* $w = 1$, *then* $\|v\|_H \approx \|v\|_{1,\Omega}$, $\forall\, v \in V_N$.

Proof. Let $v \in V_N$ and note that

$$\Delta v = 0 \text{ in } \Omega,\ \ v|_{\overline{OD}} = 0,\ \ \frac{\partial v}{\partial n}\bigg|_{\overline{OA}} = 0. \tag{4.6}$$

Using Green's formula, we have

$$|v|^2_{1,\Omega} = \iint_\Omega |\nabla v|^2 = \iint_\Omega v\Delta v + \int_{\partial\Omega} v\frac{\partial v}{\partial n}$$

and by (4.6),

$$|v|^2_{1,\Omega} = \int_{\Gamma^*} v\frac{\partial v}{dn}. \tag{4.7}$$

Now, since $w = 1$, we have from (4.5),

$$\|v\|^2_H = \int_{\Gamma^*} v\frac{\partial v}{dn} + \int_{\overline{AB}} v^2$$

and by (4.7),

$$\|v\|^2_H = |v|^2_{1,\Omega} + \|v\|^2_{0,\overline{AB}}\,.$$

The desired result follows as in the proof of Lemma 4.1.

Lemma 4.3. *For* $w \geq 1$, *there exist constants* $C_1, C_2 > 0$, *such that*

$$C_1\,\|v\|_{1,\Omega} \leq \|v\|_H \leq C_2 w\,\|v\|_{1,\Omega}\,, \qquad \forall\, v \in V_N. \tag{4.8}$$

Proof. Let Γ_1 be given by (3.9). Since $w \geq 1$, we have

$$\|v\|^2_H \geq |v|^2_{1,\Omega} + \|v\|^2_{0,\Gamma_1}\,,$$

and by Lemma 4.1,

$$\|v\|^2_H \geq C\,\|v\|^2_{1,\Omega}\,. \tag{4.9}$$

Next, we have

$$\|v\|^2_H = w^2\left\{\frac{1}{w^2}\,|v|^2_{1,\Omega} + \|v\|^2_{0,\Gamma_1}\right\} \leq Cw^2\left\{|v|^2_{1,\Omega} + \|v\|^2_{0,\Gamma_1}\right\}$$

and by Lemma 4.2,

$$\|v\|^2_H \leq Cw^2\,\|v\|^2_{1,\Omega}\,. \tag{4.10}$$

Combining (4.9) and (4.10), we get (4.8).

Now, let $u = \overline{u}_N + r_N$ with

$$\overline{u}_N = \sum_{i=1}^{N} a_i\Phi_i \tag{4.11}$$

and

$$r_N = \sum_{i=N+1}^{\infty} a_i\Phi_i \tag{4.12}$$

where a_i are the true singular coefficients and Φ_i are given by (3.8). We have the following lemma.

Lemma 4.4. *With r_N given by (4.12), we have*

$$\|r_N\|_H \le \|r_N\|_{0,\Gamma^*}^{1/2} \left\|\frac{\partial r_N}{\partial n}\right\|_{0,\Gamma^*}^{1/2} + w\,\|r_N\|_{0,\overline{AB}}\,.$$

Proof. Using (4.5) and the Cauchy-Schwartz inequality, we get

$$\|r_N\|_H^2 \le \|r_N\|_{0,\Gamma^*} \left\|\frac{\partial r_N}{\partial n}\right\|_{0,\Gamma^*} + w^2\,\|r_N\|_{0,\overline{AB}}^2\,.$$

Using the inequality $\sqrt{a^2+b^2} \le a+b$, the desired result follows.

In what follows, we make the assumption that there exists $a \in (0,1)$ such that, with r_N is given by (4.12),

$$\|r_N\|_{0,\Gamma^*} \le Ca^N, \tag{4.13}$$

$$\left\|\frac{\partial r_N}{\partial n}\right\|_{0,\Gamma^*} \le CNa^N, \tag{4.14}$$

where C is a constant independent of N.

Assumptions (4.13),(4.14) hold trivially if $r<1$ in the local solution (3.5), since then by (3.8), (4.12), and the fact that the solution u is continuous, we have

$$|r_N| \le \sum_{i=N+1}^{\infty} |a_i|\, r^{i-1/2} \le C\frac{r^{N+1/2}}{1-r} \le Ca^N$$

with $r<a<1$. In the case of $r \ge 1$, one may partition the domain Ω into subdomains in which separate approximations may be obtained, as was discussed in Section 1. The solution over the entire domain can then be composed by combining the solutions from the various subdomains and properly dealing with their interactions across the *interfaces* separating each subdomain (see, e.g., [7]).

The Penalty BAM

Using the above results, we arrive at the following theorem for the penalty BAM.

Theorem 4.1. *Let $u_N^P \in V_N$ be the solution to (3.13) and $u \in H_M^1$ the weak solution to (3.1)–(3.4). Then, there exists a positive constant C, independent of N, such that*

$$\left\|u-u_N^P\right\|_H \le C\left\{\inf_{v\in V_N}\|u-v\|_H + \frac{1}{w}\left\|\frac{\partial u}{\partial n}\right\|_{0,\overline{AB}}\right\}.$$

Proof. Note that $u_N^P \in V_N$ satisfies

$$B_1(u_N^P, v) = F_1(v), \qquad \forall\, v \in V_N, \tag{4.15}$$

where

$$B_1(u,v) = \int_{\Gamma^*} v\frac{\partial u}{\partial n} + w^2\int_{\overline{AB}} uv, \qquad F_1(v) = 500w^2\int_{\overline{AB}} v.$$

In addition, u satisfies

$$B_1(u,v) = F_1(v) + \int_{\overline{AB}} v\frac{\partial u}{\partial n}, \qquad \forall\, v \in H_M^1. \tag{4.16}$$

Combining (4.15) and (4.16), we get

$$B_1\left(u-u_N^P,v\right)=\int_{\overline{AB}} v\frac{\partial u}{\partial n}, \qquad \forall\, v\in V_N. \tag{4.17}$$

Let $\delta=(u_N^P-v)\in V_N$. Then, using (4.5) and (4.17), we obtain

$$\|\delta\|_H=B_1(\delta,\delta)=B_1\left(u_N^P-v,\delta\right)=B_1(u-v,\delta)-\int_{\overline{AB}}\delta\frac{\partial u}{\partial n}. \tag{4.18}$$

Since $|B_1(u,v)|\le C\,\|u\|_H\,\|v\|_H$, we further obtain

$$\|\delta\|_H^2\le C\,\|u-v\|_H\,\|\delta\|_H+\|\delta\|_{0,\overline{AB}}\left\|\frac{\partial u}{\partial n}\right\|_{0,\overline{AB}}$$

after using the triangle and Cauchy-Schwartz inequalities. By Lemma 4.3,

$$\|\delta\|_{0,\overline{AB}}\le C\frac{1}{w}\,\|\delta\|_H$$

hence,

$$\|\delta\|_H^2\le C\left\{\|u-v\|_H+\frac{1}{w}\left\|\frac{\partial u}{\partial n}\right\|_{0,\overline{AB}}\right\}\|\delta\|_H\,.$$

Dividing by $\|\delta\|_H$, we get

$$\|\delta\|_H\le C\left\{\|u-v\|_H+\frac{1}{w}\left\|\frac{\partial u}{\partial n}\right\|_{0,\overline{AB}}\right\}, \qquad \forall\, v\in V_N. \tag{4.19}$$

Finally,

$$\left\|u-u_N^P\right\|_H\le\|u-v\|_H+\left\|v-u_N^P\right\|_H\le C\left\{\|u-v\|_H+\frac{1}{w}\left\|\frac{\partial u}{\partial n}\right\|_{0,\overline{AB}}\right\}$$

and the proof is complete.

Corollary 4.1. *Let u be the weak solution to (3.1)–(3.4) and let u_N^P satisfy (3.13). Then, there exists a constant $C>0$, independent of N, such that*

$$\left\|u-u_N^P\right\|_H\le C\left\{\|r_N\|_{0,\Gamma^*}^{1/2}\left\|\frac{\partial r_N}{\partial n}\right\|_{0,\Gamma^*}^{1/2}+w\,\|r_N\|_{0,\overline{AB}}+\frac{1}{w}\right\}. \tag{4.20}$$

Proof. Let $v=\overline{u}_N$ and $u=\overline{u}_N+r_N$ as given by (4.11) and (4.12). Then, by Theorem 4.1

$$\left\|u-u_N^P\right\|_H\le C\left\{\inf_{v\in V_N}\|u-v\|_H+\frac{1}{w}\left\|\frac{\partial u}{\partial n}\right\|_{0,\overline{AB}}\right\}\le C\left\{\|r_N\|_H+\frac{1}{w}\left\|\frac{\partial u}{\partial n}\right\|_{0,\overline{AB}}\right\}.$$

The desired result follows from Lemma 4.4 and by noting that $\|\frac{\partial u}{\partial n}\|_{0,\overline{AB}}\le C$.

Assuming (4.13) and (4.14) hold, we may use Corollary 4.1 to obtain the *optimal* choice for the parameter $w=a^{-N/2}$ for the penalty BAM, as well as the error estimate,

$$\left\|u-u_N^P\right\|_H\le C\sqrt{N}a^{N/2}, \tag{4.21}$$

with C a constant independent of N and a.

