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Asymptotic solutions of weakly compressible Newtonian Poiseuille flows with pressure-dependent viscosity

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ABSTRACT

We consider both the axisymmetric and planar steady-state Poiseuille flows of weakly compressible Newtonian fluids, under the assumption that both the density and the shear viscosity vary linearly with pressure. The primary flow variables, i.e. the two non-zero velocity components and the pressure, as well as the mass density and viscosity of the fluid are represented as double asymptotic expansions in which the isothermal compressibility and the viscosity–pressure-dependence coefficient are taken as small parameters. A standard perturbation analysis is performed and asymptotic, analytical solutions for all the variables are obtained up to second order. These results extend the solutions of the weakly compressible flow with constant viscosity and those of the incompressible flow with pressure-dependent viscosity. The combined effects of compressibility and the pressure dependence of the viscosity are analyzed and discussed.

1. Introduction

In most isothermal flows of Newtonian liquids, the density and the viscosity are commonly assumed to be constant. Such an assumption, however, is valid only at low processing pressures and may introduce significant error when modeling flows involving high pressures or a large pressure range, such as polymer processing, crude oil and fuel oil pumping, fluid film lubrication, microfluidics, and in certain geophysical flows [1–4].

Waxy crude oil transport [5], polymer extrusion [6,7], and polymer injection molding [8] are important cases of liquid flows in long tubes where compressibility effects cannot be neglected. An exponential equation of state, relating the mass density of the fluid, ρ^*, to the total pressure, p^*, is very often used for compressible liquids [5,9]. For weakly compressible liquids, the following linear equation of state is a good approximation to the exponential equation of state:

$$\rho^* = \rho_0^* \left[1 + \varepsilon^* (p^* - p_0^*)\right] \tag{1}$$

where ε^* is the isothermal compressibility, assumed to be constant, and ρ_0^* is the mass density of the fluid at the reference pressure p_0^*. It should be noted that a superscript star throughout the text indicates a dimensional quantity.

Various numerical solutions for weakly compressible Poiseuille flows for Newtonian [10] as well as non-Newtonian fluids, such as the Carreau fluid [6], the Bingham plastic [5], and certain viscoelastic fluids [11] are available in the literature. Venerus and coworker [12,13] derived analytical perturbation solutions in terms of the compressibility for the axisymmetric and the plane isothermal Poiseuille flow of a weakly, compressible Newtonian liquid respectively, using the steamfunction/vorticity formulation and employing Eq. (1). Taliadorou et al. [14] obtained equivalent solutions using a methodology in which the perturbation is performed on the primary flow variables, i.e. on the velocity components and the pressure. Housiadas and collaborators [9,15,16] extended the primary-variable perturbation method to derive solutions of the plane and axisymmetric Poiseuille flows of a weakly compressible viscoelastic Oldroyd-B fluid.

Flows of fluids with pressure-dependent viscosity have received an increasing attention recently. The viscosity of typical liquids begins to increase substantially with pressure when pressures of the order of 1000 atm are reached [17]. In fact, under certain conditions, e.g. in elastohydrodynamics, the dependence of the viscosity on pressure may be several orders of magnitude stronger than that of density [3,17,18]. Málek and Rajagopal [19] reviewed different equations proposed in the literature in order to describe experimental observations on the pressure-dependence of the

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viscosity. The pressure-dependence of the viscosity in Poiseuille and other flows has been analyzed mathematically by various investigators [17,19–21]. Renardy [17] employed the following linear expression for the viscosity, η^*:

$$\eta^* = \eta_0^* \left[1 + \delta^*(p^* - p_0^*)\right] \tag{2}$$

where δ^* is the viscosity–pressure-dependence material constant and η_0^* is the viscosity at the reference pressure p_0^*. Recently, Kalogirou et al. [22] compiled analytical solutions for unidirectional plane, round, and annular Poiseuille flows of a Newtonian liquid assuming that the viscosity obeys Eq. (2).

In the present work, we consider the steady, isothermal Newtonian Poiseuille flows in a straight channel or slit and in a circular tube, for which both the mass density and the viscosity of the fluid depend weakly on pressure, obeying Eqs. (1) and (2), respectively. To our knowledge, studies taking into account both the compressibility and the viscosity–pressure-dependence are very scarce in the literature. Since exact analytical solutions are not possible, the objective is to obtain approximate analytical solutions for these flows by means of perturbation methods.

The rest of the paper is organized as follows. In Section 2, the governing equations and the boundary conditions are presented. In Section 3, the main steps of the perturbation method are discussed. All flow variables are expressed as double asymptotic expansions in terms of the dimensionless isothermal compressibility and the viscosity–pressure coefficient, which serve as small perturbation parameters. Perturbation solutions are then derived up to second order. The resulting analytical solutions are discussed in Section 4. It is shown in particular that at least up to second order the viscosity–pressure-dependence tends to reduce the velocity in the flow direction and to counterbalance compressibility effects on the pressure. Finally, in Section 5, concluding remarks are provided.

2. Problem and formulation

We consider the steady, weakly compressible isothermal flow of a Newtonian fluid with pressure-dependent viscosity, under zero gravity. The continuity and momentum equations can be written as follows:

$$\nabla^* \cdot (\rho^* \mathbf{u}^*) = 0 \tag{3}$$

$$\rho^* \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\nabla^* p^* + \nabla^* \cdot \boldsymbol{\tau}^* \tag{4}$$

where $\mathbf{u}^*$ is the velocity vector and $\boldsymbol{\tau}^*$ is the viscous extra-stress tensor, given by

$$\boldsymbol{\tau}^* = \eta^*(p^*) \left[\nabla^* \mathbf{u}^* + (\nabla^* \mathbf{u}^*)^T - \frac{2}{3}\mathbf{I}(\nabla^* \cdot \mathbf{u}^*)\right]. \tag{5}$$

In Eq. (5), $\nabla^* \mathbf{u}^*$ is the velocity-gradient tensor, the superscript T denotes the transpose, and $\mathbf{I}$ is the unit tensor. Substituting Eq. (5) into Eq. (4) leads to the following generalized Navier–Stokes equation:

$$\rho^* \mathbf{u}^* \cdot \nabla^* \mathbf{u}^* = -\nabla^* p^* + \eta^* \nabla^{*2} \mathbf{u}^* + \frac{\partial \eta^*}{\partial p^*} \times \left\{\nabla^* p^* \cdot \left[\nabla^* \mathbf{u}^* + (\nabla^* \mathbf{u}^*)^T\right] - \frac{2}{3}(\nabla^* \cdot \mathbf{u}^*)\nabla^* p^*\right\} + \frac{\eta^*}{3}\nabla^*(\nabla^* \cdot \mathbf{u}^*). \tag{6}$$

Two flow geometrical configurations are studied; the first is the axisymmetric Poiseuille flow in a circular tube of constant radius R^* and length L^* in cylindrical coordinates (r^*, z^*), and the second is the planar Poiseuille flow in a straight channel (or slit) of width $2H^*$ and length L^* in Cartesian coordinates (x^*, y^*) centered at the midplane. In the following, we present the axisymmetric case in more detail and provide the most important results for the planar case.

