Analysis of the Singular Function Boundary Integral Method for a Biharmonic Problem with One Boundary Singularity

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In this article, we analyze the singular function boundary integral method (SFBIM) for a two-dimensional biharmonic problem with one boundary singularity, as a model for the Newtonian stick-slip flow problem. In the SFBIM, the leading terms of the local asymptotic solution expansion near the singular point are used to approximate the solution, and the Dirichlet boundary conditions are weakly enforced by means of Lagrange multiplier functions. By means of Green's theorem, the resulting discretized equations are posed and solved on the boundary of the domain, away from the point where the singularity arises. We analyze the convergence of the method and prove that the coefficients in the local asymptotic expansion, also referred to as stress intensity factors, are approximated at an exponential rate as the number of the employed expansion terms is increased. Our theoretical results are illustrated through a numerical experiment. © 2011 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 28: 749–767, 2012

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I. INTRODUCTION

Boundary singularities appear in many problems governed by elliptic partial differential equations. These arise when there is a suddden change in the boundary conditions (e.g., domains with cracks) and/or on the boundary itself (e.g., re-entrant corners). It is well known that ignoring their presence can adversely affect the accuracy and the convergence of standard numerical methods, such as finite element, boundary element, finite difference, and spectral methods. One way to deal with singularities is to incorporate their local form into the numerical scheme, something that has been successfully done for two-dimensional Laplacian problems (see, e.g., [1, 2] and the references therein).

In the case of two-dimensional Laplacian problems with one boundary singularity, the local solution expansion is given by

$$u = \sum_{j=1}^{\infty} \alpha_j r^{\beta_j} \phi_j(\theta), \tag{1}$$

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where (r, θ) are polar coordinates centered at the singular point, $\alpha_j \in \mathbb{R}$ and β_j, ϕ_j are, respectively, the eigenvalues and eigenfunctions of the problem, which are uniquely determined by the geometry and the boundary conditions along the boundaries sharing the singular point. The α_js, called singular coefficients (or stress intensity factors if the boundary value problem arises from structural mechanics), are primary unknowns in many applications. With standard numerical schemes, such as the finite element method (FEM), the singular coefficients are calculated via a postprocessing procedure (see, e.g., [3, 4]). The singular function boundary integral method (SFBIM), belongs to the class of Trefftz methods in which the singular coefficients are calculated directly. It was originaly developed for two-dimensional Laplacian problems with boundary singularities, by Georgiou and coworkers [1, 5], and was recently extended to biharmonic problems [6–8]. See also [9–11] for reviews of Trefftz methods and recent works with applications to biharmonic problems.

The SFBIM uses the leading terms of the local asymptotic expansion to approximate the solution. The associated functions $r^{\beta_j}\phi_j(\theta)$ are used to weight the governing biharmonic equation in the Galerkin sence. This allows for the reduction of the discretized equations to boundary integrals by means of Green's theorem. Any Dirichlet boundary conditions are weakly enforced by means of Lagrange multipliers, which are calculated directly together with the unknown singular coefficients; hence, no postprocessing of the numerical solution is performed.

The implementation of the method for the solution of Laplacian and biharmonic problems with boundary singularities has given highly accurate numerical results [6–8, 12, 13]. The convergence of the SFBIM, for Laplacian problems, has been investigated theoretically in [14], where it was shown that the absolute difference between the true and approximate singular coefficients decreases at an exponential rate as the number N of the terms in the numerical approximation is increased. The main goal of this article, is to extend the analysis to the case of biharmonic problems and establish the (exponential) convergence rates observed in numerical simulations [6–8]. It should be noted that the Collocation Trefftz method also yields exponential convergence rates, when applied to biharmonic problems, as was shown in [10, 15].

The rest of this article is organized as follows: In Section II the formulation of the method for a model two-dimensional biharmonic problem with a boundary singularity is presented. In Section III the convergence analysis is carried out. Finally, in Section IV, we discuss the efficient implementation of the method and in Section V, we illustrate it through a numerical experiment. Throughout this article the usual notation $H^k(\Omega)$ will be used for spaces containing functions defined on the domain $\Omega \subset \mathbb{R}^2$ with boundary $\partial\Omega$, having k generalized derivatives in $L^2(\Omega)$. The norm and seminorm on $H^k(\Omega)$, will be denoted by $\|\cdot\|_{k,\Omega}$ and $|\cdot|_{k,\Omega}$, respectively. For the case when k is noninteger and/or negative, we utilize the definitions and concepts given in [16]. The letters C, c, with or without subscripts, will be used to denote generic positive contants, with possible different values in each occurence.

II. THE MODEL PROBLEM AND ITS FORMULATION

We consider the following model two-dimensional biharmonic problem (depicted graphically in Fig. 1): Find u such that

$$\nabla^4 u = 0 \text{ in } \Omega, \tag{2}$$

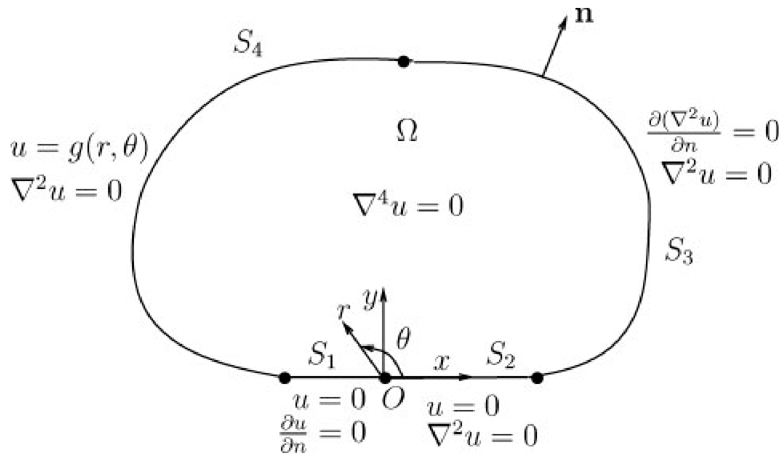

FIG. 1. The model biharmonic problem with one singular point.

with

$$\left.\begin{array}{lll} u = 0, & \dfrac{\partial u}{\partial n} = 0 & \text{on } S_1 \\ u = 0, & \nabla^2 u = 0 & \text{on } S_2 \\ \nabla^2 u = 0, & \dfrac{\partial(\nabla^2 u)}{\partial n} = 0 & \text{on } S_3 \\ u = g(r,\theta), & \nabla^2 u = 0 & \text{on } S_4 \end{array}\right\}, \tag{3}$$

where $\partial\Omega = \cup_{i=1}^{4} S_i$. A boundary singularity arises at the intersection of S_1 and S_2 (point O) due to the sudden changes in the boundary conditions. The function g is assumed to be smooth enough and such that no other boundary singularities arise (at the endpoints of S_4). We also assume that the only singularity present is the one at the point O. The above boundary value problem models the so-called Newtonian stick-slip flow problem [6].