4.2. The Hybrid BAM

In the hybrid BAM, we seek $u_N^H \in V_N$ such that (3.16) holds $\forall\, v \in V_N$ (with u_N replaced by u_N^H). Note that u_N^H also satisfies

$$B_2(u_N^H, v) = F_2(v), \qquad \forall\, v \in V_N, \tag{4.22}$$

where

$$B_2(u,v) \doteq \int_{\Gamma^*} u\frac{\partial v}{\partial n} - \int_{\overline{AB}} v\frac{\partial u}{\partial n} - \int_{\overline{AB}} u\frac{\partial v}{\partial n} = \int_{\overline{BC}\cup\overline{CD}} v\frac{\partial u}{\partial n} + \int_{\overline{AB}} u\frac{\partial v}{\partial n} \tag{4.23}$$

and

$$F_2(v) = -500\int_{\overline{AB}} \frac{\partial v}{\partial n}. \tag{4.24}$$

We have the following theorem.

THEOREM 4.2. *Let $u_N^H \in V_N$ satisfy (4.22) and $u \in H_M^1$ be the weak solution to (3.1)–(3.4). Then,*

$$\left|u - u_N^H\right|_{1,\Omega} \le 2 \inf_{v\in V_N} |u-v|_{1,\Omega}\,.$$

PROOF. Note that

$$B_2(v,v) = \int_{\Gamma^*} v\frac{\partial v}{\partial n} = \iint_\Omega |\nabla v|^2 = |v|_{1,\Omega}^2\,, \qquad \forall\, v \in V_N.$$

Moreover, with $u \in H_M^1$ the solution to (3.1)–(3.4), we have

$$B_2(u - u_N^H, v) = 0, \qquad \forall\, v \in V_N. \tag{4.25}$$

Let $\delta = (u_N^H - v) \in V_N$ with $v \in V_N$ arbitrary. Then,

$$|\delta|_{1,\Omega}^2 = B_2(\delta,\delta) = B_2\left(u_N^H - v, \delta\right),$$

so that using (4.25),

$$\begin{aligned} |\delta|_{1,\Omega}^2 &= B_2\left(u_N^H - v,\delta\right) + B_2(u - u_N^H,\delta) = B_2(u-v,\delta) \\ &\le \left[B_2\left(u-v,u-v\right)B_2\left(\delta,\delta\right)\right]^{1/2} = |u-v|_{1,\Omega}\,|\delta|_{1,\Omega}\,, \end{aligned}$$

which gives

$$|\delta|_{1,\Omega} \le |u-v|_{1,\Omega}\,.$$

Thus, with $v \in V_N$,

$$\left|u-u_N^H\right|_{1,\Omega} \le |u-v|_{1,\Omega} + \left|v-u_N^H\right|_{1,\Omega} = |u-v|_{1,\Omega} + |\delta|_{1,\Omega} \le 2\,|u-v|_{1,\Omega}\,,$$

from which the desired result follows.

COROLLARY 4.2. *Let the assumptions of Theorem 4.2 as well as (4.13),(4.14) hold. Then,*

$$\left\|u-u_N^H\right\|_{1,\Omega} \le C\sqrt{N}a^N,$$

where C is a constant independent of N and $a \in (0,1)$.

PROOF. By Poincaré's inequality and Theorem 4.2,

$$\left\|u-u_N^H\right\|_{1,\Omega} \le C(\Omega)\left|u-u_N^H\right|_{1,\Omega} < 2C(\Omega)\inf_{v\in V_N}|u-v|_{1,\Omega}\,.$$

Letting $v = \overline{u}_N$, $u = \overline{u}_N + r_N$, and using (4.7), (4.13), and (4.14), we get

$$\left\|u-u_N^H\right\|_{1,\Omega} < C\,|r_N|_{1,\Omega} \le CNa^N$$

as desired.

Comparing the above result with the error bound (4.21), we see that the hybrid BAM converges at an optimal rate.

4.3. The Penalty/Hybrid BAM

Recall that u_N^{PH} is obtained from

$$I_{PH}\left(u_N^{PH}\right) = \min_{v \in V_N} I_{PH}(v) \tag{4.26}$$

where $I_{PH}(v)$ is defined by (3.17). Equivalently, we may seek $u_N^{PH} \in V_N$, such that

$$B_3\left(u_N^{PH}, v\right) = F_3\left(v\right), \qquad \forall\, v \in V_N, \tag{4.27}$$

where

$$B_3\left(u, v\right) = \int_{\Gamma^*} u \frac{\partial v}{\partial n} + 2w^2 \int_{\overline{AB}} uv - \alpha \int_{\overline{AB}} \left(\frac{\partial u}{\partial n} v + u \frac{\partial v}{\partial n} \right), \tag{4.28}$$

$$F_3\left(v\right) = 2w^2 \int_{\overline{AB}} 500v - \alpha \int_{\overline{AB}} 500 \frac{\partial v}{\partial n}. \tag{4.29}$$

First, let us consider how to choose the two parameters α and w above. The value of w must be chosen in such a way that the first two integrals in (3.17) are balanced. To this end, let us, for simplicity, restrict our consideration to a semicircular domain,

$$S_R = \left\{ (r, \theta) : 0 \leq r \leq R,\ 0 \leq \theta \leq \pi \right\}, \tag{4.30}$$

with boundary

$$\ell_R = \left\{ (R, \theta) : 0 \leq \theta \leq \pi \right\}, \tag{4.31}$$

for which the following result holds.

LEMMA 4.5. *Let ℓ_R be given by (4.31). Then, for any $v \in V_N$,*

$$\int_{\ell_R} v \frac{\partial v}{\partial n} \leq \frac{N+1}{R} \int_{\ell_R} v^2, \tag{4.32}$$

$$\int_{\ell_R} \left(\frac{\partial v}{\partial n} \right)^2 \leq \frac{N+1}{R^2} \int_{\ell_R} v^2. \tag{4.33}$$

PROOF. Since $v \in V_N$, we have

$$v = \sum_{i=1}^{N} \beta_i r^{(2i-1)/2} \cos\left[\left(\frac{2i-1}{2} \right) \theta \right] \tag{4.34}$$

with $\beta_i \in \mathbb{R}$. By direct calculation, using the orthogonality of trigonometric functions, we obtain

$$\int_{\ell_R} v^2 = \frac{\pi}{2} \sum_{i=1}^{N} \left(\beta_i\right)^2 R^{2i}, \tag{4.35}$$

$$\int_{\ell_R} v \frac{\partial v}{\partial n} = \frac{\pi}{2} \sum_{i=1}^{N} \left(\beta_i\right)^2 (2i-1) R^{2i-1}, \tag{4.36}$$

$$\int_{\ell_R} \left(\frac{\partial v}{\partial n} \right)^2 = \frac{\pi}{2} \sum_{i=1}^{N} \left(\beta_i\right)^2 (2i-1)^2 R^{2i-2}. \tag{4.37}$$

From (4.36), we get

$$\int_{\ell_R} v \frac{\partial v}{\partial n} = \frac{\pi}{2} \sum_{i=1}^{N} \left(\beta_i\right)^2 (2i-1) R^{2i-1} = \frac{1}{R} \frac{\pi}{2} \sum_{i=1}^{N} \left(\beta_i\right)^2 (2i-1) R^{2i} \leq \frac{N+1}{R} \frac{\pi}{2} \sum_{i=1}^{N} \left(\beta_i\right)^2 R^{2i},$$

which along with (4.35) gives (4.32). Similarly, from (4.37),

$$\begin{aligned}\int_{\ell_R}\left(\frac{\partial v}{\partial n}\right)^2 &= \frac{\pi}{2}\sum_{i=1}^{N}(\beta_i)^2(2i-1)^2R^{2i-2}\\ &= \frac{1}{R^2}\,\frac{\pi}{2}\sum_{i=1}^{N}(\beta_i)^2(2i-1)R^{2i}\\ &\le \frac{N+1}{R^2}\,\frac{\pi}{2}\sum_{i=1}^{N}(\beta_i)^2R^{2i},\end{aligned}$$

which along with (4.35) gives (4.33), and the proof is complete.

Guided by (4.32) in the above lemma, we return to our problem and choose $w^2 = C^*(N+1)$, where $C^* \in \mathbb{R}^+$ will be determined shortly. Moreover, in view of (4.33), we make the following assumption, $\exists\, C \in \mathbb{R}$ independent of N, such that

$$\left\|\frac{\partial v}{\partial n}\right\|_{0,\overline{AB}} \le C\,(N+1)\,\|v\|_{0,\overline{AB}}\,, \qquad \forall\, v \in V_N. \tag{4.38}$$

In what follows, we will obtain error bounds for this method in the norm

$$\begin{aligned}\|v\|_* &= \left(|v|^2_{1,\Omega} + w^2\,\|v\|^2_{0,\overline{AB}}\right)^{1/2}\\ &= \left(|v|^2_{1,\Omega} + C^*\,(N+1)\,\|v\|^2_{0,\overline{AB}}\right)^{1/2}.\end{aligned} \tag{4.39}$$

We have the following lemma.