2.1. Axisymmetric flow

For the flow in a circular tube, the governing equations are rendered dimensionless scaling r^* by R^*, z^* by L^*, u_z^* by U^*, u_r^* by U^*R^*/L^*, and $p^* - p_0^*$ by $8\eta_0^* L^* U^*/R^{*2}$, where U^* is the mean velocity at the tube exit. The mass density and the viscosity are scaled by ρ_0^* and η_0^*, respectively. Thus, the two components of the momentum equation (6), the continuity equation (3), the equation of state (1), and the equation for the shear viscosity (2) become:

$$\alpha Re \rho \left(u_r \frac{\partial u_z}{\partial r} + u_z \frac{\partial u_z}{\partial z}\right) = -8\frac{\partial p}{\partial z} + \frac{\eta}{3}\left[\frac{3}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_z}{\partial r}\right) + 4\alpha^2 \frac{\partial^2 u_z}{\partial z^2} + \alpha^2 \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right)\right] + \frac{2\alpha^2}{3}\frac{\partial \eta}{\partial z}\left[2\frac{\partial u_z}{\partial z} - \frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right] + \frac{\partial \eta}{\partial r}\left(\frac{\partial u_z}{\partial r} + \alpha^2 \frac{\partial u_r}{\partial z}\right) \tag{7}$$

$$\alpha^3 Re\, \rho \left(u_r \frac{\partial u_r}{\partial r} + u_z \frac{\partial u_r}{\partial z}\right) = -8\frac{\partial p}{\partial r} + \alpha^2 \eta \left[\frac{4}{3}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(ru_r)\right) + \frac{1}{3}\frac{\partial^2 u_z}{\partial r \partial z} + \alpha^2 \frac{\partial^2 u_r}{\partial z^2}\right] + 2\alpha^2 \frac{\partial \eta}{\partial r}\left[\frac{\partial u_r}{\partial r} - \frac{1}{3r}\frac{\partial}{\partial r}(ru_r) - \frac{1}{3}\frac{\partial u_z}{\partial z}\right] + \alpha^2 \frac{\partial \eta}{\partial z}\left(\alpha^2 \frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) \tag{8}$$

$$\frac{\partial(\rho r u_r)}{\partial r} + \frac{\partial(\rho r u_z)}{\partial z} = 0 \tag{9}$$

$$\rho = 1 + \varepsilon p \tag{10}$$

$$\eta = 1 + \delta p \tag{11}$$

where the Reynolds number, Re, the aspect ratio of the tube, α, the dimensionless compressibility number, ε, and the viscosity pressure-dependence number, δ, are respectively defined by:

$$Re \equiv \frac{\rho_0^* U^* R^*}{\eta_0^*}, \qquad \alpha \equiv \frac{R^*}{L^*}, \qquad \varepsilon \equiv \frac{8\varepsilon^* \eta_0^* L^* U^*}{R^{*2}}, \qquad \delta \equiv \frac{8\delta^* \eta_0^* L^* U^*}{R^{*2}}. \tag{12}$$

The system of equations (7)–(11) closes with appropriate boundary conditions. Along the axis of symmetry, symmetry conditions are applied:

$$\frac{\partial u_z}{\partial r}(0, z) = u_r(0, z) = 0, \quad 0 \le z \le 1. \tag{13}$$

Also, no-slip and no-penetration are imposed along the tube wall:

$$u_r(1, z) = u_z(1, z) = 0, \quad 0 \le z \le 1. \tag{14}$$

Moreover, the pressure datum is set at the tube exit,

$$p(1, 1) = 0 \tag{15}$$

and the dimensionless mass flow rate is unity at any distance $z \in [0, 1]$ from the inlet plane:

$$2\int_0^1 \rho u_z r dr = 1. \tag{16}$$

2.2. Planar flow

The governing equations are rendered dimensionless by scaling x^* by L^*, y^* by H^*, u_x^* by U^*, u_y^* by U^*H^*/L^*, and $p^* - p_0^*$ by $3\eta_0^* L^* U^*/H^{*2}$, where U^* is the mean velocity (per unit width) at the channel exit. The dimensionless equations governing the flow are:

$$\alpha Re\rho\left(u_x\frac{\partial u_x}{\partial x}+u_y\frac{\partial u_x}{\partial y}\right)$$
$$=-3\frac{\partial p}{\partial x}+\eta\left(\frac{4\alpha^2}{3}\frac{\partial^2 u_x}{\partial x^2}+\frac{\partial^2 u_x}{\partial y^2}+\frac{\alpha^2}{3}\frac{\partial^2 u_y}{\partial x\partial y}\right)$$
$$+\frac{2\alpha^2}{3}\frac{\partial\eta}{\partial x}\left(2\frac{\partial u_x}{\partial x}-\frac{\partial u_y}{\partial y}\right)+\frac{\partial\eta}{\partial y}\left(\frac{\partial u_x}{\partial y}+\alpha^2\frac{\partial u_y}{\partial x}\right) \quad (17)$$

$$\alpha^3 Re\rho\left(u_x\frac{\partial u_y}{\partial x}+u_y\frac{\partial u_y}{\partial y}\right)$$
$$=-3\frac{\partial p}{\partial y}+\alpha^2\eta\left(\alpha^2\frac{\partial^2 u_y}{\partial x^2}+\frac{4}{3}\frac{\partial^2 u_y}{\partial y^2}+\frac{1}{3}\frac{\partial^2 u_x}{\partial x\partial y}\right)$$
$$+\alpha^2\frac{\partial\eta}{\partial x}\left(\alpha^2\frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y}\right)+\frac{2\alpha^2}{3}\frac{\partial\eta}{\partial y}\left(2\frac{\partial u_y}{\partial y}-\frac{\partial u_x}{\partial x}\right) \quad (18)$$

$$\frac{\partial(\rho u_x)}{\partial x}+\frac{\partial(\rho u_y)}{\partial y}=0. \quad (19)$$

The equation of state and that of the viscosity are the same as in the axisymmetric flow (Eqs. (10) and (11)). The definitions of the dimensionless numbers are analogous to those of Eq. (12) for the axisymmetric case, with H^* replacing R^* and 3 replacing 8. The boundary conditions are also similar to Eqs. (13)–(16): $\partial u_x/\partial y(x,0)=u_y(x,0)=0$ along the symmetry plane, $u_x(x,1)=u_y(x,1)=0$ along the slit wall, $p(1,1)=0$, and the dimensionless flow rate (per unit length in the neutral direction) is unity, i.e. $\int_0^1 \rho u_x dy=1$ at any distance x from the inlet plane.