For two-dimensional biharmonic problems, the solution in the neighbourhood of the boundary singularity is given by an asymptotic expansion of the form [6]:

$$u(r,\theta) = \sum_{j=1}^{\infty} \alpha_j r^{\mu_j+1} f_1(\theta,\mu_j) + \sum_{j=1}^{\infty} \beta_j r^{\rho_j+1} f_2(\theta,\rho_j), \tag{4}$$

where α_j and β_j are the unknown singular coefficients, μ_j and ρ_j are the two sets of singularity powers (i.e., the eigenvalues of the problem) arranged in ascending order, and the functions $f_1(\theta,\mu_j)$ and $f_2(\theta,\rho_j)$ represent the θ-dependence of the eigensolution. The functions $r^{\mu_j+1} f_1(\theta,\mu_j)$ and $r^{\rho_j+1} f_2(\theta,\rho_j)$ are called singular functions. As we are considering a model for the stick-slip problem where S_1 and S_2 meet at an angle π, the eigenvalues μ_j, ρ_j are real and the functions $f_1(\theta,\mu_j)$, $f_2(\theta,\rho_j)$ are even and odd, respectively [17, 18]. In fact, in this setting, one finds that

$$f_1(\theta,\mu_j) = \cos(\mu_j+1)\theta - \cos(\mu_j-1)\theta,\ \mu_j = j - \frac{1}{2},\ j = 1,2,\dots \tag{5}$$

$$f_2(\theta,\rho_j) = (\rho_j-1)\sin(\rho_j+1)\theta - (\rho_j+1)\sin(\rho_j-1)\theta,\ \rho_j = j+1,\ j = 1,2,\dots \tag{6}$$

Now, suppose v is a function which satisfies

$$\left.\begin{aligned} &\nabla^4 v = 0 \text{ in } \Omega \\ &v = 0, \frac{\partial v}{\partial n} = 0 \text{ on } S_1 \\ &v = 0, \nabla^2 v = 0 \text{ on } S_2 \end{aligned}\right\}. \tag{7}$$

One choice for v is

$$v = \gamma_j r^{\mu_j+1} f_1(\theta,\mu_j) + \delta_j r^{\rho_j+1} f_2(\theta,\rho_j),$$

for some constants γ_j and δ_j. Multiplying the governing biharmonic equation by v, integrating over Ω and using Green's formula, we obtain

$$-\iint_\Omega \nabla v \cdot \nabla(\nabla^2 u) + \int_{\partial\Omega} v\frac{\partial(\nabla^2 u)}{\partial n} = 0.$$

Using Green's formula once again, the above expression becomes:

$$\iint_\Omega \nabla^2 v \nabla^2 u - \int_{\partial\Omega} \nabla^2 u \frac{\partial v}{\partial n} + \int_{\partial\Omega} v\frac{\partial(\nabla^2 u)}{\partial n} = 0.$$

Considering the boundary conditions in (3) and (7), we find that

$$\iint_\Omega \nabla^2 v \nabla^2 u + \int_{S_4} v\frac{\partial(\nabla^2 u)}{\partial n} = 0. \tag{8}$$

Now, on S_4 we have $u = g$ and thus

$$\int_{S_4} (u-g)\frac{\partial(\nabla^2 v)}{\partial n} = 0,$$

which added to (8) gives

$$\iint_\Omega \nabla^2 v \nabla^2 u + \int_{S_4} v\frac{\partial(\nabla^2 u)}{\partial n} + \int_{S_4} u\frac{\partial(\nabla^2 v)}{\partial n} = \int_{S_4} g\frac{\partial(\nabla^2 v)}{\partial n}.$$

Letting

$$\lambda = \frac{\partial(\nabla^2 u)}{\partial n}|_{S_4}, \mu = \frac{\partial(\nabla^2 v)}{\partial n}|_{S_4}, \tag{9}$$

we get

$$\iint_\Omega \nabla^2 v \nabla^2 u + \int_{S_4} v\lambda + \int_{S_4} u\mu = \int_{S_4} g\mu, \tag{10}$$

which leads to the following variational formulation: Find $(u, \lambda) \in V_1 \times V_2$ such that $\forall (v, \mu) \in V_1 \times V_2$

$$B(u, v) + b(u, v; \lambda, \mu) = F(v, \mu), \tag{11}$$

where

$$\left.\begin{aligned} B(u, v) &= \iint_\Omega \nabla^2 v \nabla^2 u \\ b(u, v; \lambda, \mu) &= \int_{S_4} u\mu + \int_{S_4} v\lambda \\ F(v, \mu) &= \int_{S_4} g\mu \end{aligned}\right\}. \tag{12}$$

The spaces V_1 and V_2 are defined as

$$V_1 = H^2_*(\Omega) = \left\{ v \in H^2(\Omega) : v|_{S_1 \cup S_2} = 0, \frac{\partial v}{\partial n}\bigg|_{S_1} = 0 \right\}, V_2 = H^{-\frac{3}{2}}(S_4). \tag{13}$$

Remark 1. *The above formulation will be used in the analysis of the method. As described in Section IV, an equivalent formulation will be used for the implementation, which will involve only one-dimensional integrations along the parts of the boundary that are away from the singular point.*

III. DISCRETIZATION AND ERROR ANALYSIS

To describe the discrete analog of (11), boundary part S_4 is divided into sections Γ_i, with $i = 1, \ldots, n$ such that $S_4 = \cup_{i=1}^n \Gamma_i$. Let $h_i = |\Gamma_i|$ and set $h = \max_{1 \le i \le n} h_i$. Now, let

$$v_j^{(1)} = r^{\mu_j + 1} f_1(\theta, \mu_j) \text{ and } v_j^{(2)} = r^{\rho_j + 1} f_2(\theta, \rho_j)$$

denote the singular functions, and define the following finite dimensional space:

$$V_1{}^N = span\{v_j^{(1)}\} \cup span\{v_j^{(2)}\}, j = 1, 2, \ldots, N. \tag{14}$$

We assume that for each segment Γ_i, there exists an invertible mapping $\mathcal{F}_i : I = [-1, 1] \to \Gamma_i$ and define the space

$$V_2^h = \left\{ \lambda_h : \lambda_h|_{\Gamma_i} \circ \mathcal{F}_i^{-1} \in P_k(I), i = 1, \ldots, n \right\}, \tag{15}$$

where $P_k(I)$ denotes the set of polynomials of degree $\le k$ on I. Then the discrete version of (11) reads: Find $(u_N, \lambda_h) \in [V_1{}^N \times V_2{}^h] \subset [V_1 \times V_2]$ such that

$$B(u_N, v_N) + b(u_N, v_N; \lambda_h, \mu_h) = F(v_N, \mu_h) \forall (v_N, \mu_h) \in V_1^N \times V_2^h, \tag{16}$$

with $B(u, v)$, $b(u, v; \lambda, \mu)$ and $F(v, \mu)$ given by (12).

We have the following result, which is a generalization of Theorem 4.5 from [19].

Theorem 1. *Let (u, λ) and (u_N, λ_h) be the solutions to (11) and (16), respectively. Suppose there exist positive constants c_0, c, β^* and γ, independent of N and h such that the following three conditions hold:*

$$B(v_N, v_N) \geq c_0 \|v_N\|^2{}_{2,\Omega} \text{ and } |B(u, v_N)| \leq c\|u\|_{2,\Omega}\|v_N\|_{2,\Omega} \ \forall\ v_N \in {V_1}^N, \tag{17}$$

$$\exists\, 0 \neq w_N \in {V_1}^N \ s.t. \ \left|\int_{S_4} \mu_h w_N\right| \geq \beta^* \|\mu_h\|_{-\frac{3}{2},S_4} \|w_N\|_{2,\Omega} \ \forall\ \mu_h \in {V_2}^h, \tag{18}$$

$$\left|\int_{S_4} \lambda v_N\right| \leq \gamma \|\lambda\|_{-\frac{3}{2},S_4} \|v_N\|_{2,\Omega} \ \forall\ v_N \in {V_1}^N. \tag{19}$$

Then,

$$\|u - u_N\|_{2,\Omega} + \|\lambda - \lambda_h\|_{-\frac{3}{2},S_4} \leq C\Big\{ \inf_{v_N \in {V_1}^N} \|u - v_N\|_{2,\Omega} + \inf_{\eta_h \in {V_2}^h} \|\lambda - \eta_h\|_{-\frac{3}{2},S_4} \Big\}, \tag{20}$$

with $C \in \mathbb{R}^+$ independent of N and h.