LEMMA 4.6. *Suppose (4.38) holds. Then, for $\alpha \in (0,1]$ there exists $C^* \in \mathbb{R}$ independent of N, such that*

$$B_3\,(v,v) \ge \|v\|^2_*\,, \qquad \forall\, v \in V_N, \tag{4.40}$$

and

$$|B_3\,(u,v)| \le C\,\|u\|_*\,\|v\|_*\,, \qquad \forall\, u,v \in V_N, \tag{4.41}$$

with $C \in \mathbb{R}$ independent of N.

PROOF. Note that $B_3(u,v)$ given by (4.28) may be written as

$$B_3\,(u,v) = \iint_\Omega \nabla u\cdot\nabla v + 2C^*\,(N+1)\int_{\overline{AB}} uv - \alpha\int_{\overline{AB}}\left(\frac{\partial u}{\partial n}v + u\frac{\partial v}{\partial n}\right) \tag{4.42}$$

so that

$$B_3\,(v,v) = \iint_\Omega |\nabla v|^2 + 2C^*\,(N+1)\int_{\overline{AB}} v^2 - 2\alpha\int_{\overline{AB}}\frac{\partial v}{\partial n}v. \tag{4.43}$$

Using the Cauchy-Schwartz inequality and (4.38),

$$\int_{\overline{AB}}\frac{\partial v}{\partial n}v \le \left\|\frac{\partial v}{\partial n}\right\|_{0,\overline{AB}}\|v\|_{0,\overline{AB}} \le C\,(N+1)\,\|v\|^2_{0,\overline{AB}}\,. \tag{4.44}$$

Hence,

$$B_3\,(v,v) \ge \iint_\Omega |\nabla v|^2 + 2\,(C^* - C\alpha)\,(N+1)\,\|v\|^2_{0,\overline{AB}}\,, \tag{4.45}$$

where $C \in \mathbb{R}$ is the constant in (4.44). Choosing $C^* \in \mathbb{R}$ to satisfy $2\left(C^* - C\alpha\right) \geq C^*$, i.e.,

$$C^* \geq 2C\alpha \tag{4.46}$$

gives

$$B_3\left(v, v\right) \geq \iint_\Omega \left|\nabla v\right|^2 + C^*\left(N+1\right)\left\|v\right\|_{0,\overline{AB}}^2 = \left\|v\right\|_*^2, \tag{4.47}$$

which is precisely (4.40).

Next, we have

$$\left|B_3\left(u, v\right)\right| \leq \left|\iint_\Omega \nabla u \cdot \nabla v\right| + 2C^*\left(N+1\right)\left|\int_{\overline{AB}} uv\right| + \alpha\left(\left|\int_{\overline{AB}} \frac{\partial u}{\partial n} v\right| + \left|\int_{\overline{AB}} u\frac{\partial v}{\partial n}\right|\right). \tag{4.48}$$

Moreover,

$$\left|\int_{\overline{AB}} \frac{\partial u}{\partial n} v\right| \leq C\left\|u\right\|_{1,\Omega}\left\|v\right\|_{1,\Omega} \leq C\left\|u\right\|_*\left\|v\right\|_* \tag{4.49}$$

and similarly,

$$\left|\int_{\overline{AB}} \frac{\partial v}{\partial n} u\right| \leq C\left\|u\right\|_*\left\|v\right\|_*. \tag{4.50}$$

Combining (4.48)–(4.50), we get

$$\begin{aligned}\left|B_3\left(u, v\right)\right| &\leq \left|\iint_\Omega \nabla u \cdot \nabla v\right| + 2C^*\left(N+1\right)\left|\int_{\overline{AB}} uv\right| + C\alpha\left\|u\right\|_*\left\|v\right\|_* \\ &\leq \left|u\right|_{1,\Omega}\left|v\right|_{1,\Omega} + 2C^*\left(N+1\right)\left\|u\right\|_{0,\overline{AB}}\left\|v\right\|_{0,\overline{AB}} + C\alpha\left\|u\right\|_*\left\|v\right\|_* \\ &\leq \left(1 + C\alpha\right)\left\|u\right\|_*\left\|v\right\|_*,\end{aligned}$$

from which (4.41) follows.

Using the above lemma, we obtain the following result.

THEOREM 4.3. *Let $u_N^{PH} \in V_N$ satisfy (4.27) and $u \in H_M^1$ be the weak solution to (3.1)–(3.4). Assuming (4.38) holds, there exists a constant C, independent of N, such that*

$$\left\|u - u_N^{PH}\right\|_* \leq C\left\{\inf_{v \in V_N}\left\|u - v\right\|_* + \frac{\left|1-\alpha\right|}{\sqrt{C^*\left(N+1\right)}}\left\|\frac{\partial u}{\partial n}\right\|_{0,\overline{AB}}\right\}. \tag{4.51}$$

PROOF. With $u \in H_*^1$ the weak solution to (3.1)–(3.4), we have from (4.28),(4.29),

$$B_3(u, v) = (1-\alpha)\int_{\overline{AB}} \frac{\partial u}{\partial n} v + F_3(v), \tag{4.52}$$

so that using (4.27) and (4.39),

$$\begin{aligned}B_3(u - u_N^{PH}, v) &= (1-\alpha)\int_{\overline{AB}} \frac{\partial u}{\partial n} v \\ &\leq \left|1-\alpha\right|\left\|\frac{\partial u}{\partial n}\right\|_{0,\overline{AB}}\left\|v\right\|_{0,\overline{AB}} \\ &\leq C\frac{\left|1-\alpha\right|}{\sqrt{C^*\left(N+1\right)}}\left\|\frac{\partial u}{\partial n}\right\|_{0,\overline{AB}}\left\|v\right\|_*.\end{aligned}$$

With $v \in V_N$, let $\delta = \left(v - u_N^{PH}\right) \in V_N$. By Lemma 4.6,

$$\begin{aligned}\left\|\delta\right\|_*^2 \leq B_3(\delta, \delta) &= B_3(v - u_N^{PH}, \delta) = B_3(u - v, \delta) - (1-\alpha)\int_{\overline{AB}} \frac{\partial u}{\partial n}\delta \\ &\leq C\left\{\left\|u - v\right\|_*\left\|\delta\right\|_* + \frac{\left|1-\alpha\right|}{\sqrt{C^*\left(N+1\right)}}\left\|\frac{\partial u}{\partial n}\right\|_{0,\overline{AB}}\left\|\delta\right\|_*\right\}.\end{aligned}$$

Hence,

$$\|\delta\|_* \le C\left\{\|u-v\|_* + |1-\alpha|\,\frac{|1-\alpha|}{\sqrt{C^*(N+1)}}\left\|\frac{\partial u}{\partial n}\right\|_{0,\overline{AB}}\right\}. \tag{4.53}$$

Therefore, by the triangle inequality and (4.53),

$$\begin{aligned}\left\|u-u_N^{PH}\right\|_* &\le \|u-v\|_* + \left\|v-u_N^{PH}\right\|_* = \|u-v\|_* + \|\delta\|_* \\ &\le \|u-v\|_* + C\left\{\|u-v\|_* + \frac{|1-\alpha|}{\sqrt{C^*(N+1)}}\left\|\frac{\partial u}{\partial n}\right\|_{0,\overline{AB}}\right\},\end{aligned}$$

and the proof is complete.

Based on Theorem 4.3, we could choose the parameter $\alpha = 1$ in order to raise the accuracy of the method. In this case, we have the following theorem.

THEOREM 4.4. *Let the assumptions of Theorem 4.3 and (4.13),(4.14) hold, and choose $\alpha = 1$. Then, there exists a constant C independent of N, such that*

$$\left\|u-u_N^{PH}\right\|_* \le C\sqrt{N}a^N, \tag{4.54}$$

with $a \in (0,1)$.

PROOF. With $\alpha = 1$, we have from Theorem 4.3,

$$\left\|u-u_N^{PH}\right\|_* \le C \inf_{v\in V_N} \|u-v\|_*. \tag{4.55}$$

Letting $u = \overline{u}_N + r_N$ with $\overline{u}_N$ and r_N given by (4.11) and (4.12), we further have

$$\left\|u-u_N^{PH}\right\|_* \le C\|r_N\|_* = C\left(|r_N|_{1,\Omega}^2 + C^*(N+1)\|r_N\|_{0,\overline{AB}}^2\right)^{1/2}, \tag{4.56}$$

and by (4.13),

$$\left\|u-u_N^{PH}\right\|_* \le C\left(|r_N|_{1,\Omega}^2 + Na^{2N}\right)^{1/2} \le C\left(|r_N|_{1,\Omega} + \sqrt{N}a^N\right). \tag{4.57}$$

It remains to bound $|r_N|_{1,\Omega}$ in (4.57). By (4.7), (4.13), and (4.14), we have

$$|r_N|_{1,\Omega}^2 = \int_{\Gamma^*}\frac{\partial v}{\partial n}v \le C\left\|\frac{\partial r_N}{\partial n}\right\|_{0,\overline{AB}}\|r_N\|_{0,\overline{AB}} \le CNa^{2N} \tag{4.58}$$

so that combined with (4.57) gives the desired result.

We should point out that the parameter $w^2 = C^*(N+1)$ includes the constant C^* satisfying (4.46); in practice it turn outs that simply choosing $C^* = 1$ suffices, as observed in the numerical computations of Section 5.