Proving the existence and uniqueness of the solution to the problem under consideration is an open and substantially difficult task, which is out of the scope of the present work. The reader is referred, for example, to the work of Lions [23] and to the papers by Lanzendörfer [24] and Jesslé et al. [25] for relevant discussions. However, it should be pointed out that the asymptotic solutions derived in Section 3 are unique, due to the fact that the governing equations that result from the perturbation procedure are linear.

3. Perturbation method

Under the assumption that the compressibility number, ε, and the viscosity–pressure coefficient, δ, are small, the primary flow variables can be expressed as a double asymptotic expansion in ε and δ:

$$X=X^{(00)}+\varepsilon X^{(10)}+\delta X^{(01)}+\varepsilon^2X^{(20)}$$
$$+\delta^2X^{(02)}+\varepsilon\delta X^{(11)}+h.o.t. \quad (20)$$

where X is any primary variable, e.g. $X\in\{u_z,u_r,p,\rho,\eta\}$ in the axisymmetric case, the superscript (ij) denotes a variable at order $O(\varepsilon^i\delta^j)$ and $h.o.t.$ stands for "*higher order terms*", which includes terms of $O(\varepsilon^3,\delta^3,\varepsilon^2\delta,\varepsilon\delta^2)$ and higher. Substituting the expansions of all primary variables into the governing equations and collecting terms of the same order, we derive a sequence of systems of partial differential equations at each order (ij) that is solved along with the boundary conditions at the same order. More details are given in [26].

There are three possible balances between ε and δ: $\varepsilon\sim\delta$, $\varepsilon\ll\delta$, and $\varepsilon\gg\delta$. We focus on the first case and derive the perturbation solution up to second order, i.e. for the zero-order and the orders ε, δ, ε^2, δ^2, and $\varepsilon\delta$. Since the derivation of the zero-order solution is trivial, the solutions at orders ε and ε^2 are given in [14], and the solutions at orders δ and δ^2 are Taylor expansions of the analytical solution provided in [22], the lengthy but straightforward derivation of the perturbation solution is briefly discussed below.

3.1. Axisymmetric flow

The zero-order solution corresponds to the classical incompressible Newtonian Poiseuille flow, i.e.

$$u_z^{(00)}=2(1-r^2),\qquad u_r^{(00)}=0,$$
$$p^{(00)}=\tilde{z},\qquad \rho^{(00)}=1,\qquad \eta^{(00)}=1 \quad (21)$$

where

$$\tilde{z}\equiv 1-z\geq 0. \quad (22)$$

The deviations from the incompressible parabolic solution, at orders ε and ε^2 have been derived by Taliadorou et al. [14] (also confirmed in [13] for the weakly compressible flow with constant viscosity, since at these orders the viscosity–pressure dependence has no effect on the solution. The deviations are due to fluid inertia and to geometric effects, represented by the Reynolds number, *Re*, and the tube aspect ratio, α, respectively. In the current notation we have in addition $\eta^{(10)}=\eta^{(20)}=0$.

In order to derive the solution at order δ, we assume that $u_r^{(01)}=u_r^{(01)}(r)$. From Eqs. (10) and (11) we respectively get $\rho^{(01)}=0$ and $\eta^{(10)}=p^{(00)}$. Then, from the continuity equation at order δ we find that $u_z^{(01)}=F(r)$, where F is a function to be determined. Similarly, substituting all expressions into the r-momentum equation and integrating with respect to r, we find that $p^{(01)}=G(z)-\alpha^2(1-r^2)/4$ where G is another unknown function to be determined. Substituting all the known quantities into the r- and z-momentum equations, and separating variables we find three ordinary differential equations for $u_r^{(01)}$, F and G which are solved analytically. Finally, applying the boundary conditions we find that $u_r^{(01)}=0$, $F=0$ and $G=\tilde{z}^2/2$. Therefore the solution at order δ is:

$$u_z^{(01)}=0,\qquad u_r^{(01)}=0,\qquad p^{(01)}=\frac{\tilde{z}^2}{2}-\frac{\alpha^2}{4}(1-r^2),$$
$$\rho^{(01)}=0,\qquad \eta^{(01)}=\tilde{z}. \quad (23)$$

There are no contributions to the velocity at this order nor any compressibility effects. We observe, however, that there is a quadratic correction to the pressure profile, which introduces dependence on the radial coordinate r and increases with α^2 (geometric effect).

For the solution of order δ^2, we assume that $u_r^{(02)}=u_r^{(02)}(r)$. From the equation of state we easily deduce that $\rho^{(02)}=0$, and from Eq. (11) that $\eta^{(02)}=p^{(01)}$. Then, substituting all the known quantities into the continuity, z-momentum, and r-momentum equations at order δ^2, carrying out suitable integrations, using separation of variables, and applying the boundary conditions, we eventually find that $u_r^{(02)}$ vanishes. It turns out that the solution at order δ^2 is:

$$u_z^{(02)}=\frac{\alpha^2}{24}(1-r^2)(1-3r^2),\qquad u_r^{(02)}=0,$$
$$p^{(02)}=\frac{\tilde{z}^3}{6}-\frac{\alpha^2}{12}(2-3r^2)\tilde{z},\qquad \rho^{(02)}=0, \quad (24)$$
$$\eta^{(02)}=\frac{\tilde{z}^2}{2}-\frac{\alpha^2}{4}(1-r^2).$$

To obtain the solution at $O(\varepsilon\delta)$, which represents the coupling between compressibility and viscosity–pressure-dependence effects, we first assume that $u_r^{(11)}=u_r^{(11)}(r)$. For the density and the viscosity we easily find that $\rho^{(11)}=p^{(01)}$ and $\eta^{(11)}=p^{(10)}$. As described in [26], substituting into the remaining equations at this order and following similar steps as above, we find again that the transverse velocity vanishes and eventually get the following solution:

$$\left.\begin{aligned} u_z^{(11)} &= -(1-r^2)\tilde{z}^2 + \frac{\alpha^2}{36}(1-r^2)(11+3r^2) \\ u_r^{(11)} &= 0 \\ p^{(11)} &= -\frac{2}{3}\tilde{z}^3 + \frac{\alpha Re}{4}\tilde{z}^2 + \frac{\alpha^2}{36}(31-24r^2)\tilde{z} \\ &\quad - \frac{\alpha^3 Re}{144}(7-9r^4+2r^6) \\ \rho^{(11)} &= \frac{\tilde{z}^2}{2} - \frac{\alpha^2}{4}(1-r^2) \\ \eta^{(11)} &= -\frac{\tilde{z}^2}{2} + \frac{\alpha Re}{4}\tilde{z} + \frac{\alpha^2}{12}(1-r^2) \end{aligned}\right\}. \tag{25}$$

It is clear that $\rho^{(11)}$ decreases monotonically (decompression) with the axial distance z, while $\eta^{(11)}$ increases monotonically provided that $\alpha \ll 1$.