Proof. Obviously, $\forall\ (v, \mu) \in V_1 \times V_2$ we have

$$B(u - u_N, v) = -b(u - u_N, v; \lambda - \lambda_h, \mu) = -\int_{S_4} (u - u_N)\mu - \int_{S_4} (\lambda - \lambda_h)v.$$

Since $u = g$ on S_4 and $\int_{S_4} \mu_h (u_N - g) = 0\ \forall\ \mu_h \in V_2^h$, we have

$$\int_{S_4} \mu_h (u_N - u) = 0\ \forall\ \mu_h \in V_2^h, \tag{21}$$

and

$$B(u - u_N, v_N) = -\int_{S_4} (\lambda - \lambda_h) v_N\ \forall\ v_N \in V_1^N. \tag{22}$$

Letting $w_N = u_N - v_N \in V_1^N$ we obtain

$$\begin{aligned} B(v_N - u_N, w_N) &= B(u - u_N, w_N) + B(v_N - u, w_N) \\ &= B(v_N - u, w_N) - \int_{S_4} (\lambda - \lambda_h) w_N \\ &= B(v_N - u, w_N) - \int_{S_4} (\lambda - \eta_h) w_N - \int_{S_4} (\eta_h - \lambda_h) w_N, \end{aligned}$$

with $\eta_h \in V_2^h$ arbitrary. Using the definition of w_N and (21) with $\mu_h = \lambda_h - w_h \in V_2^h$, we further have

$$\begin{aligned} B(v_N - u_N, w_N) &= B(v_N - u, w_N) - \int_{S_4} (\lambda - \eta_h) w_N - \int_{S_4} (u_N - v_N)(\eta_h - \lambda_h) \\ &= B(v_N - u, w_N) - \int_{S_4} (\lambda - \eta_h) w_N - \int_{S_4} u_N(\eta_h - \lambda_h) + \int_{S_4} v_N(\eta_h - \lambda_h) \\ &= B(v_N - u, w_N) - \int_{S_4} (\lambda - \eta_h) w_N - \int_{S_4} u(\eta_h - \lambda_h) + \int_{S_4} v_N(\eta_h - \lambda_h) \\ &= B(v_N - u, w_N) - \int_{S_4} (\lambda - \eta_h) w_N - \int_{S_4} (u - v_N)(\eta_h - \lambda_h). \end{aligned}$$

This along with Eqs. (17) and (19) give

$$\begin{aligned} c_0 \|w_N\|_{2,\Omega}^2 &\le |B(w_N, w_N)| \le |B(u_N - v_N, w_N)| \\ &\le |B(v_N - u, w_N)| + \left| \int_{S_4} (\lambda - \eta_h) w_N \right| + \left| \int_{S_4} (\eta_h - \lambda_h)(u - v_N) \right| \\ &\le c\|v_N - u\|_{2,\Omega} \|w_N\|_{2,\Omega} + \gamma \|\lambda - \eta_h\|_{-\frac{3}{2},S_4} \|w_N\|_{2,\Omega} + \gamma \|\eta_h - \lambda_h\|_{-\frac{3}{2},S_4} \|u - v_N\|_{2,\Omega} \\ &\le C_1 \big\{ \big(\|v_N - u\|_{2,\Omega} + \|\lambda - \eta_h\|_{-\frac{3}{2},S_4} \big) \|w_N\|_{2,\Omega} + \|\eta_h - \lambda_h\|_{-\frac{3}{2},S_4} \|u - v_N\|_{2,\Omega} \big\}, \end{aligned}$$

with $C_1 \in \mathbb{R}$ satsifying $C_1 \ge \max\{c, \gamma\}$. This is an inequality of order 2: $c_0 x^2 \le bx + d$, where

$$x = \|w_N\|_{2,\Omega},\ b = C_1 \big\{ \|v_N - u\|_{2,\Omega} + \|\lambda - \eta_h\|_{-\frac{3}{2},S_4} \big\},\ d = C_1 \|\eta_h - \lambda_h\|_{-\frac{3}{2},S_4} \|u - v_N\|_{2,\Omega}.$$

For any $\epsilon > 0$, we have

$$d \le \frac{C_1}{2} \left\{ \frac{1}{\epsilon} \|u - v_N\|_{2,\Omega} + \epsilon \|\eta_h - \lambda_h\|_{-\frac{3}{2},S_4} \right\}^2.$$

Therefore, we have the bound

$$x \le \frac{b + \sqrt{b^2 + 4c_0\, d}}{2c_0},$$

or, equivalently,

$$\|w_N\|_{2,\Omega} \le C_2 \left\{ \|v_N - u\|_{2,\Omega} + \|\lambda - \eta_h\|_{-\frac{3}{2},S_4} + \frac{1}{\epsilon} \|v_N - u\|_{2,\Omega} \right\} + C_2 \epsilon \|\lambda_h - \eta_h\|_{-\frac{3}{2},S_4}, \quad (23)$$

with $C_2 \ge \frac{1}{c_0} \max\{C_1, \sqrt{c_0 C_1/2}\}$. Next, using Eq. (18) with $\mu_h = \lambda_h - \eta_h$ we have that there exists a nonzero $v_N \in V_1^N$ such that

$$\|\lambda_h - \eta_h\|_{-\frac{3}{2},S_4} \le \frac{1}{\beta} \frac{\left| \int_{S_4} (\lambda_h - \eta_h) v_N \right|}{\|v_N\|_{2,\Omega}}. \quad (24)$$

Also, it follows from (22) that

$$\left|\int_{S_4}(\lambda_h-\eta_h)v_N\right| = \left|\int_{S_4}(\lambda_h-\lambda)v_N + \int_{S_4}(\lambda-\eta_h)v_N\right|$$
$$\leq |B(u-u_N,v_N)| + \left|\int_{S_4}(\lambda-\eta_h)v_N\right|$$
$$\leq c\|u-u_N\|_{2,\Omega}\|v_N\|_{2,\Omega} + \gamma\|\lambda-\eta_h\|_{-\frac{3}{2},S_4}\|v_N\|_{2,\Omega}.$$

Hence, (24) becomes

$$\|\lambda_h-\eta_h\|_{-\frac{3}{2},S_4} \leq C_3\{\|u-u_N\|_{2,\Omega} + \|\lambda-\eta_h\|_{-\frac{3}{2},S_4}\}$$
$$\leq C_3\{\|u-v_N\|_{2,\Omega} + \|v_N-u_N\|_{2,\Omega} + \|\lambda-\eta_h\|_{-\frac{3}{2},S_4}\},$$

with $C_3 \geq \frac{1}{\beta}\max\{c,\gamma\}$. Since $\|v_N-u_N\|_{2,\Omega} = \|w_N\|_{2,\Omega}$, using (23) leads to

$$\|\lambda_h-\eta_h\|_{-\frac{3}{2},S_4} \leq C_3(1+C_2/\epsilon)\|u-v_N\|_{2,\Omega} + C_3(C_2+1)\|\lambda-\eta_h\|_{-\frac{3}{2},S_4} + C_3C_2\epsilon\|\lambda-\eta_h\|_{-\frac{3}{2},S_4}.$$