4.4. The BAM with Lagrange Multipliers

When the Dirichlet condition $u|_{\overline{AB}} = 500$ is regarded as a constraint, the solution to the Motz problem may be obtained by minimizing the (not positive definite) functional $I_L(v)$ given by (3.19), or equivalently by solving the variational problem given by (3.20). While for the implementation of the method (3.20) is used, for the analysis it is often convenient to state the variational problem as follows. Find $(u,\lambda) \in H_M^1(\Omega)\times H^{-1/2}(\overline{AB})$, such that

$$B(u,v) + G(u,v;\lambda,\mu) = 0, \qquad \forall\,(v,\mu) \in H_M^1(\Omega)\times H^{-1/2}\left(\overline{AB}\right), \tag{4.59}$$

where

$$B(u,v) = \iint_{\Omega} \nabla v \cdot \nabla u, \tag{4.60}$$

$$G(u,v;\lambda,\mu) = -\int_{\overline{AB}} v\lambda - \int_{\overline{AB}} \mu(u-500). \tag{4.61}$$

For the discretization, we divide $\overline{AB}$ into sections Γ_i, $i = 1,\ldots,n$, such that

$$\overline{AB} = \bigcup_{i=1}^{n} \Gamma_i, \qquad h_i = |\Gamma_i|, \quad h = \max_{1\le i\le n} h_i. \tag{4.62}$$

With $\mathcal{P}_k(\overline{AB})$ the space of polynomials of degree $\le k$ on $\overline{AB}$, we define

$$\Lambda_h = \{\lambda_h : \lambda_h|_{\Gamma_i} \in \mathcal{P}_k(\Gamma_i),\ i = 1,\ldots,n\}. \tag{4.63}$$

Then, the discrete version of (4.59) reads. Find $(u_N^L, \lambda_h) \in V_N \times \Lambda_h$, such that

$$B(u_N^L, v) + G(u_N^L, v; \lambda_h, \mu) = 0, \qquad \forall\ (v,\mu) \in V_N \times \Lambda_h. \tag{4.64}$$

The present method was first introduced in [6] and was subsequently used to efficiently solve Laplacian problems in domains with boundary singularities (cf., [4,5]). Below, we give a brief justification for the method, as it pertains to the Motz problem. We begin with the following theorem from [1].

Theorem 4.5. *Let (u,λ) and (u_N^L, λ_h) be the solutions to (4.59) and (4.64), respectively. Suppose there exist positive constants c_0, c, β, and γ, independent of N and h, such that the following assumptions hold,*

$$B(v,v) \ge c_0 \|v\|_{1,\Omega}^2 \qquad \text{and} \qquad |B(u,v)| \le c\, \|u\|_{1,\Omega} \|v\|_{1,\Omega}, \quad \forall\, v \in V_N, \tag{4.65}$$

$$\exists\, 0 \ne v_N \in V_N, \qquad \text{such that } \left| \int_{\overline{AB}} \mu_h v_N \right| \ge \beta \|\mu_h\|_{-1/2,\overline{AB}} \|v_N\|_{1,\Omega}, \quad \forall\, \mu_h \in \Lambda_h, \tag{4.66}$$

$$\left| \int_{\overline{AB}} \lambda v \right| \le \gamma \|\lambda\|_{-1/2,\overline{AB}} \|v\|_{1,\Omega}, \qquad \forall\, v_N \in V_N. \tag{4.67}$$

Then,

$$\left\|u - u_N^L\right\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-1/2,\overline{AB}} \le C \left\{ \inf_{v\in V_N} \|u - v\|_{1,\Omega} + \inf_{\eta\in\Lambda_h} \|\lambda - \eta\|_{-1/2,\overline{AB}} \right\}, \tag{4.68}$$

with $C \in \mathbb{R}^+$ independent of N and h.

Proof. For a proof, see Theorem 6.1 in [1].

Let us verify that (4.67)–(4.67) hold for our problem. First, note that $B(v,v) = |v|_{1,\Omega}^2$ so that, by Poincaré's inequality,

$$B(v,v) \ge c_0 \|v\|_{1,\Omega}^2, \qquad \forall\, v \in H_M^1(\Omega). \tag{4.69}$$

By the Cauchy-Schwartz inequality,

$$B(u,v) \le c\, \|u\|_{1,\Omega} \|v\|_{1,\Omega}, \qquad \forall\, u,v \in H_M^1(\Omega), \tag{4.70}$$

so that (4.69) and (4.70) give (4.65).

To verify (4.66), consider the following auxiliary problem. Find $w \in H_M^1(\Omega)$, such that

$$\Delta w = 0 \text{ in } \Omega, \tag{4.71}$$

$$\frac{\partial w}{\partial n} = \mu_h \text{ on } \overline{AB}, \tag{4.72}$$

$$w = 0 \text{ on } \overline{OD}, \tag{4.73}$$

$$\frac{\partial w}{\partial n} = 0 \text{ on } \overline{O}A \cup \overline{BC} \cup \overline{CD}, \tag{4.74}$$

where $\mu_h \in \Lambda_h$ in (4.72). From (4.71)–(4.72) we obtain, using Green's formula and Poincaré's inequality,

$$\int_{\overline{AB}} \mu_h w \equiv \int_{\overline{AB}} w \frac{\partial w}{\partial n} = \iint_\Omega w \Delta w + \iint_\Omega |\nabla w|^2 = |w|_{1,\Omega}^2 \geq c_0 \|w\|_{1,\Omega}^2 , \tag{4.75}$$

with $c_0 \in \mathbb{R}^+$. Also,

$$\|\mu_h\|_{-1/2,\overline{AB}} = \left\| \frac{\partial w}{\partial n} \right\|_{-1/2,\overline{AB}} \leq C \|w\|_{1,\Omega} , \tag{4.76}$$

so that by (4.75),(4.76),

$$\int_{\overline{AB}} \mu_h w \geq c_0 \|w\|_{1,\Omega}^2 \geq \beta \|w\|_{1,\Omega} \|\mu_h\|_{-1/2,\overline{AB}} , \tag{4.77}$$

with $\beta \in \mathbb{R}^+$ independent of w and h. Now, let $w_N \in V_N$ be such that $w = w_N + r_N$ with $r_N \in H_M^1(\Omega)$ the remainder (see (4.11)–(4.14)). We have

$$\int_{\overline{AB}} \mu_h w_N = \int_{\overline{AB}} \mu_h w - \int_{\overline{AB}} \mu_h r_N \tag{4.78}$$

and also

$$\int_{\overline{AB}} \mu_h r_N \leq \|\mu_h\|_{-1/2,\overline{AB}} \|r_N\|_{1/2,\overline{AB}} \leq C_1 \|\mu_h\|_{-1/2,\overline{AB}} \|r_N\|_{1,\Omega} , \tag{4.79}$$

so that combining (4.77)–(4.79), we get

$$\int_{\overline{AB}} \mu_h w_N \geq \beta \|w\|_{1,\Omega} \|\mu_h\|_{-1/2,\overline{AB}} - C_1 \|\mu_h\|_{-1/2,\overline{AB}} \|r_N\|_{1,\Omega} . \tag{4.80}$$

Now, using

$$\|w\|_{1,\Omega} = \|w_N + r_N\|_{1,\Omega} \geq \|w_N\|_{1,\Omega} - \|r_N\|_{1,\Omega}$$

along with (4.80), we obtain

$$\begin{aligned} \int_{\overline{AB}} \mu_h w_N &\geq \beta \left(\|w_N\|_{1,\Omega} - \|r_N\|_{1,\Omega} \right) \|\mu_h\|_{-1/2,\overline{AB}} - C_1 \|\mu_h\|_{-1/2,\overline{AB}} \|r_N\|_{1,\Omega} \\ &\geq \beta \|w_N\|_{1,\Omega} \|\mu_h\|_{-1/2,\overline{AB}} - (C_1 + \beta) \|\mu_h\|_{-1/2,\overline{AB}} \|r_N\|_{1,\Omega} . \end{aligned} \tag{4.81}$$

Since, by assumptions (4.13),(4.14), w_N converges to w exponentially, we have

$$0 < \frac{\|r_N\|_{1,\Omega}}{\|w_N\|_{1,\Omega}} < 1.$$

For N sufficiently large, we may write

$$\frac{\|r_N\|_{1,\Omega}}{\|w_N\|_{1,\Omega}} \leq \frac{\beta}{2(C_1 + \beta)}, \tag{4.82}$$

where C_1 and β are the positive constants from above. Combining (4.81) and (4.82), we have

$$\int_{\overline{AB}} \mu_h w_N \geq \frac{\beta}{2} \|\mu_h\|_{-1/2,\overline{AB}} \|w_N\|_{1,\Omega} ,$$

which gives (4.66) once we replace w_N by v_N and $\beta/2$ by β.

Condition (4.67) follows in a similar fashion; see, e.g., (4.79). The preceding discussion leads to the following theorem.

THEOREM 4.6. *Let (u, λ) and (u_N^L, λ_h) be the solutions to (4.59) and (4.64), respectively, and suppose (4.13) and (4.14) hold. Then, if $\lambda \in H^{k+1}(\overline{AB})$, there exists $C \in \mathbb{R}^+$ independent of N and h, such that*

$$\left\|u - u_N^L\right\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-1/2,\overline{AB}} \leq C\left\{\sqrt{N}a^N + h^{k+1}\right\},$$

where $a \in (0,1)$ and h is given by (4.62).