Combining all solutions up to second order we get the following perturbation solution:

$$\begin{aligned} u_z \approx (1-r^2)\Bigg\{ & 2 + \varepsilon\left[-2\tilde{z} - \frac{\alpha Re}{18}\left(2-7r^2+2r^4\right)\right] \\ & + \frac{\delta^2\alpha^2}{24}\left(1-3r^2\right) + \varepsilon\delta\left[-\tilde{z}^2 + \frac{\alpha^2}{36}(11+3r^2)\right] \\ & + \varepsilon^2\bigg[3\tilde{z}^2 - \frac{\alpha Re}{6}(1+7r^2-2r^4)\tilde{z} + \frac{\alpha^2}{72}(1-27r^2) \\ & + \frac{\alpha^2 Re^2}{21600}\left(43-957r^2+2343r^4-1257r^6+168r^8\right)\bigg]\Bigg\} \end{aligned} \tag{26}$$

$$u_r \approx \varepsilon^2 \frac{\alpha Re}{36} r\left(1-r^2\right)^2\left(4-r^2\right) \tag{27}$$

$$\begin{aligned} p \approx\ & \tilde{z} + \frac{\varepsilon}{2}\left[-\tilde{z}^2 + \frac{\alpha Re}{2}\tilde{z} + \frac{\alpha^2}{6}(1-r^2)\right] \\ & + \frac{\delta}{2}\left[\tilde{z}^2 - \frac{\alpha^2}{2}(1-r^2)\right] + \frac{\delta^2}{6}\left[\tilde{z}^3 - \frac{\alpha^2}{2}\left(2-3r^2\right)\tilde{z}\right] \\ & + \frac{\varepsilon^2}{2}\bigg[\tilde{z}^3 - \alpha Re\,\tilde{z}^2 + \frac{\alpha^2}{18}\left(9r^2-29\right)\tilde{z} \\ & + \frac{2\alpha^2 Re^2}{27}\tilde{z} + \frac{\alpha^3 Re}{216}(1-r^2)\left(19-35r^2+10r^4\right)\bigg] \\ & + \varepsilon\delta\bigg[-\frac{2}{3}\tilde{z}^3 + \frac{\alpha Re}{4}\tilde{z}^2 + \frac{\alpha^2}{36}(31-24r^2)\tilde{z} \\ & - \frac{\alpha^3 Re}{144}(7-9r^4+2r^6)\bigg] \end{aligned} \tag{28}$$

$$\begin{aligned} \rho \approx\ & 1 + \varepsilon\tilde{z} + \frac{\varepsilon^2}{2}\left[-\tilde{z}^2 + \frac{\alpha Re}{2}\tilde{z} + \frac{\alpha^2}{6}(1-r^2)\right] \\ & + \frac{\varepsilon\delta}{2}\left[\tilde{z}^2 - \frac{\alpha^2}{2}(1-r^2)\right] \end{aligned} \tag{29}$$

$$\begin{aligned} \eta \approx\ & 1 + \delta\tilde{z} + \frac{\delta^2}{2}\left[\tilde{z}^2 - \frac{\alpha^2}{6}(1-r^2)\right] \\ & + \frac{\varepsilon\delta}{2}\left[-\tilde{z}^2 + \frac{\alpha\, Re}{2}\tilde{z} + \frac{\alpha^2}{6}(1-r^2)\right]. \end{aligned} \tag{30}$$

3.2. Planar flow

The zero-order solution is the standard Poiseuille flow solution. The deviations from the latter solution at orders ε and ε^2 have been derived by various researchers [13–15] for the weakly compressible flow with constant viscosity. The procedure followed to derive higher-order solutions is similar to that described in Section 3.1 for the axisymmetric flow. Combining all solutions up to second order, one gets:

$$\begin{aligned} u_x \approx\ & \left(1-y^2\right)\Bigg\{\frac{3}{2} + \frac{3\varepsilon}{2}\left[-\tilde{x} + \frac{\alpha Re}{140}\left(-5+28y^2-7y^4\right)\right] \\ & + \frac{\delta^2\alpha^2}{20}\left(1-5y^2\right) + \frac{\varepsilon\delta}{4}\left[-3\tilde{x}^2 + \frac{\alpha^2}{10}\left(23+5y^2\right)\right] \\ & + \frac{\varepsilon^2}{4}\bigg[9\tilde{x}^2 - \frac{9\alpha Re}{70}\left(19+28y^2-7y^4\right)\tilde{x} \\ & - \frac{\alpha^2}{2}\left(1+3y^2\right) - \frac{3\alpha^2 Re^2}{107800} \\ & \times\left(2193-9163y^2-6853y^4+5159y^6-616y^8\right)\bigg]\Bigg\} \end{aligned} \tag{31}$$

$$u_y \approx \varepsilon^2 \frac{3\alpha Re}{140} y\left(1-y^2\right)^2\left(5-y^2\right) \tag{32}$$

$$\begin{aligned} p \approx\ & \tilde{x} + \frac{\varepsilon}{2}\left[-\tilde{x}^2 + \frac{36\alpha Re}{35}\tilde{x} + \frac{\alpha^2}{3}\left(1-y^2\right)\right] \\ & + \frac{\delta}{2}\left[\tilde{x}^2 - \alpha^2\left(1-y^2\right)\right] + \frac{\delta^2}{2}\left[\frac{\tilde{x}^3}{3} - \frac{\alpha^2}{5}\left(3-5y^2\right)\tilde{x}\right] \\ & + \varepsilon^2\bigg[\frac{\tilde{x}^3}{2} - \frac{36\alpha Re}{35}\tilde{x}^2 - \frac{\alpha^2}{6}\left(11-3y^2\right)\tilde{x} + \frac{3044\alpha^2 Re^2}{13475}\tilde{x} \\ & + \frac{\alpha^3 Re}{840}\left(1-y^2\right)\left(97-140y^2+35y^4\right)\bigg] \\ & + \varepsilon\delta\bigg[-\frac{2}{3}\tilde{x}^3 + \frac{18\alpha Re}{35}\tilde{x}^2 + \frac{\alpha^2}{15}\left(27-20y^2\right)\tilde{x} \\ & - \frac{\alpha^3 Re}{280}\left(1-y^2\right)\left(67+28y^2-7y^4\right)\bigg] \end{aligned} \tag{33}$$

$$\begin{aligned} \rho \approx\ & 1 + \varepsilon\tilde{x} + \frac{\varepsilon^2}{2}\left[-\tilde{x}^2 + \frac{36\alpha Re}{35}\tilde{x} + \frac{\alpha^2}{3}(1-y^2)\right] \\ & + \frac{\varepsilon\delta}{2}\left[\tilde{x}^2 - \alpha^2(1-y^2)\right] \end{aligned} \tag{34}$$

$$\begin{aligned} \eta \approx\ & 1 + \delta\tilde{x} + \frac{\delta^2}{2}\left[\tilde{x}^2 - \alpha^2\left(1-y^2\right)\right] \\ & + \frac{\varepsilon\delta}{2}\left[-\tilde{x}^2 + \frac{36\alpha R\,e}{35}\tilde{x} + \frac{\alpha^2}{3}\left(1-y^2\right)\right] \end{aligned} \tag{35}$$

where $\tilde{x} \equiv 1-x \geq 0$.