Choosing $\epsilon = 1/(2C_3C_2)$ we get, for some constant $C_4 > \max\{C_2, C_3\}$,

$$\|\lambda_h-\eta_h\|_{-\frac{3}{2},S_4} \leq C_4\left\{\|u-v_N\|_{2,\Omega} + \|\lambda-\eta_h\|_{-\frac{3}{2},S_4}\right\},$$

and using the triangle inequality we have

$$\|\lambda_h-\lambda\|_{-\frac{3}{2},S_4} \leq \|\lambda_h-\eta_h\|_{-\frac{3}{2},S_4} + \|\eta_h-\lambda\|_{-\frac{3}{2},S_4} \leq C\left\{\|u-v\|_{2,\Omega} + \|\lambda-\eta_h\|_{-\frac{3}{2},S_4}\right\}.$$

Similarly, using the above inequality and (23), we finally get

$$\|u-u_N\| \leq \|u-v_N\|_{2,\Omega} + \|v_N-u_N\|_{2,\Omega}$$
$$\leq \|u-v_N\|_{2,\Omega} + \|w_N\|_{2,\Omega}$$
$$\leq C\left\{\|u-v_N\|_{2,\Omega} + \|\lambda-\eta_h\|_{-\frac{3}{2},S_4}\right\},$$

which gives the desired result. ■

Before verifying that (17)–(19) hold for our problem, consider the following: For any

$$w = \sum_{j=1}^{\infty}\alpha_j v_j^{(1)} + \sum_{j=1}^{\infty}\beta_j v_j^{(2)}$$

we can always write

$$w = w_N + r_N, \tag{25}$$

where

$$w_N = \sum_{j=1}^{N}\alpha_j v_j^{(1)} + \sum_{j=1}^{N}\beta_j v_j^{(2)} \in V_1^N, r_N = \sum_{j=N+1}^{\infty}\alpha_j v_j^{(1)} + \sum_{j=N+1}^{\infty}\beta_j v_j^{(2)}, \tag{26}$$

with α_j and β_j the real singular coefficients. We will assume that there exists a constant $\alpha \in (0, 1)$ such that for $\ell = 0, 1, 2$

$$\left|\frac{\partial^\ell(r_N)}{\partial r^\ell}\right| \leq CN^\ell \alpha^N. \tag{27}$$

Note that when $r < 1$, assumption (27) can be replaced by the assumption that the singular coefficients are bounded, since then, due to the fact that $f_1(\theta, \mu_j)$ and $f_2(\theta, \rho_j)$ are biharmonic, we have

$$|r_N| \leq \sum_{j=N+1}^{\infty} |\alpha_j| r^{\mu_j+1} + \sum_{j=N+1}^{\infty} |\beta_j| r^{\rho_j+1} \leq C_1 \frac{r^{\mu_{N+1}+1}}{1-r} + C_2 \frac{r^{\rho_{N+1}+1}}{1-r} \leq C\alpha^N,$$

with $r < \alpha < 1$ and $C \in \mathbb{R}^+$ independent of α and N. Similarly,

$$\begin{aligned}
\left|\frac{\partial r_N}{\partial r}\right| &\leq \sum_{j=N+1}^{\infty} |\alpha_j|(\mu_j+1) r^{\mu_j} + \sum_{j=N+1}^{\infty} |\beta_j|(\rho_j+1) r^{\rho_j} \\
&= \sum_{j=N+1}^{\infty} |\alpha_j|(\mu_j+1)\left\{\frac{d}{dr}\int_0^r \xi^{\mu_j} d\xi\right\} + \sum_{j=N+1}^{\infty} |\beta_j|(\rho_j+1)\left\{\frac{d}{dr}\int_0^r \xi^{\rho_j} d\xi\right\} \\
&= \frac{d}{dr}\left(\sum_{j=N+1}^{\infty} |\alpha_j|(\mu_j+1)\left\{\int_0^r \xi^{\mu_j} d\xi\right\} + \sum_{j=N+1}^{\infty} |\beta_j|(\rho_j+1)\left\{\int_0^r \xi^{\rho_j} d\xi\right\}\right) \\
&\leq \frac{d}{dr}\left(\sum_{j=N+1}^{\infty} |\alpha_j| r^{\mu_j+1} + \sum_{j=N+1}^{\infty} |\beta_j| r^{\rho_j+1}\right) \\
&\leq C_1 \frac{d}{dr}\left(\frac{r^{\mu_{N+1}+1}}{1-r}\right) + C_2 \frac{d}{dr}\left(\frac{r^{\rho_{N+1}+1}}{1-r}\right) \\
&\leq CN\alpha^N.
\end{aligned}$$

(The case $\ell = 2$ follows in a similar fashion.)

In the case $r \geq 1$ one may partition the domain Ω into subdomains in which separate approximations may be used, including one (near the singular point O) that is valid for $r < 1$. The solution over the entire domain can then be composed by combining solutions from each subdomain and properly dealing with their interactions across the interfaces separating them (see, e.g., [20] where this idea was applied to a Laplacian problem).

We are now ready to verify that (17)–(19) hold for the problem (16). We have (see, e.g. [21]),

$$B(v, v) = \iint_\Omega \nabla^2 v \nabla^2 v = \iint_\Omega |\nabla^2 v|^2 \geq C_0 \|v\|^2_{2,\Omega} \;\forall\, v \in V_1$$

and $\exists\, c \in \mathbb{R}^+$ such that

$$|B(u, v)| \leq c\|u\|_{2,\Omega}\|v\|_{2,\Omega} \;\forall\, u, v \in V_1,$$

therefore (17) is verified.

To verify (18) we consider the following auxiliary problem:

$$\nabla^4 w = 0, \text{ in } \Omega, \tag{28}$$

with the boundary conditions

$$\left.\begin{array}{lll} w = 0, & \dfrac{\partial w}{\partial n} = 0 & \text{on } S_1 \\ w = 0, & \nabla^2 w = 0 & \text{on } S_2 \\ \dfrac{\partial(\nabla^2 w)}{\partial n} = 0, & \nabla^2 w = 0 & \text{on } S_3 \\ \nabla^2 w = 0, & \dfrac{\partial(\nabla^2 w)}{\partial n} = \mu_h & \text{on } S_4 \end{array}\right\}, \tag{29}$$

where $\mu_h \in V_2^h$ in (29). By using Green's formula we obtain

$$\begin{aligned} \left|\int_{S_4} w\mu_h\right| &= \left|\int_{S_4} w\frac{\partial(\nabla^2 w)}{\partial n}\right| = \left|\iint_\Omega w\nabla^4 w + \iint_\Omega \nabla w \cdot \nabla(\nabla^2 w)\right| \\ &= \left|-\iint_\Omega \nabla^2 w \nabla^2 w + \int_{\partial\Omega} \nabla^2 w \frac{\partial w}{\partial n}\right| \\ &= \left|\iint_\Omega \nabla^2 w \nabla^2 w\right| = \iint_\Omega \left(\nabla^2 w\right)^2 \\ &\geq C_0 \|w\|_{2,\Omega}^2. \end{aligned} \tag{30}$$