PROOF. From Theorem 4.5, we have

$$\left\|u - u_N^L\right\|_{1,\Omega} + \|\lambda - \lambda_h\|_{-1/2,\overline{AB}} \leq C\left\{\inf_{v \in V_N} \|u - v\|_{1,\Omega} + \inf_{\eta \in \Lambda_h} \|\lambda - \eta\|_{-1/2,\overline{AB}}\right\}. \tag{4.83}$$

Now,

$$\inf_{v \in V_N} \|u - v\|_{1,\Omega} \leq \|u - \overline{u}_N\|_{1,\Omega} = \|r_N\|_{1,\Omega},$$

with $\overline{u}_N$, r_N given by (4.11) and (4.12), respectively. Using (4.13) and (4.14), we get

$$\inf_{v \in V_N} \|u - v\|_{1,\Omega} \leq C\sqrt{N}a^N, \tag{4.84}$$

with $C \in \mathbb{R}^+$ independent of N.

Next, let $\lambda_I \in \Lambda_h$ be the k^{th}-order interpolant of λ. Then, since $\lambda \in H^{k+1}\left(\overline{AB}\right)$, we have

$$\inf_{\eta \in \Lambda_h} \|\lambda - \eta\|_{-1/2,\overline{AB}} \leq \|\lambda - \lambda_I\|_{-1/2,\overline{AB}} \leq C\,\|\lambda - \lambda_I\|_{0,\overline{AB}} \leq Ch^{k+1}$$

which along with (4.84) gives the desired result.

Based on the above theorem, one may obtain the *optimal matching* between N and h, i.e., the relationship between the number of singular functions and the number of Lagrange multipliers used in the method, by choosing $h^{k+1} \sim \sqrt{N}a^N$. This leads to the following approximate expression for N,

$$N \approx (k+1)\left|\frac{\ln h}{\ln a}\right|. \tag{4.85}$$

5. NUMERICAL RESULTS AND DISCUSSION

In this section, new numerical results for the Motz problem (3.1)–(3.4) obtained using the hybrid and the penalty/hybrid BAMs are presented and discussed in connection with the error analyses of Section 4. Comparisons are also made with the results obtained with the classic BAM of Li *et al.* [7] and the BAM with Lagrange multipliers of Georgiou *et al.* [6]. Due to the low efficiency of the penalty BAM (see Section 4.1), no results of this method are presented.

Obtaining accurate estimates of the leading singular coefficients, a_i, is the main goal of all these special methods. Tables 1–3 list the singular coefficients a_i^{35}, $i = 1, \ldots, 35$, obtained using the classic, hybrid, and penalty/hybrid BAMs, respectively (with $N = 35$). For comparison purposes, we list in Table 4 the most accurate singular coefficients calculated by the BAM with Lagrange multipliers in [6], using a much larger number of singular functions, i.e., $N = 75$, and 33 discrete Lagrange multipliers, i.e., $N_\lambda = 33$. Note that, in this method, N should be much greater than N_λ in order to obtain satisfactory convergence of the leading singular coefficients. In Table 5, we list the numbers of the converged significant digits of the leading 19 singular coefficients for all four methods, as calculated by Li and Lu [18], using the conformal transformation method (CTM) of Whiteman and Papamichael [19]. We observe that the four BAMs yield very accurate estimates of the singular coefficients. For $i = 1, 2, 3$, the classic BAM gives one less significant digit than the other three BAMs, while for the higher coefficients all methods yield about the same number of converged significant digits. The BAM with Lagrange multipliers has a slight advantage as i

Table 1. Computed singular coefficients with the classic BAM for $N = 35$.

i	a_i^N	i	a_i^N
1	$0.40116245374497 \times 10^{3}$	19	$0.11534855091605 \times 10^{-4}$
2	$0.87655920195502 \times 10^{2}$	20	$-0.52932746412879 \times 10^{-5}$
3	$0.17237915079248 \times 10^{2}$	21	$0.22897323500171 \times 10^{-5}$
4	$-0.80712152596499 \times 10^{1}$	22	$0.10624097261554 \times 10^{-5}$
5	$0.14402727170434 \times 10^{1}$	23	$0.53073158247781 \times 10^{-6}$
6	$0.33105488588606 \times 10^{0}$	24	$-0.24510085058588 \times 10^{-6}$
7	$0.27543734452816 \times 10^{0}$	25	$0.10862672983328 \times 10^{-6}$
8	$-0.86932994509462 \times 10^{-1}$	26	$0.51043248247979 \times 10^{-7}$
9	$0.33604878399124 \times 10^{-1}$	27	$0.25407074732821 \times 10^{-7}$
10	$0.15384374465022 \times 10^{-1}$	28	$-0.11054833875475 \times 10^{-7}$
11	$0.73023016452998 \times 10^{-2}$	29	$0.49285560339473 \times 10^{-8}$
12	$-0.31841136217467 \times 10^{-2}$	30	$0.23304869676739 \times 10^{-8}$
13	$0.12206458571187 \times 10^{-2}$	31	$0.11523150093507 \times 10^{-8}$
14	$0.53096530065606 \times 10^{-3}$	32	$-0.34653285095421 \times 10^{-9}$
15	$0.27151202841413 \times 10^{-3}$	33	$0.15243365277043 \times 10^{-9}$
16	$-0.12004506715157 \times 10^{-3}$	34	$0.72493901550694 \times 10^{-10}$
17	$0.50538906322972 \times 10^{-4}$	35	$0.35291922501256 \times 10^{-10}$
18	$0.23166270362346 \times 10^{-4}$		

Table 2. Computed singular coefficients with the hybrid BAM for $N = 35$.

i	a_i^N	i	a_i^N
1	$0.401162453745250 \times 10^{3}$	19	$0.115343772789621 \times 10^{-4}$
2	$0.876559201951038 \times 10^{2}$	20	$-0.529380676633001 \times 10^{-5}$
3	$0.172379150794574 \times 10^{2}$	21	$0.228969115585334 \times 10^{-5}$
4	$-0.807121525969505 \times 10^{1}$	22	$0.106202202610555 \times 10^{-5}$
5	$0.144027271701729 \times 10^{1}$	23	$0.530229339048478 \times 10^{-6}$
6	$0.331054885909148 \times 10^{0}$	24	$-0.245459749591207 \times 10^{-6}$
7	$0.275437344500486 \times 10^{0}$	25	$0.108590887362510 \times 10^{-6}$
8	$-0.869329945171928 \times 10^{-1}$	26	$0.508138311029889 \times 10^{-7}$
9	$0.336048783999441 \times 10^{-1}$	27	$0.251496766940829 \times 10^{-7}$
10	$0.153843744418389 \times 10^{-1}$	28	$-0.111642374722729 \times 10^{-7}$
11	$0.730230161393995 \times 10^{-2}$	29	$0.491554865658322 \times 10^{-8}$
12	$-0.318411372788438 \times 10^{-2}$	30	$0.226743542107491 \times 10^{-8}$
13	$0.122064584771336 \times 10^{-2}$	31	$0.109000401834271 \times 10^{-8}$
14	$0.530965184801430 \times 10^{-3}$	32	$-0.358701765271215 \times 10^{-9}$
15	$0.271511819668155 \times 10^{-3}$	33	$0.150813240028775 \times 10^{-9}$
16	$-0.120045429073067 \times 10^{-3}$	34	$0.660571911959434 \times 10^{-10}$
17	$0.505388854473519 \times 10^{-4}$	35	$0.296216590328091 \times 10^{-10}$
18	$0.231659564580221 \times 10^{-4}$		

Table 3. Computed singular coefficients with the penalty/hybrid BAM for $N = 35$.

i	a_i^N	i	a_i^N
1	$0.401162453745202 \times 10^{3}$	19	$0.1153491708827968 \times 10^{-4}$
2	$0.876559201951031 \times 10^{2}$	20	$-0.5293654843369576 \times 10^{-5}$
3	$0.172379150794664 \times 10^{2}$	21	$0.2290138896618886 \times 10^{-5}$
4	$-0.807121525968356 \times 10^{1}$	22	$0.1062509190385607 \times 10^{-5}$
5	$0.144027271701729 \times 10^{1}$	23	$0.5308058949628783 \times 10^{-6}$
6	$0.331054885895757 \times 10^{0}$	24	$-0.2453536905374091 \times 10^{-6}$
7	$0.275437344521521 \times 10^{0}$	25	$0.1088807854053806 \times 10^{-6}$
8	$-0.8693299450621651 \times 10^{-1}$	26	$0.5111717330707535 \times 10^{-7}$
9	$0.3360487842325408 \times 10^{-1}$	27	$0.2545239239069238 \times 10^{-7}$
10	$0.1538437441454227 \times 10^{-1}$	28	$-0.1112961949757686 \times 10^{-7}$
11	$0.7302301661989898 \times 10^{-2}$	29	$0.5001877506926354 \times 10^{-8}$
12	$-0.3184113682966637 \times 10^{-2}$	30	$0.2353670227283211 \times 10^{-8}$
13	$0.1220645960584796 \times 10^{-2}$	31	$0.1165476462446361 \times 10^{-8}$
14	$0.5309652666820730 \times 10^{-3}$	32	$-0.3545390456290663 \times 10^{-9}$
15	$0.2715120554917799 \times 10^{-3}$	33	$0.1603064746407727 \times 10^{-9}$
16	$-0.1200453186155349 \times 10^{-3}$	34	$0.7511467779109671 \times 10^{-10}$
17	$0.5053921174507389 \times 10^{-4}$	35	$0.3672896632385438 \times 10^{-10}$
18	$0.2316630831563956 \times 10^{-4}$		