4. Discussion

In this section we discuss the perturbation solutions found in Section 3, along with some interesting quantities which are usually reported for internal, pressure driven flows, i.e. the volumetric flow rate, Q, the average pressure drop required to drive the flow $\overline{\Delta p} \equiv \bar{p}(0) - \bar{p}(1)$ (where the overbar means averaging in the transverse direction), and the average Darcy friction factor, $\bar{f}$, defined as follows:

$$Q \equiv \begin{cases} 2\int_0^1 u_z(r,z)rdr, & \text{axisymmetric case} \\ \int_0^1 u_x(x,y)dy, & \text{planar case} \end{cases} \tag{36}$$

$$\overline{\Delta p} \equiv \begin{cases} 2\int_0^1 [p(r,0)-p(r,1)]\, rdr, & \text{axisymmetric case} \\ \int_0^1 [p(0,y)-p(1,y)]\, dy, & \text{planar case} \end{cases} \tag{37}$$

$$\bar{f} \equiv \begin{cases} -\dfrac{8}{Re}\displaystyle\int_0^1 \eta(1,z)\frac{\partial u_z}{\partial r}(1,z)\, dz, & \text{axisymmetric case} \\ -\dfrac{8}{Re}\displaystyle\int_0^1 \eta(x,1)\frac{\partial u_x}{\partial y}(x,1)\, dx, & \text{planar case.} \end{cases} \tag{38}$$

4.1. Incompressible flow ($\varepsilon = 0$, $\delta > 0$)

First, it is interesting to consider the incompressible case, i.e. when $\varepsilon = 0$, for which the relation between the shear viscosity and the pressure is given by Eq. (11).

Incompressible axisymmetric flow

For the axisymmetric case the (dimensionless) analytical solution is given by:

$$u_z = \frac{64}{A\alpha^2\delta^2}\ln\left[\frac{I_0(A\delta a/8)}{I_0(A\delta a\, r/8)}\right], \qquad u_r = 0,$$
$$p = \frac{1}{\delta}\left[\frac{I_0(A\delta a\, r/8)}{I_0(A\delta a/8)}e^{A\delta\tilde{z}/8} - 1\right], \qquad \eta = \frac{I_0(A\delta ar/8)}{I_0(A\delta a/8)}e^{A\delta\tilde{z}/8} \tag{39}$$

where I_0 is the modified Bessel function of zero order. The above solution coincides with that derived by Kalogirou et al. [22] who considered the flow domain $-1 \le z \le 0$ and imposed a zero datum pressure at the center of the tube exit; as a result, the expression of the pressure (and the viscosity) is slightly different. In Eq. (39), A is a constant that can be found by demanding that the dimensionless mean velocity is unity. It turns out that A is a root of the following equation

$$2\int_0^1 \ln[I_0(A\delta a\, r/8)]\, rdr - \ln[I_0(A\delta a/8)] + \frac{A\delta^2\alpha^2}{64} = 0. \tag{40}$$

Results for A as a function of δa have been presented in [22]. In particular, it has been shown that A is an increasing function of δa. For the average pressure gradient we find

$$\overline{\Delta p} = \frac{16(e^{A\delta/8}-1)I_1(A\alpha\delta/8)}{A\alpha\delta^2 I_0(A\delta a/8)} \tag{41}$$

where I_1 is the modified Bessel function of first order. It is deduced from Eq. (41) that for $\alpha\delta < 1$, $\overline{\Delta p}$ increases with $\alpha\delta$. Given that the flow is incompressible $\bar{f} = 32\overline{\Delta p}/Re$.

Next, we proceed with the derivation of asymptotic expressions for the solution (39). Indeed, when $\delta \ll 1$ we can assume that $X = X^{(0)} + \delta X^{(1)} + \delta^2 X^{(2)} + \cdots$, where $X \in \{u_z, p, \eta, A\}$. Substituting in Eqs. (39) and (40) and collecting terms of the same order, we get asymptotic expressions for all variables. These expressions up to $O(\delta^4)$ are:

$$u_z \approx (1-r^2)\left\{2 + \frac{\delta^2\alpha^2}{24}(1-3r^2) + \frac{\delta^4\alpha^4}{288}(1-r^2)(1-4r^2)\right\} \tag{42}$$

$$\begin{aligned} p \approx\ & \tilde{z} + \frac{\delta}{2}\left[\tilde{z}^2 - \frac{\alpha^2}{2}(1-r^2)\right] + \frac{\delta^2}{6}\left[\tilde{z}^3 - \frac{\alpha^2}{2}(2-3r^2)\tilde{z}\right] \\ & + \frac{\delta^3}{24}\left[\tilde{z}^4 - \alpha^2(1-3r^2)\tilde{z}^2 + \frac{\alpha^4}{8}(1-r^2)(1-3r^2)\right] \\ & + \frac{\delta^4}{24}\left[\frac{\tilde{z}^5}{5} + \alpha^2 r^2\tilde{z}^3 - \frac{\alpha^4}{8}(1-3r^4)\tilde{z}\right] \end{aligned} \tag{43}$$

$$A \approx 8 + \frac{2\delta^2\alpha^2}{3} + \frac{\delta^4\alpha^4}{12}. \tag{44}$$

The expression for the viscosity η can be found up to $O(\delta^5)$ using Eqs. (11) and (43). It is easily verified that when $\varepsilon = 0$ Eqs. (42) and (43) agree with Eqs. (26) and (28) up to $O(\delta^2)$.

The asymptotic expression for $\overline{\Delta p}$ up to $O(\delta^4)$ is:

$$\overline{\Delta p} \approx 1 + \frac{\delta}{2} + \delta^2\left(\frac{1}{6} - \frac{\alpha^2}{24}\right) + \delta^3\left(\frac{1}{24} + \frac{\alpha^2}{48}\right) + \delta^4\left(\frac{1}{120} + \frac{\alpha^2}{48}\right). \tag{45}$$

It is clear that for typical values of the aspect ratio α, $\overline{\Delta p}$ is an increasing function of δ.