Note that (see, e.g., [22])

$$\|\mu_h\|_{-\frac{3}{2},S_4}^2 = \left\|\frac{\partial(\nabla^2 w)}{\partial n}\right\|_{-\frac{3}{2},S_4}^2 \leq C\|w\|_{2,\Omega}^2,\ C \in \mathbb{R}^+, \tag{31}$$

so, by (30),

$$\left|\int_{S_4} \mu_h w\right| \geq \beta \|w\|_{2,\Omega} \|\mu_h\|_{-\frac{3}{2},S_4}, \tag{32}$$

with $\beta \in \mathbb{R}^+$ independent of w and h. Now, let $w_N \in V_1^N$ be such that $w = w_N + r_N$, as given by (25)–(26). We have

$$\left|\int_{S_4} \mu_h w_N\right| = \left|\int_{S_4} \mu_h w - \int_{S_4} \mu_h r_N\right| \geq \left|\int_{S_4} \mu_h w\right| - \left|\int_{S_4} \mu_h r_N\right| \tag{33}$$

and

$$\left|\int_{S_4} \mu_h r_N\right| \leq C_1 \|\mu_h\|_{-\frac{3}{2},S_4} \|r_N\|_{2,\Omega},\ C_1 \in \mathbb{R}^+. \tag{34}$$

Now, combining (31)–(33) we obtain

$$\left|\int_{S_4} \mu_h w_N\right| \geq \beta\|w\|_{2,\Omega}\|\mu_h\|_{-\frac{3}{2},S_4} - C_1\|\mu_h\|_{-\frac{3}{2},S_4}\|r_N\|_{2,\Omega}. \tag{35}$$

Also, from the reverse triangle inequality,

$$\|w\|_{2,\Omega} = \|w_N + r_N\|_{2,\Omega} \geq \|w_N\|_{2,\Omega} - \|r_N\|_{2,\Omega},$$

and by (34), we get

$$\left|\int_{S_4} \mu_h w_N\right| \geq \beta(\|w_N\|_{2,\Omega} - \|r_N\|_{2,\Omega})\|\mu_h\|_{-\frac{3}{2},S_4} - C_1\|\mu_h\|_{-\frac{3}{2},S_4}\|r_N\|_{2,\Omega}. \tag{36}$$

Therefore,

$$\left|\int_{S_4} \mu_h w_N\right| \geq \beta\|w_N\|_{2,\Omega}\|\mu_h\|_{-\frac{3}{2},S_4} - (C_1 + \beta)\|\mu_h\|_{-\frac{3}{2},S_4}\|r_N\|_{2,\Omega}. \tag{37}$$

Since by assumption (27), r_N converges to zero exponentially (or, equivalently w_N converges to w exponentially), we have

$$\lim_{N\to\infty} \frac{\|r_N\|_{2,\Omega}}{\|w_N\|_{2,\Omega}} = 0,$$

which means that for any $\varepsilon > 0$ there exists N^* such that $\frac{\|r_N\|_{2,\Omega}}{\|w_N\|_{2,\Omega}} < \varepsilon$ whenever $N > N^*$. Hence, for N sufficiently large we may write

$$\frac{\|r_N\|_{2,\Omega}}{\|w_N\|_{2,\Omega}} \leq \frac{\beta}{2(C_1 + \beta)}.$$

Combining (36) with (37) yields

$$\left|\int_{S_4} \mu_h w_N\right| \geq \frac{\beta}{2}\|\mu_h\|_{-\frac{3}{2},S_4}\|w_N\|_{2,\Omega}.$$

By replacing $\frac{\beta}{2}$ by β^*, inequality (18) is obtained. Finally, condition (19) follows from (see, e.g., [22])

$$\int_{S_4} \lambda v_N \leq \gamma\|\lambda\|_{-\frac{3}{2},S_4}\|v_N\|_{2,\Omega}\ \forall\ v_N \in V_1^N, \text{with } \gamma \in \mathbb{R}^+.$$

The above analysis leads to the following theorem.

Theorem 2. *Let (u,λ) and (u_N,λ_h) be the solutions to (11) and (16), respectively. If $\lambda \in H^k(S_4)$, for some $k \geq 1$, then there exists a positive constant C, independent of N and h, such that as $N \to \infty$*

$$\|u - u_N\|_{2,\Omega} + \|\lambda - \lambda_h\|_{-\frac{3}{2},S_4} \leq C\{N^2\alpha^N + h^{k+1}\},$$

with $\alpha \in (0, 1)$.

Proof. From Theorem 1 we have

$$\|u - u_N\|_{2,\Omega} + \|\lambda - \lambda_h\|_{-\frac{3}{2},S_4} \le C\Big\{ \inf_{v_N \in V_1{}^N} \|u - v_N\|_{2,\Omega} + \inf_{\eta_h \in V_2{}^h} \|\lambda - \eta_h\|_{-\frac{3}{2},S_4} \Big\}, \tag{38}$$

with $C \in \mathbb{R}^+$ independent of N and h. Note that by (25) and (26)

$$\inf_{v_N \in V_1{}^N} \|u - v_N\|_{2,\Omega} \le \|u - w_N\|_{2,\Omega} = \|r_N\|_{2,\Omega}.$$

Using assumption (27) we get

$$\inf_{v_N \in V_1{}^N} \|u - v_N\|_{2,\Omega} \le C N^2 \alpha^N, \tag{39}$$

where the constant $C > 0$ is independent of N and α. Next let λ_I be the k^{th}-order interpolant of λ. Then, since $\lambda \in H^k(S_4)$ and λ_h is the best approximation, we have

$$\|\lambda - \lambda_h\|_{-\frac{3}{2},S_4} \le \|\lambda - \lambda_h\|_{0,S_4} \le \|\lambda - \lambda_I\|_{0,S_4} \le h^{k+1}\|\lambda\|_{k,S_4} \le C h^{k+1},$$

which, along with (38)–(39) gives the desired result. ■

The approximation of the singular coefficients is given by the following.

Corollary 1. *Let*

$$u = \sum_{j=1}^{\infty} \alpha_j r^{\mu_j+1} f_1(\theta, \mu_j) + \sum_{j=1}^{\infty} \beta_j r^{\rho_j+1} f_2(\theta, \rho_j) \tag{40}$$

and

$$u_N = \sum_{j=1}^{N} \alpha_j^N r^{\mu_j+1} f_1(\theta, \mu) + \sum_{j=1}^{N} \beta_j^N r^{\rho_j+1} f_2(\theta, \rho_j) \tag{41}$$

satisfy (11) and (16), respectively, with α_j, β_j and α_j^N, β_j^N denoting the true and approximate singular coefficients. Then, there exists a positive constant $C \in \mathbb{R}^+$, independent of N and α, but depending on j, such that as $N \to \infty$

$$\left|\left(\alpha_j - \alpha_j^N\right)\right| + \left|\left(\beta_j - \beta_j^N\right)\right| \le C N^2 \alpha^N. \tag{42}$$