Table 4. Computed singular coefficients with the BAM with Lagrange multipliers [6] for $N = 75$ and $N_\lambda = 33$ (only the first 36 coefficients are listed).

i	a_i^N	i	a_i^N
1	$.401162453745234 \times 10^{3}$	19	$.115352825403054 \times 10^{-4}$
2	$.876559201950877 \times 10^{2}$	20	$-.529575461575406 \times 10^{-5}$
3	$.172379150794469 \times 10^{2}$	21	$.229103011774740 \times 10^{-5}$
4	$-.807121525969814 \times 10^{1}$	22	$.106349634823553 \times 10^{-5}$
5	$.144027271702291 \times 10^{1}$	23	$.531399419800137 \times 10^{-6}$
6	$.331054885920656 \times 10^{0}$	24	$-.247423064850164 \times 10^{-6}$
7	$.275437344509193 \times 10^{0}$	25	$.108706636458335 \times 10^{-6}$
8	$-.869329945252286 \times 10^{-1}$	26	$.529296106984506 \times 10^{-7}$
9	$.336048784263123 \times 10^{-1}$	27	$.264253479339111 \times 10^{-7}$
10	$.153843744820525 \times 10^{-1}$	28	$-.120550254504250 \times 10^{-7}$
11	$.730230167439347 \times 10^{-2}$	29	$.116026519978975 \times 10^{-8}$
12	$-.318411391508881 \times 10^{-2}$	30	$.622763895228202 \times 10^{-8}$
13	$.122064610746985 \times 10^{-2}$	31	$.332311983973516 \times 10^{-8}$
14	$.530965479850461 \times 10^{-3}$	32	$.554937941399033 \times 10^{-9}$
15	$.271512187507913 \times 10^{-3}$	33	$-.107137722721491 \times 10^{-7}$
16	$-.120046373993572 \times 10^{-3}$	34	$.719736757310813 \times 10^{-8}$
17	$.505398053367447 \times 10^{-4}$	35	$.432710661454326 \times 10^{-8}$
18	$.231668535028465 \times 10^{-4}$	36	$.405044840445786 \times 10^{-8}$

Table 5. Numbers of converged significant digits in a_i^N, $i = 1, \dots, 19$ for four BAMs.

i	Classic $N = 35$	Hybrid $N = 35$	Penalty/Hybrid $N = 35$	Lagrange multipliers $N = 75$, $N_\lambda = 33$
1	12	13	13	13
2	11	12	12	12
3	11	12	12	12
4	11	12	11	11
5	11	12	11	11
6	10	10	10	10
7	10	11	10	10
8	9	9	9	9
9	9	9	10	9
10	8	9	9	9
11	8	8	8	8
12	7	7	7	8
13	7	7	7	8
14	6	6	6	7
15	6	6	6	7
16	5	5	5	6
17	5	5	5	5
18	5	5	5	5
19	5	4	5	5

increases, but it should be kept in mind that the number of singular functions is much higher ($N = 75$ instead of 35). Moreover, the implementation of the method is more difficult.

In addition to the convergence of the singular coefficients, we have also investigated the effect of the number N of the singular functions on the error,

$$\epsilon = u - u_N,$$

where u corresponds to a reference solution calculated using the extremely accurate results in [18] and u_N denotes the approximate solution, and on the condition number of the matrix associated with the linear system arising from each method. The following error norms have been considered,

$$|\epsilon|_{0,\Omega} \doteq \left\{ \iint_\Omega (u - u_N)^2 ds \right\}^{1/2}, \tag{5.1}$$

$$|\epsilon|_{1,\Omega} \doteq \left\{ \iint_\Omega |\nabla(u - u_N)|^2 ds \right\}^{1/2} = \int_{\Gamma^*} \left\{ (u - u_N) \frac{\partial(u - u_N)}{\partial n} d\ell \right\}^{1/2}, \tag{5.2}$$

$$|\epsilon|_{\infty,\overline{AB}} \doteq \max_{\overline{AB}} |\epsilon|, \tag{5.3}$$

$$\left|\frac{\partial \epsilon}{\partial n}\right|_{\infty,\overline{BC}} \doteq \max_{\overline{BC}} \left|\frac{\partial \epsilon}{\partial n}\right|, \tag{5.4}$$

$$\left|\frac{\partial \epsilon}{\partial n}\right|_{\infty,\overline{CD}} \doteq \max_{\overline{CD}} \left|\frac{\partial \epsilon}{\partial n}\right|. \tag{5.5}$$

For the computations with the penalty/hybrid BAM, we choose $\alpha = 1$ and $w^2 = C^*(N+1)$ with $C^* = 1$; (cf., (4.46) and the discussion at the end of Section 4.3). The matrix $A_{PH} \in \mathbb{R}^{N \times N}$ of the linear system that arises from (4.27), is symmetric and positive definite (provided (4.26) holds), and its condition number, κ, is given as the ratio of the maximum to the minimum eigenvalue,

$$\kappa\,(A_{PH}) = \frac{\lambda_{\max}(A_{\mathrm{PH}})}{\lambda_{\min}(A_{\mathrm{PH}})}. \tag{5.6}$$

Table 6. Error norms and condition numbers for the classic, the hybrid, and the penalty/hybrid BAMs for different values of N.

Classic BAM						
N	$\left\|\frac{\partial\epsilon}{\partial n}\right\|_{\infty,\overline{BC}}$	$\left\|\frac{\partial\epsilon}{\partial n}\right\|_{\infty,\overline{CD}}$	$\|\epsilon\|_{\infty,\overline{AB}}$	$\|\epsilon\|_{0,\Omega}$	$\|\epsilon\|_{1,\Omega}$	$\kappa(F)$
11	0.327×10^{0}	0.296×10^{0}	0.795×10^{-2}	0.216×10^{-1}	0.936×10^{-1}	0.106×10^{2}
19	0.328×10^{-2}	0.313×10^{-2}	0.658×10^{-4}	0.288×10^{-3}	0.901×10^{-3}	0.225×10^{4}
27	0.354×10^{-4}	0.366×10^{-4}	0.606×10^{-6}	0.761×10^{-5}	0.114×10^{-4}	0.431×10^{5}
35	0.387×10^{-7}	0.445×10^{-7}	0.596×10^{-8}	0.248×10^{-7}	0.175×10^{-6}	0.787×10^{6}
Hybrid BAM						
N	$\left\|\frac{\partial\epsilon}{\partial n}\right\|_{\infty,\overline{BC}}$	$\left\|\frac{\partial\epsilon}{\partial n}\right\|_{\infty,\overline{CD}}$	$\|\epsilon\|_{\infty,\overline{AB}}$	$\|\epsilon\|_{0,\Omega}$	$\|\epsilon\|_{1,\Omega}$	$\kappa(A_H)$
11	0.400×10^{0}	0.551×10^{0}	0.397×10^{-1}	0.176×10^{-1}	0.759×10^{-1}	0.753×10^{3}
19	0.524×10^{-2}	0.675×10^{-2}	0.258×10^{-3}	0.280×10^{-3}	0.844×10^{-3}	0.184×10^{6}
27	0.719×10^{-4}	0.850×10^{-4}	0.222×10^{-5}	0.759×10^{-5}	0.125×10^{-4}	0.464×10^{8}
35	0.883×10^{-7}	0.110×10^{-6}	0.196×10^{-7}	0.286×10^{-7}	0.210×10^{-6}	0.118×10^{11}
Penalty/Hybrid BAM						
N	$\left\|\frac{\partial\epsilon}{\partial n}\right\|_{\infty,\overline{BC}}$	$\left\|\frac{\partial\epsilon}{\partial n}\right\|_{\infty,\overline{CD}}$	$\|\epsilon\|_{\infty,\overline{AB}}$	$\|\epsilon\|_{0,\Omega}$	$\|\epsilon\|_{1,\Omega}$	$\kappa(A_PH)$
11	0.461×10^{0}	0.512×10^{0}	0.143×10^{-1}	0.175×10^{-1}	0.361×10^{-1}	0.104×10^{4}
19	0.605×10^{-2}	0.604×10^{-2}	0.120×10^{-3}	0.281×10^{-3}	0.680×10^{-3}	0.251×10^{6}
27	0.815×10^{-4}	$.749 \times 10^{-4}$	0.123×10^{-5}	0.760×10^{-5}	0.139×10^{-4}	0.628×10^{8}
35	0.101×10^{-5}	0.944×10^{-6}	0.141×10^{-7}	0.177×10^{-7}	0.302×10^{-6}	0.159×10^{11}

In the hybrid BAM, the matrix $A_H \in \mathbb{R}^{N\times N}$ of the linear system arising from (4.22) is positive definite, but not symmetric. Hence, the condition number is calculated as follows,

$$\kappa\left(A_H\right) = \frac{\sqrt{\lambda_{\max}\left(A_H^T A_H\right)}}{\sqrt{\lambda_{\min}\left(A_H^T A_H\right)}}. \tag{5.7}$$