Incompressible planar flow

The solution for the incompressible planar flow is [22]:

$$u_x = \frac{9}{A\alpha^2\delta^2}\ln\left[\frac{\cosh(A\delta a/3)}{\cosh(A\delta a\, y/3)}\right], \qquad u_y = 0,$$
$$p = \frac{1}{\delta}\left[\frac{\cosh(A\delta a\, y/3)}{\cosh(A\delta a/3)}e^{A\delta\tilde{x}/3} - 1\right],$$
$$\eta = \frac{\cosh(A\delta a\, y/3)}{\cosh(A\delta a/3)}e^{A\delta\tilde{x}/3} \tag{46}$$

where A is the root of

$$\int_0^1 \ln[\cosh(A\delta a\, y/3)]\, dy - \ln[\cosh(A\delta a/3)] + \frac{A\delta^2\alpha^2}{9} = 0. \tag{47}$$

The above solution is the same as that of Kalogirou et al. [22], if the differences in the definitions of the axial coordinate and the datum pressure are taken into account.

The average pressure drop is given by:

$$\overline{\Delta p} = \frac{3(e^{A\delta/3}-1)}{A\delta^2 a}\tanh(A\delta a/3). \tag{48}$$

In the case of incompressible planar flow, $\bar{f} = 24\overline{\Delta p}/Re$.

The corresponding expansions of the solution and all quantities of interest up to $O(\delta^4)$ are:

$$u_x \approx \frac{1}{2}(1-y^2)\left[3 + \frac{\delta^2\alpha^2}{10}(1-5y^2) + \frac{\delta^4\alpha^4}{1050}(23 - 175y^2 + 140y^4)\right] \tag{49}$$

$$\begin{aligned} p \approx\ & \tilde{x} + \frac{\delta}{2}\left[\tilde{x}^2 - \alpha^2(1-y^2)\right] + \frac{\delta^2}{2}\left[\frac{\tilde{x}^3}{3} - \frac{\alpha^2}{5}\left(3-5y^2\right)\tilde{x}\right] \\ & + \frac{\delta^3}{120}\left[5\tilde{x}^4 - 6\alpha^2(1-5y^2)\tilde{x}^2 + \alpha^4\left(1+6y^2+5y^4\right)\right] \\ & + \frac{\delta^4}{4200}\left[35\tilde{x}^5 + \alpha^2(1+10y^2)\tilde{x}^3 + \alpha^4\left(-121 + 210y^2 + 175y^4\right)\tilde{x}\right] \end{aligned} \tag{50}$$

$$A \approx 3\left(1 + \frac{\delta^2\alpha^2}{5} + \frac{11\delta^4\alpha^4}{175}\right) \tag{51}$$

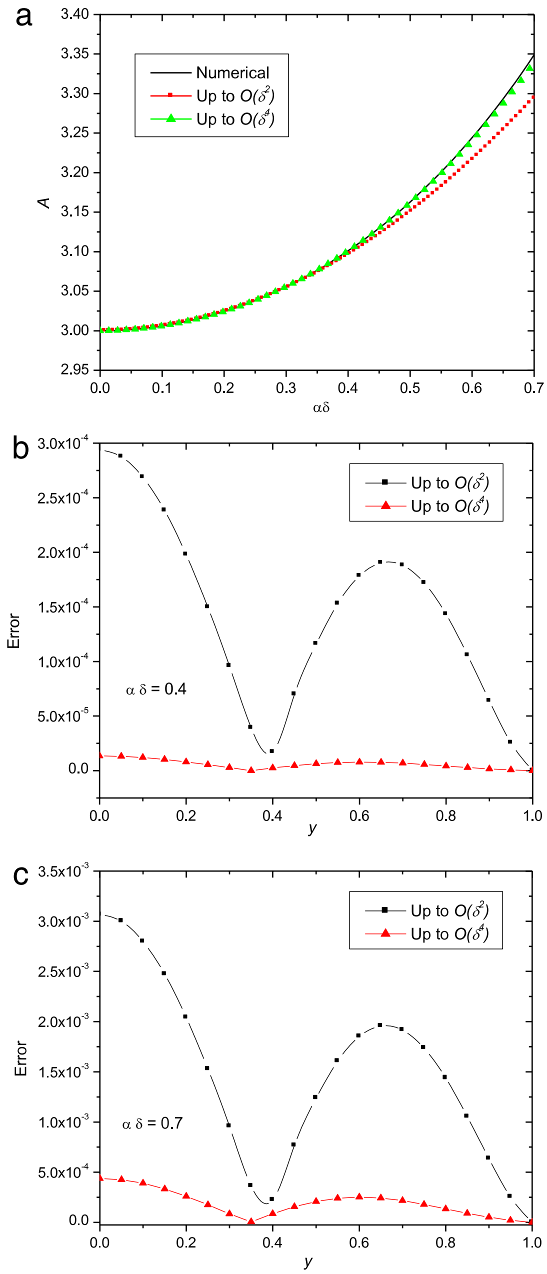

Fig. 1. Incompressible case ($\varepsilon = 0$): (a) Comparison between the numerical solution and the perturbation solutions up to $O(\delta^2)$ and $O(\delta^4)$ for the A parameter, (b) the absolute error in u_x between the analytical solution and the perturbation solution up to $O(\delta^2)$ and $O(\delta^4)$ for $\alpha\,\delta = 0.4$, (c) the absolute error in u_x between the analytical solution and the perturbation solution up to $O(\delta^2)$ and $O(\delta^4)$ for $\alpha\,\delta = 0.7$.

$$\overline{\Delta p} \approx 1 + \frac{\delta}{2} + \delta^2\left(\frac{1}{6} - \frac{2\alpha^2}{15}\right) + \delta^3\left(\frac{1}{24} + \frac{\alpha^2}{30}\right) + \delta^4\left(\frac{1}{120} + \frac{2\alpha^2}{45} - \frac{2\alpha^4}{525}\right). \quad (52)$$

Let us compare the exact solution in Eq. (46) with the perturbation solution in Eq. (49), for various values of the dimensionless viscosity pressure-dependence number δ. The variation of the parameter A with $\alpha\delta$ is illustrated in Fig. 1(a), where the exact solution is plotted along with the perturbation solutions up to $O(\alpha^2\delta^2)$ and $O(\alpha^4\delta^4)$. It should be noted that in general $\alpha \ll 1$ and $\delta < 1$, which implies that $\alpha\delta \ll 1$. It is seen that the second-order accurate solution is a very good approximation to the numerical solution up to $\alpha\delta \approx 0.4$, while the agreement for the fourth order accurate solution is excellent up to $\alpha\delta \approx 0.7$. The absolute errors in the axial velocity profile for $\alpha\delta = 0.4$ and 0.7 are plotted in Fig. 1(b) and (c), respectively. It is clear that the perturbation solution approximates the exact solution very well; as expected, the fourth-order accurate solution is much more accurate than the second-order solution. Therefore, there is strong indication that the asymptotic solutions found in the present paper are very good approximations of the full solution.

4.2. Compressible flow ($\varepsilon, \delta > 0$)

In this subsection we consider the general case in which both ε and δ are nonzero.