Proof. We begin by noting the following (which can be obtained by elementary calculations):

$$\int_0^{2\pi} f_1(\theta, \mu_j) f_1(\theta, \mu_k) d\theta = 2\pi \delta_{j,k} \tag{43}$$

$$\int_0^{2\pi} f_1(\theta, \mu_j) f_2(\theta, \rho_k) d\theta = 0 \ \forall j, k = 1, 2, \ldots \tag{44}$$

$$\int_0^{2\pi} f_2(\theta, \rho_j) f_2(\theta, \rho_k) d\theta = 2\pi \frac{4k^2 - 4k + 5}{4k^2 + 4k + 1} \delta_{j,k} \tag{45}$$

where f_1, f_2 are given by (5)–(6) and $\delta_{j,k}$ is the Kronecker delta. Now, in (40) take a fixed $r = r_0 < 1$, multiply by $f_1(\theta, \mu_k)$ and integrate from $\theta = 0$ to $\theta = 2\pi$. Using (43) and (44) we find that

$$\int_0^{2\pi} u(r_0, \theta) f_1(\theta, \mu_k) d\theta = 2\pi r_0^{\mu_k+1} \alpha_k. \tag{46}$$

Next, multiply (40) by $f_2(\theta, \rho_k)$ and integrate from $\theta = 0$ to $\theta = 2\pi$, to get with the aid of (44) and (45),

$$\int_0^{2\pi} u(r_0, \theta) f_2(\theta, \rho_k) d\theta = 2\pi r_0^{\rho_k+1} \frac{4k^2 - 4k + 5}{4k^2 + 4k + 1} \beta_k. \tag{47}$$

Similarly, one obtains expressions like (46), (47) corresponding to the approximate coefficients α_k^N, β_k^N, i.e. Eqs. (46), (47) with u replaced by u_N and α_k, β_k replaced by α_k^N, β_k^N, respectively. Therefore, we have

$$\left|\alpha_k - \alpha_k^N\right| \leq \frac{1}{2\pi r_0^{\mu_k+1}} \int_0^{2\pi} |u - u_N||f_1| d\theta \leq \hat{C}_k \|u - u_N\|_{0,\Omega},$$

$$\left|\beta_k - \beta_k^N\right| \leq \frac{4k^2 + 4k + 1}{2\pi r_0^{\rho_k+1}(4k^2 - 4k + 5)} \int_0^{2\pi} |u - u_N||f_2| d\theta \leq \tilde{C}_k \|u - u_N\|_{0,\Omega},$$

where the Cauchy-Schwartz inequality and the smoothness of f_1, f_2 were used. The positive constants $\hat{C}_k, \tilde{C}_k$ depend only on k (and r_0).

The result then follows from (39) and the fact that $\|u - u_N\|_{0,\Omega} \leq \|u - u_N\|_{2,\Omega}$. ■

Note that the above corollary establishes the exponential convergence of the SFBIM, in the case of the biharmonic problem shown in Fig. 1; the term N^2 in (42) can be absorbed in the exponentially decaying term α^N. This result is analogous to the one obtained in [14] for 2D Laplacian problems.

IV. IMPLEMENTATION

We now give a description of the implementation of the method, as mentioned in Remark 1. Recall the discrete problem given by (16), which may be rewritten in mixed form as follows: Find $(u_N, \lambda_h) \in [V_1^N \times V_2^h] \subset [V_1 \times V_2]$ such that

$$\iint_\Omega \nabla^2 v_N \nabla^2 u_N + \int_{S_4} v_N \lambda_h = 0 \;\forall\; v_N \in V_1^N, \tag{48}$$

$$\int_{S_4} \mu_h u_N = \int_{S_4} \mu_h g \;\forall\; \mu_h \in V_2^h. \tag{49}$$

We may reduce the double integral in (48) using Green's second identity and the boundary conditions in (3) and (7), as follows:

$$\iint_{\Omega} \nabla^2 v_N \nabla^2 u_N = \int_{\partial\Omega} \left(\nabla^2 v_N \frac{\partial u_N}{\partial n} - u_N \frac{\partial(\nabla^2 v_N)}{\partial n} \right)$$
$$= \int_{S_3 \cup S_4} \left(\nabla^2 v_N \frac{\partial u_N}{\partial n} - u_N \frac{\partial(\nabla^2 v_N)}{\partial n} \right). \tag{50}$$

Hence, the problem (48)–(49) becomes: Find $(u_N, \lambda_h) \in [V_1^N \times V_2^h] \subset [V_1 \times V_2]$ such that

$$\int_{S_3 \cup S_4} \left(\nabla^2 v_N \frac{\partial u_N}{\partial n} - u_N \frac{\partial(\nabla^2 v_N)}{\partial n} \right) + \int_{S_4} v_N \lambda_h = 0 \; \forall \; v_N \in V_1^N, \tag{51}$$

$$\int_{S_4} \mu_h u_N = \int_{S_4} \mu_h g \; \forall \; \mu_h \in V_2^h. \tag{52}$$

Obviously, if $(u_N, \lambda_h) \in [V_1^N \times V_2^h] \subset [V_1 \times V_2]$ solves (48)–(49) (or (16)), then it also solves (51)–(52). Now suppose that $(u_N, \lambda_h) \in [V_1^N \times V_2^h] \subset [V_1 \times V_2]$ solves (51)–(52). We have from (50) that

$$\int_{S_3 \cup S_4} \left(\nabla^2 v_N \frac{\partial u_N}{\partial n} - u_N \frac{\partial(\nabla^2 v_N)}{\partial n} \right) = \iint_{\Omega} \nabla^2 v_N \nabla^2 u_N,$$

hence, adding Eqs. (51)–(52) and using the above fact, we find that

$$\iint_{\Omega} \nabla^2 v_N \nabla^2 u_N + \int_{S_4} v_N \lambda_h + \int_{S_4} \mu_h u_N = \int_{S_4} \mu_h g,$$

which shows that (u_N, λ_h) solves (16). Equations (51)–(52) are used in the implementation, since they are posed only on the boundary of the domain away from the singular point. This reduces the dimension of the problem by one and leads to significant computational savings.

Now, to obtain a linear system of equations corresponding to (51)–(52), we approximate u and λ by means of

$$u_N = \sum_{i=1}^{N} \alpha_i^N v_i^{(1)} + \sum_{i=1}^{N} \beta_i^N v_i^{(2)} \in V_1^N, \tag{53}$$

and

$$\lambda_h = \sum_{k=1}^{M} \gamma_k \psi_k \in V_2^h(S_4), \tag{54}$$

with α_i^N, β_i^N and γ_k the unknowns in the system, and $V_1^N = span\{v_i^{(1)}\}_{i=1}^N \cup span\{v_i^{(2)}\}_{i=1}^N$, $V_2^h = span\{\psi_k\}_{k=1}^M$. Upon inserting (53) and (54) into (51)–(52), a $(2N + M) \times (2N + M)$ linear system of the following composite form is obtained:

$$\begin{bmatrix} K_{11} & K_{12} & \Lambda_1 \\ K_{21} & K_{22} & \Lambda_2 \\ \Lambda_1^T & \Lambda_2^T & 0 \end{bmatrix} \begin{bmatrix} \vec{\alpha} \\ \vec{\beta} \\ \vec{\gamma} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{0} \\ \vec{G} \end{bmatrix}, \tag{55}$$

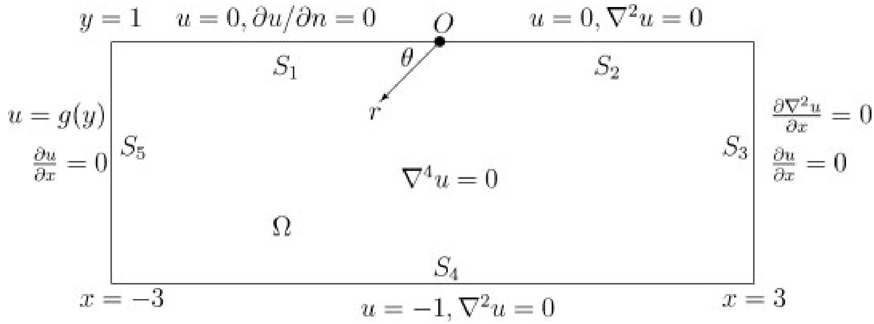

FIG. 2. Stick-slip problem; $g(y) = \frac{1}{2}y(3 - y^2) - 1$.