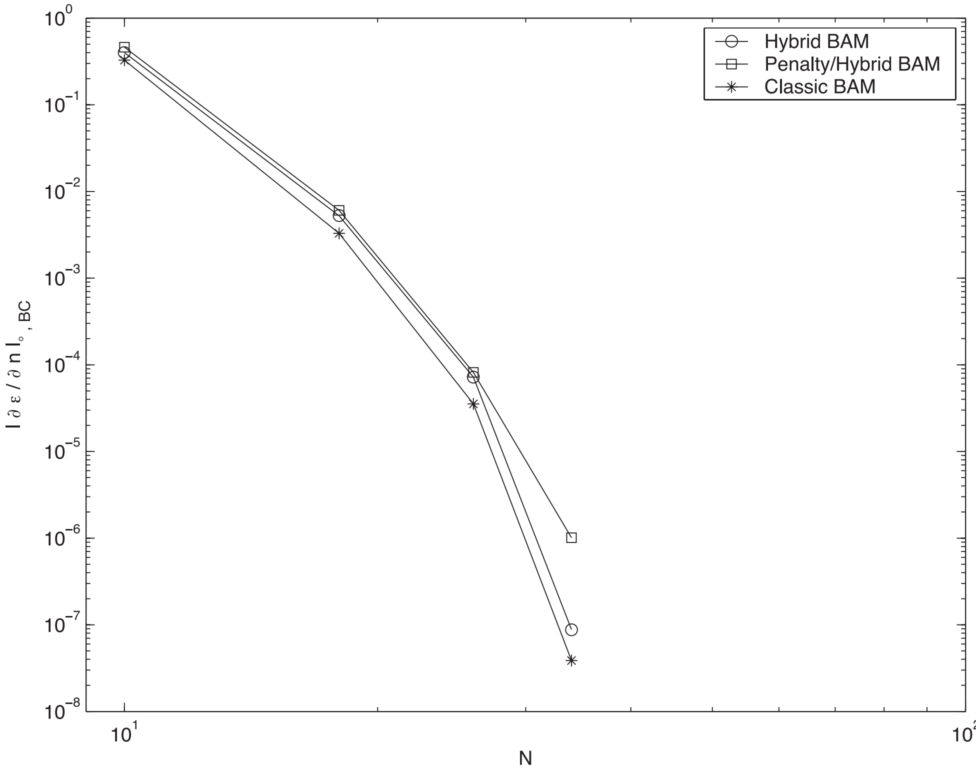
As described in [7], in the classic BAM, the side $\overline{AB}$ is divided into M equally spaced pieces of width $h = 1/M$. The direct collocation method is used to impose the boundary conditions (3.2),(3.3). The condition number of the matrix $F \in \mathbb{R}^{(4M)\times(N+1)}$ of the resulting linear system is given by (5.7), with F replacing A_H. Note that since $4M \gg N+1$, the least squares method is used to solve the linear system.

The variations of the error norms (5.1)–(5.5) with N, as well as the condition numbers obtained using the classic, the hybrid, and the penalty/hybrid BAMs, are tabulated in Table 6. These are presented graphically in Figure 2, where the exponential convergence rates established in Section 4 are readily visible. Upon careful examination of the numbers given in Table 6, we see that for the classic BAM, we have

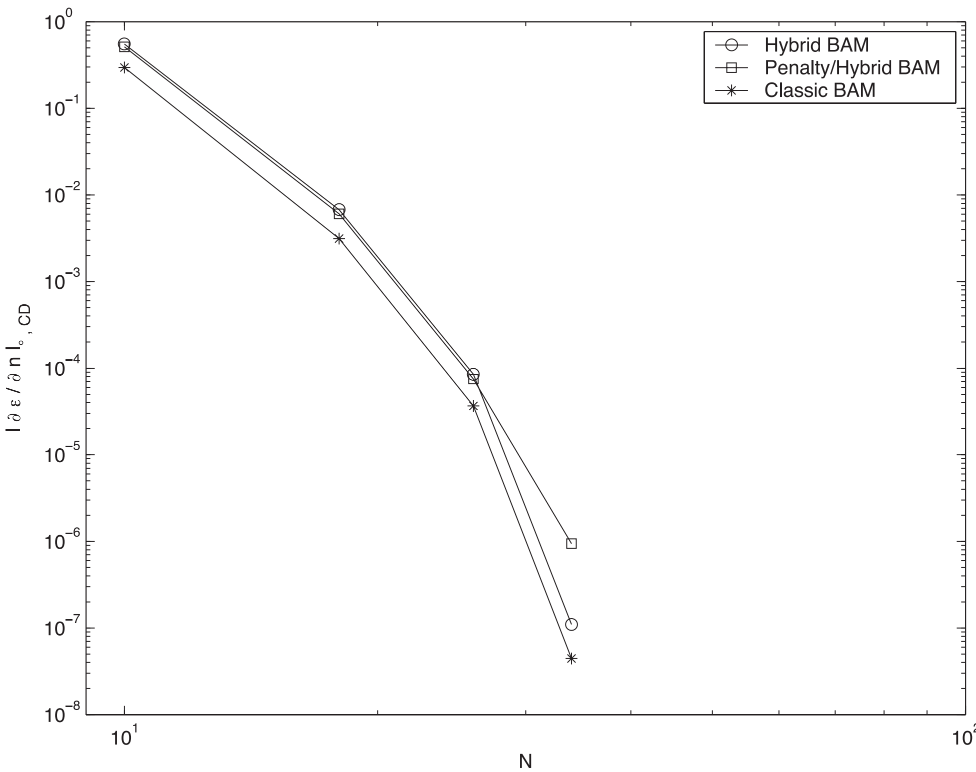
$$\begin{aligned} |u-u_N|_{\infty,\overline{AB}} &\to 2.8 \times 0.55^N, \\ |\epsilon|_{0,\Omega} &\to 5.0 \times 0.57^N, \\ |\epsilon|_{1,\Omega} &\to 19.5 \times 0.58^N, \end{aligned}$$

while for the hybrid BAM,

$$\begin{aligned} |u-u_N|_{\infty,\overline{AB}} &\to 3.5 \times 0.55^N, \\ |\epsilon|_{0,\Omega} &\to 4.9 \times 0.57^N, \\ |\epsilon|_{1,\Omega} &\to 3.5 \times 0.59^N, \end{aligned}$$

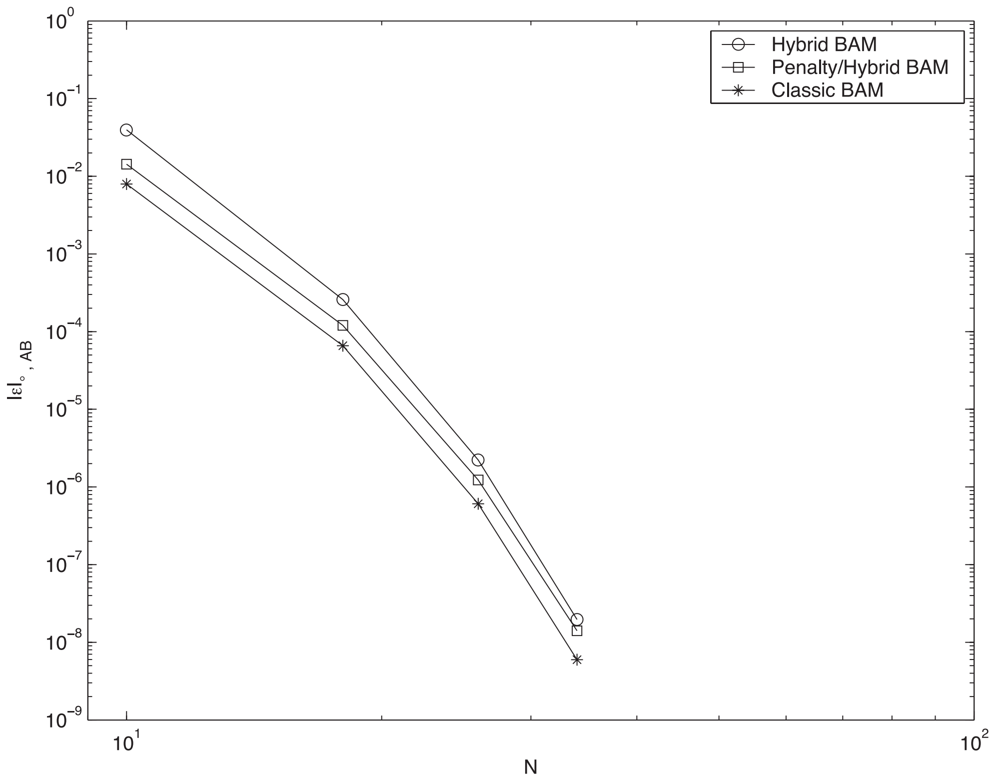


(a)