Compressible axisymmetric flow

The perturbation solutions in Eqs. (26)–(28) reveal some interesting features of the steady compressible Newtonian Poiseuille flow with pressure-dependent viscosity. The radial velocity u_r, which is zero by assumption at first order in ε, is always positive at second order, depends only on r (and not on z), varies linearly with α and Re, and is independent of δ at second order. On the other hand, the solutions $u_z^{(10)}$ and $u_z^{(20)}$ of the velocity component in the flow direction depend on both z and r. These may be positive or negative depending on the values of α, Re, z and r. Also, $u_z^{(01)}$ is zero so the pressure-dependence of the viscosity does not affect the velocity at this order; $u_z^{(02)}$ depends only on r, changes sign in the flow domain, and is proportional to α^2. Finally, $u_z^{(11)}$ can be either positive or negative depending on the values of α and r.

In the limit $\alpha \to 0$ (infinitely long tube), the axial velocity is simplified to

$$u_z \approx (1 - r^2)\left(2 - 2\varepsilon\tilde{z} - \varepsilon\delta\tilde{z}^2 + 3\varepsilon^2\tilde{z}^2\right). \quad (53)$$

We observe that $u_z^{(02)}$ vanishes, $u_z^{(10)} \leq 0$, and $u_z^{(20)} \geq 0$ everywhere in the tube. All components are independent of inertia effects. Also, $u_z^{(11)}$ is everywhere negative, i.e. the combination of compressibility and viscosity pressure-dependence reduces the velocity in the flow direction. It should be noted that here, and in all subsequent discussions, when we consider the limit $\alpha \to 0$, the Reynolds number is assumed to be sufficiently small. More specifically, in order for the assumptions of the asymptotic expansions to be satisfied, the coefficients in all orders should be of order 1. Given that long tubes are of interest, α is small, and therefore the terms involving only α do not impose a threat to the validity of asymptotic expansions (26)–(28). Therefore, considering the terms that involve also the Reynolds number Re, we derived various constraints on Re, and upon choosing the most stringent one, we found that, approximately, $Re < 1/\alpha$, that is, the longer the tube the larger the value of Re allowed.

For the volumetric flow rate we get

$$Q(z) \approx 1 - \varepsilon\tilde{z} + \frac{\varepsilon^2}{2}\left(3\tilde{z}^2 - \frac{\alpha Re}{2}\tilde{z} - \frac{\alpha^2}{9}\right) + \frac{\varepsilon\delta}{2}\left(-\tilde{z}^2 + \frac{\alpha^2}{3}\right). \quad (54)$$

The above equation shows that the major correction to the volumetric flow rate is due to the compressibility of the fluid, and is negative (naturally, the volumetric flow rate is reduced upstream due to mass conservation). The combination of compressibility and pressure-dependent viscosity reduces further the volumetric flow rate.

In Fig. 2, we show the contours of u_z for $\delta = 0.01$, $\alpha = 0.1$, $Re = 0$, and various values of ε. The value of α was intentionally chosen to be high (short tube), in order to facilitate

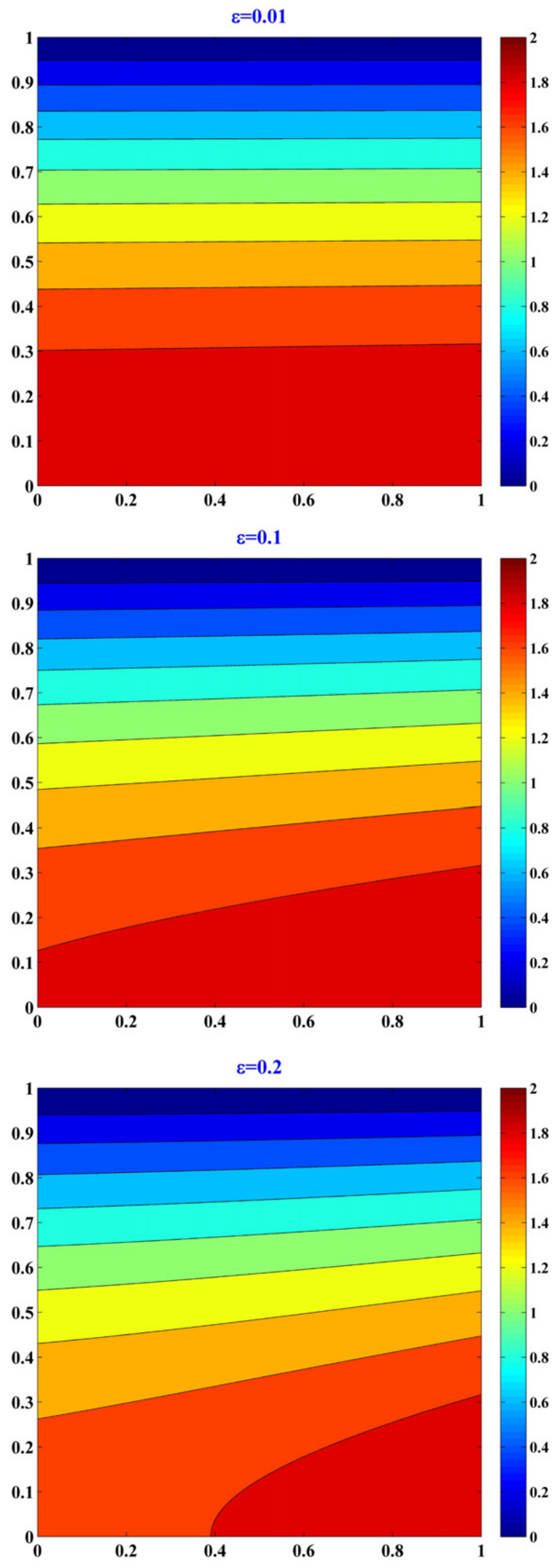

Fig. 2. Contours of u_z for $\delta = 0.01, \alpha = 0.1, Re = 0$: (top) $\varepsilon = 0.01$; (middle) $\varepsilon = 0.1$; (bottom) $\varepsilon = 0.2$.

the visualization of compressibility effects. The contour lines are obviously horizontal for low values of ε, since the dependence on z is weak. At higher values of ε, this dependence increases; the fluid accelerates downstream, and thus the contour lines tend to bend towards the axis of symmetry. The larger deviation here is that of $u_z^{(10)}$ which increases linearly with z, and this causes a more pronounced bending of the contours close to the symmetry axis. In Fig. 3, the contours of u_z for the higher value $\delta = 0.1$ are shown. Again compressibility has a more pronounced effect on the velocity. The viscosity–pressure-dependence only contributes slightly to the acceleration of the fluid downstream.

The pressure at zero order, $p^{(00)}$, is independent of r, being simply the standard linear distribution of steady, incompressible, Poiseuille flow with constant viscosity. Both the compressibility and the pressure dependence of viscosity introduce dependence on r. For very long channels ($\alpha \to 0$), we observe that $p^{(10)} \approx -\tilde{z}^2/2$, i.e. $p^{(10)}$ is *negative* for almost all values of z. Therefore, compressibility causes an order-ε reduction to the pressure.