where $\overrightarrow{\alpha} = [\alpha_1^N, \ldots, \alpha_N^N]^T$, $\overrightarrow{\beta} = [\beta_1^N, \ldots, \beta_N^N]^T$, $\overrightarrow{\gamma} = [\gamma_1, \ldots, \gamma_M]^T$, and

$$[K_{11}]_{i,j} = \int_{S_3 \cup S_4} \left\{ \nabla^2 v_j^{(1)} \frac{\partial v_i^{(1)}}{\partial n} - v_i^{(1)} \frac{\partial}{\partial n} \left(\nabla^2 v_j^{(1)} \right) \right\}, \ i, j = 1, \ldots, N,$$

$$[K_{12}]_{i,j} = \int_{S_3 \cup S_4} \left\{ \nabla^2 v_j^{(1)} \frac{\partial v_i^{(2)}}{\partial n} - v_i^{(2)} \frac{\partial}{\partial n} \left(\nabla^2 v_j^{(1)} \right) \right\}, \ i, j = 1, \ldots, N,$$

$$[K_{21}]_{i,j} = \int_{S_3 \cup S_4} \left\{ \nabla^2 v_j^{(2)} \frac{\partial v_i^{(1)}}{\partial n} - v_i^{(1)} \frac{\partial}{\partial n} (\nabla^2 v_j^{(2)}) \right\}, \ i, j = 1, \ldots, N,$$

$$[K_{22}]_{i,j} = \int_{S_3 \cup S_4} \left\{ \nabla^2 v_j^{(2)} \frac{\partial v_i^{(2)}}{\partial n} - v_i^{(2)} \frac{\partial}{\partial n} (\nabla^2 v_j^{(2)}) \right\}, \ i, j = 1, \ldots, N,$$

$$[\Lambda_1]_{k,j} = \int_{S_4} \psi_k \, v_j^{(1)}, \ k = 1, \ldots, M, \ j = 1, \ldots, N,$$

$$[\Lambda_2]_{k,j} = \int_{S_4} \psi_k \, v_j^{(2)}, \ k = 1, \ldots, M, \ j = 1, \ldots, N,$$

$$[\overrightarrow{G}]_\ell = \int_{S_4} g \, \psi_\ell, \ \ell = 1, \ldots, M.$$

It is easily shown that the coefficient matrix in (55) is nonsingular provided $2N > M$. Hence, N should be chosen larger than $M/2$, but not too large since for excessively large values of N the linear system (55) becomes ill-conditioned and the results obtained are unreliable. As a final remark, we should point out that all integrals involved in the determination of the coefficient matrix (and right hand side) in (55) are along the parts of the domain boundaries that do not contain the singularity. These are one dimensional and can be approximated by standard techniques, such as Gaussian quadrature.

V. NUMERICAL RESULTS

In this section, we illustrate the main theoretical findings through one numerical experiment, as described below. As the method is proposed for the efficient approximation of the singular

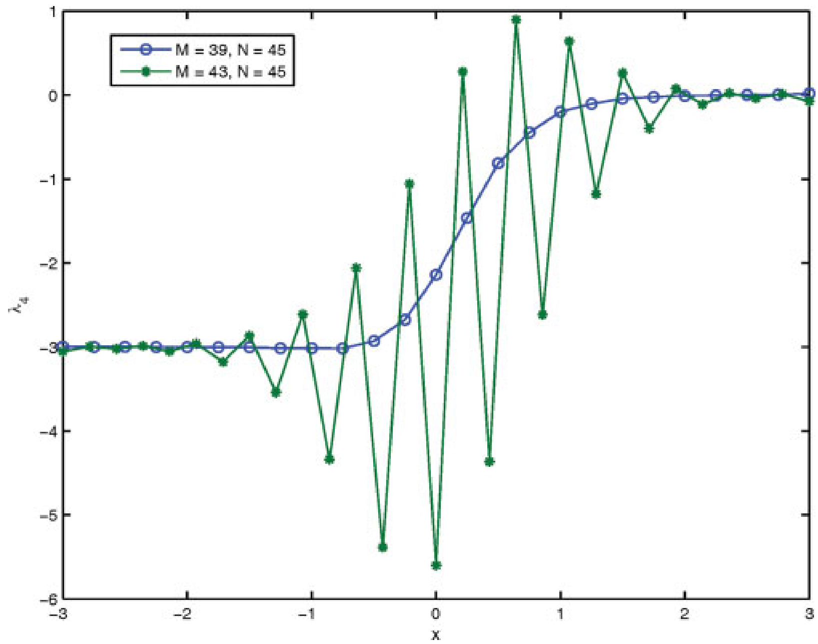

FIG. 3. Approximation of Lagrange multipler function along S_4. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

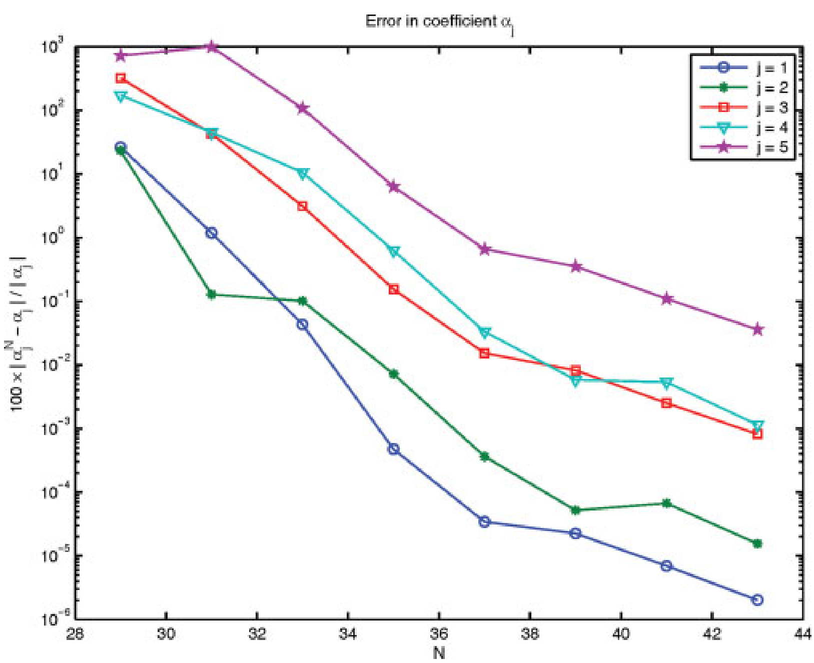

FIG. 4. Error in coefficient α_j^N. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