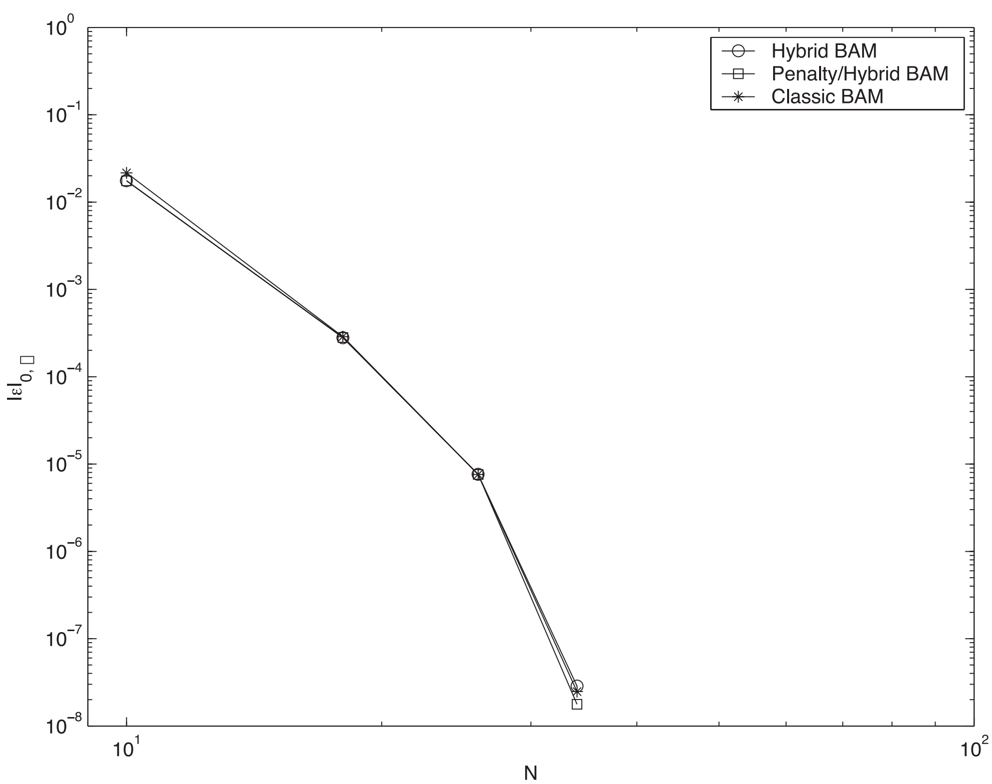


(b)

Figure 2. Convergence and variation of the condition numbers with N when using the hybrid, the penalty/hybrid, and the classic BAMs. The error estimates are defined by (5.1)–(5.5).

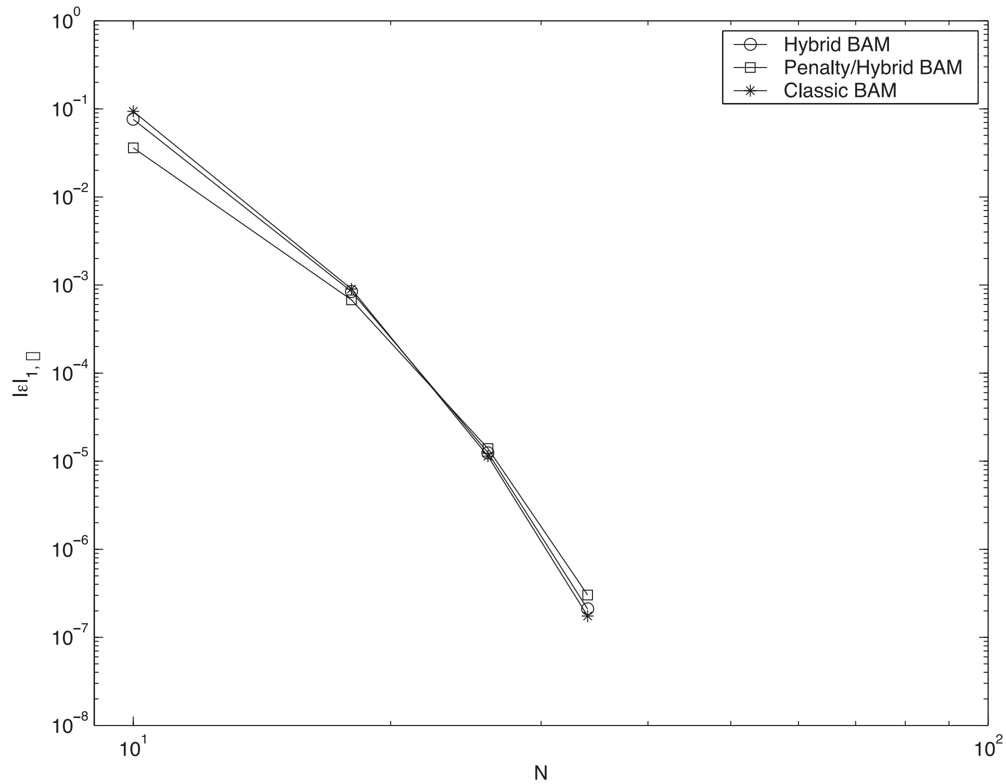


(c)

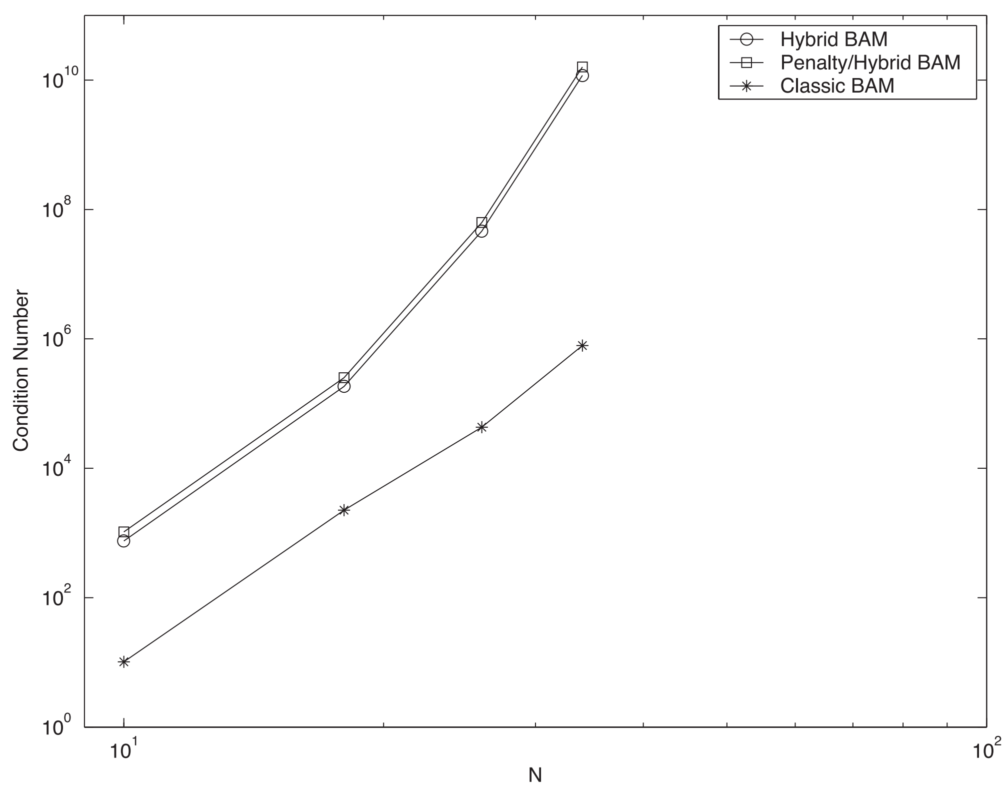


(d)

Figure 2. (cont.)



(e)



(f)

Figure 2. (cont.)

and for for the penalty/hybrid BAM,

$$|u - u_N|_{\infty,\overline{AB}} \to 6.7 \times 0.54^N,$$
$$|\epsilon|_{0,\Omega} \to 5.8 \times 0.56^N,$$
$$|\epsilon|_{1,\Omega} \to 5.1 \times 0.61^N.$$

It appears that the classic BAM slightly outperforms the other two, when the errors (5.1)–(5.5) are of interest.

As for the condition numbers, we have

$$\kappa(F) \to 0.09 \times 1.60^N,$$
$$\kappa(A_H) \to 0.7 \times 1.99^N,$$
$$\kappa(A_{\rm PH}) \to 1.1 \times 1.99^N.$$

Therefore, the condition number for the classic BAM grows at a *significantly slower* rate than those of the other two BAMs, which is also evident in Figure 2. Hence, in terms of numerical stability, the classic BAM is to be preferred.

In summary, when compared to the classic BAM, the hybrid and the penalty/hybrid BAMs may yield slightly more accurate estimates for the singular coefficients, but their performance is slightly worse in terms of the error norms (5.1)–(5.5), due to the ill-conditioning of the matrices associated with the corresponding linear systems.

Finally, we wish to make a short remark on the choice $N = 35$ in our numerical experiments. Take, for example, the hybrid BAM for which we have $|\epsilon|_{1,\Omega} \approx (|\epsilon|^2_{0,\Omega} + |\epsilon|^2_{1,\Omega})^{1/2} = 0.211 \times 10^{-6}$ (see Table 6). Since the true solution satisfies $|u|_{1,\Omega} = O(10^2)$, the relative errors in the H^1 norm reach $O(10^{-9})$, whereas the condition number reaches $O(10^{10})$! It is clear that 16-decimal-digit accuracy allowed by the double-precision arithmetic is reached when $N = 35$. For $N > 35$, the increasing condition number causes a loss of accuracy.

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