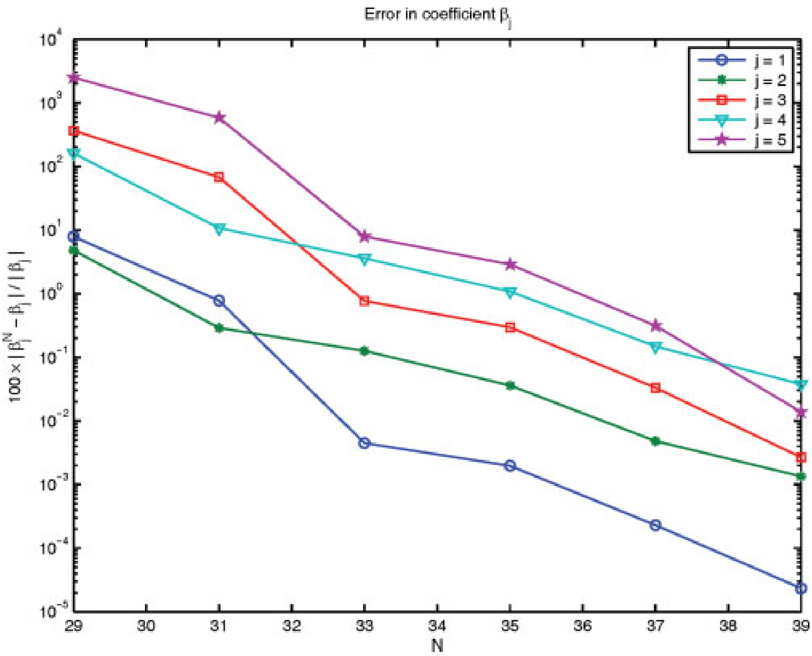

FIG. 5. Error in coefficient β_j^N. [Color figure can be viewed in the online issue, which is available at wileyonlinelibrary.com.]

cofficients, the numerical results shown below correspond to how fast (and accurately) these coefficients are approximated. The interested reader is referred to [6–8] for additional numerical results obtained with the SFBIM for biharmonic problems arising in Stokes flow and in fracture mechanics.

We are considering the boundary value problem depicted graphically in Fig. 2, which is the classical stick-slip flow problem from fluid mechanics [6]. We note that the boundary of the domain consists of five parts, with S_4 and S_5 being the portions of $\partial\Omega$ where Lagrange multipliers will be applied, since Dirichlet boundary conditions are prescribed there.

We implemented our method, as explained in Section IV, using piecewise quadratic polynomials for the approximation of the Lagrange multiplier functions, on a subdivision of S_4 and S_5 characterized by a meshwidth h – for simplicity a uniform subdivision of the same meshwidth h was used for both portions of the boundary. All integrals involved were approximated by a 15-point Gaussian quadrature on each element. Systematic runs were performed to find the "optimal" combination of N and h (or M), which ultimately was chosen as the one that gave the "smoothest" approximation to

$$\lambda_4 := \left.\frac{\partial \nabla^2 u}{\partial n}\right|_{S_4}.$$

This is shown in Fig. 3 which shows that for $M = 39$ and $N = 45$ the approximation to the Lagrange multipler function on S_4 is free of oscillations. Using this pair of values, the constant α in (42) is calculated by "balancing" the error estimate of Theorem 2, i.e.,

$$N^2\alpha^N \approx h^{k+1}.$$

We find that $\alpha \approx 0.87$, from which subsequent "optimal" pairs of N and M may be found.

Figures 4 and 5 show the (percentage relative) error in the approximation of the first five coefficients $\alpha_j, \beta_j, j = 1, \ldots, 5$, in a semilogarithmic scale, as N is increased. The exponential convergence is clearly visible, as the curves are (essentially) straight lines, even for small values of N. We should mention that for α_1 there is an exact answer [23], while for the rest we used a reference value for the computations.

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References

1. G. Georgiou, L. Olson, and G. Smyrlis, A singular function boundary integral method for the Laplace equation, Commun Numer Meth Eng 12 (1996), 127–134.
2. Z. C. Li, Penalty combinations of Ritz-Galerkin and finite-difference methods for singularity problems, J Comput Appl Math 81 (1997), 1–17.
3. I. Babuška and A. Miller, The post-processing approach in the finite element method. II. The calculation of stress intensity factors, Int J Numer Meth Eng 20 (1984), 1111–1129.
4. B. Szabó and Z. Yosibash, Numerical analysis of singularities in two dimensions. II. Computation of generalized flux/stress intensity factors, Int J Numer Meth Eng 39 (1996), 409–434.
5. G. Georgiou, A. Boudouvis, and A. Poullikkas, Comparison of two methods for the computation of singular solutions in elliptic problems, J Comp Appl Math 79 (1997), 277–290.
6. M. Elliotis, G. Georgiou, and C. Xenophontos, Solution of the planar Newtonian stick-slip problem with a singular function boundary integral method, Int J Numer Methods Fluids 48 (2005), 1000–1021.
7. M. Elliotis, G. Georgiou, and C. Xenophontos, The singular function boundary integral method for a two-dimensional fracture problem, Eng Anal Bound Elem 30 (2006), 100–106.
8. M. Elliotis, G. Georgiou, and C. Xenophontos, The singular function boundary integral method for biharmonic problems with crack singularities, Eng Anal Bound Elem 31 (2007), 209–215.
9. Z. C. Li, T. T. Lu, and H. Y. Hu, The collocation Trefftz method for biharmonic equations with crack singularities, Eng Anal Bound Elem 28 (2004), 79–96.
10. Z. C. Li, T. T. Lu, H. Y. Hu, and A. H. D. Cheng, Trefftz and collocation Methods, WIT Press, Southampton, Boston, 2008.
11. T. T. Lu, C. M. Chang, H. T. Huang, and Z. C. Li, Stability analysis of Trefftz methods for the stick slip problem, Eng Anal Bound Elem 33 (2009), 474–484.
12. M. Elliotis, G. Georgiou, and C. Xenophontos, The solution of a Laplacian problem over an L-shaped domain with a singular function boundary integral method, Comm Numer Meth Eng 18 (2002), 213–222.
13. M. Elliotis, G. Georgiou, and C. Xenophontos, Solving Laplacian problems with boundary singularities: A comparison of a singular function boundary integral method with the p/hp version of the finite element method, Appl Math Comput 169 (2005), 485–499.
14. C. Xenophontos, M. Elliotis, and G. Georgiou, A singular function boundary integral method for Laplacian problems with singularities, SIAM J Sci Comp 28 (2006), 517–532.
15. Z. C. Li, Combined methods for elliptic equations with singularities, interfaces and infinities, Kluwer Academic Publishers, Dordrecht, 1998.
16. J. L. Lions and E. Magenes, Non-homogeneous boundary value problems and applications, Vol. I, Springer-Verlang, New York-Heidelberg, 1972.
17. D. H. Michael, The separation of a viscous fluid, Mathematica 5 (1958), 82–84.
18. T. Papanastasiou, G. Georgiou, and A. Alexandrou, Viscous fluid flow, CRC Press, Boca Raton, 1999.

19. Z. C. Li, Y. L. Chan, G. Georgiou, and C. Xenophontos, Special boundary approximation methods for Laplace equation problems with boundary singularities—applications to the Motz problem, Comp Math Appl 51 (2006), 115–142.

20. Z. C. Li, R. Mathon, and P. Sermer, Boundary methods for solving elliptic problems with singularities and interfaces, SIAM J Numer Anal 24 (1987), 487–498.

21. J. Claes, Numerical solution of partial differential equations by the finite element method, Cambridge University Press, 1987.

22. J. Włoka, Partial differential equations, Cambridge University Press, Cambridge, 1987.

23. S. Richardson, A stick-slip problem related to the motion of a free jet at low Reynolds number, Proc Cambridge Philos Soc 67 (1970), 477–489.