At order δ, $p^{(01)}$ varies quadratically with both z and r and the r-dependence becomes stronger as α increases. When $\alpha \to 0$, $p^{(01)} \approx \tilde{z}^2/2$. The effects of the pressure-dependence of viscosity and of compressibility compete with each other. At first order, $p^{(01)}$ and $p^{(10)}$ cancel each other out when $\varepsilon \sim \delta$. Similarly, at the second order and for long tubes, we find that the contributions of $p^{(20)}$ and $p^{(02)}$ are canceled by $p^{(11)}$. The competition of compressibility and viscosity pressure-dependence effects is also present when α is finite. It also appears at the second order for u_z, ρ, and η.

The mean pressure drop for the axisymmetric case is:

$$\overline{\Delta p} \approx 1 + \frac{\delta}{2} + \delta^2\left(\frac{1}{6} - \frac{\alpha^2}{24}\right) + \varepsilon\left(-\frac{1}{2} + \frac{\alpha Re}{4}\right) + \frac{\varepsilon^2}{2}\left(1 - \frac{49\alpha^2}{36} - \alpha Re + \frac{2\alpha^2 Re^2}{27}\right) + \varepsilon\delta\left(-\frac{2}{3} + \frac{19\alpha^2}{36} + \frac{\alpha Re}{4}\right). \quad (55)$$

For $\delta = 0$, the above expression reduces to that derived previously by Housiadas et al. [16]. The net effect of the pressure-dependent shear viscosity is strictly positive, since both the $O(\delta)$ and $O(\delta^2)$ terms in Eq. (55) are positive. Indeed, the analytical solution up to second order shows an enhancement of the shear viscosity with the increase of the distance from the inlet plane. Consequently, a higher pressure difference is required to drive the flow. This is also illustrated in Fig. 4, where the mean pressure drop for $\delta = 0$ and 0.02, $Re = 0$, and $\alpha = 0.1$ is plotted versus the compressibility number ε. The dashed line corresponds to the predictions of Eq. (55) for $\delta = 0.02$ and $Re = 1$. The difference from its Re=0 counterpart is due mainly to the $O(\varepsilon)$ term.

The average Darcy friction factor is found to be

$$\frac{Re\bar{f}}{32} \approx 1 + \frac{\delta}{2} + \delta^2\left(\frac{1}{6} - \frac{\alpha^2}{24}\right) + \varepsilon\left(-\frac{1}{2} + \frac{\alpha Re}{12}\right) + \frac{\varepsilon^2}{2}\left(1 - \frac{13\alpha^2}{36} - \frac{\alpha Re}{2} + \frac{17\alpha^2 Re^2}{1080}\right) + \varepsilon\,\delta\left(-\frac{2}{3} + \frac{7\alpha^2}{36} + \frac{\alpha Re}{6}\right). \quad (56)$$

For an incompressible fluid, i.e. for $\varepsilon = 0$, Eq. (56) shows that $Re\bar{f}$ increases with δ. In the general case, i.e. for $\varepsilon, \delta > 0$, and due to the fact that the geometric effects are very small, $\alpha^2 \ll 1$, both the compressibility of the fluid and the pressure-dependence of the shear viscosity result in the reduction of the average Darcy friction factor.

Compressible planar flow

The volumetric flow rate is given by

$$Q(x) \approx 1 - \varepsilon\tilde{x} + \varepsilon^2\left(\frac{3}{2}\tilde{x}^2 - \frac{18}{35}\alpha Re\,\tilde{x} - \frac{2\alpha^2}{15}\right) + \varepsilon\delta\left(-\frac{\tilde{x}^2}{2} + \frac{2\alpha^2}{5}\right) \quad (57)$$

and the average pressure drop by

$$\overline{\Delta p} \approx 1 + \varepsilon \left(-\frac{1}{2} + \frac{18}{35}\alpha Re \right) + \frac{\delta}{2} + \varepsilon^2 \left(\frac{1}{2} - \frac{5}{3}\alpha^2 - \frac{36}{35}\alpha Re + \frac{3044}{13475}\alpha^2 Re^2 \right) + \delta^2 \left(\frac{1}{6} - \frac{2\alpha^2}{15} \right) + \varepsilon\delta \left(-\frac{2}{3} + \frac{61}{45}\alpha^2 + \frac{18}{35}\alpha Re \right). \quad (58)$$

Given that $\alpha Re \ll 1$, the mean pressure drop increases with δ and decreases with ε. Moreover, it is easily verified that when $\alpha \to 0$, $Re = 0$, and $\varepsilon \sim \delta$, the viscosity–pressure-dependence and compressibility effects cancel each other.

Finally, the average Darcy friction factor is:

$$\frac{Re\bar{f}}{24} \approx 1 + \frac{\delta}{2} + \delta^2 \left(\frac{1}{6} - \frac{2\alpha^2}{15} \right) + \varepsilon \left(-\frac{1}{2} + \frac{4\alpha Re}{35} \right) + \varepsilon^2 \left(\frac{1}{2} - \frac{\alpha^2}{3} - \frac{3\alpha Re}{7} + \frac{116\alpha^2 Re^2}{2695} \right) + \varepsilon\,\delta \left(-\frac{2}{3} + \frac{7\alpha^2}{15} + \frac{11\alpha Re}{35} \right). \quad (59)$$

5. Conclusions

Asymptotic solutions for the steady, planar and axisymmetric, Poiseuille flows of a weakly compressible Newtonian fluid with viscosity that is weakly dependent on the pressure have been obtained. Both the density and the viscosity are assumed to vary linearly with the pressure, and the primary flow variables are perturbed in terms of the compressibility number ε and the viscosity–pressure coefficient δ. The perturbation solution, derived up to the second order in terms of ε and δ, reveals the following:

(a) The transverse velocity is only affected by compressibility at second order;
(b) When ε and δ are of the same order, the horizontal velocity component is reduced;
(c) The pressure field is affected by compressibility and the viscosity–pressure-dependence at both the first order and the second order; these two effects counterbalance each other when ε and δ are of the same order. A similar competition is also observed for the main velocity component;
(d) The mean pressure drop increases with viscosity–pressure-dependence and decreases with compressibility.

The expressions for the average pressure drop and the Darcy friction factor have also been derived. These are important for the design and control of fluid transport in channels and tubes and other processes involving high pressures at which compressibility and viscosity–pressure dependent effects are important. They can also be used to study various heat transfer problems which are of significance in a variety of practical and industrial applications.

Note that the most popular formula for the dependence of the viscosity on the pressure is the exponential law proposed by Barus [27], which can be viewed as a generalization of the linear equation (2) that we have used. If Barus law is employed instead, the differences in the solutions will appear at $O(\delta^2)$ and higher. We have chosen to use the linear law, not only because of the complexity of the governing equations but also in order to be able to check the validity of the asymptotic solutions derived here with the available analytical solution of the full governing equations for unidirectional flow.

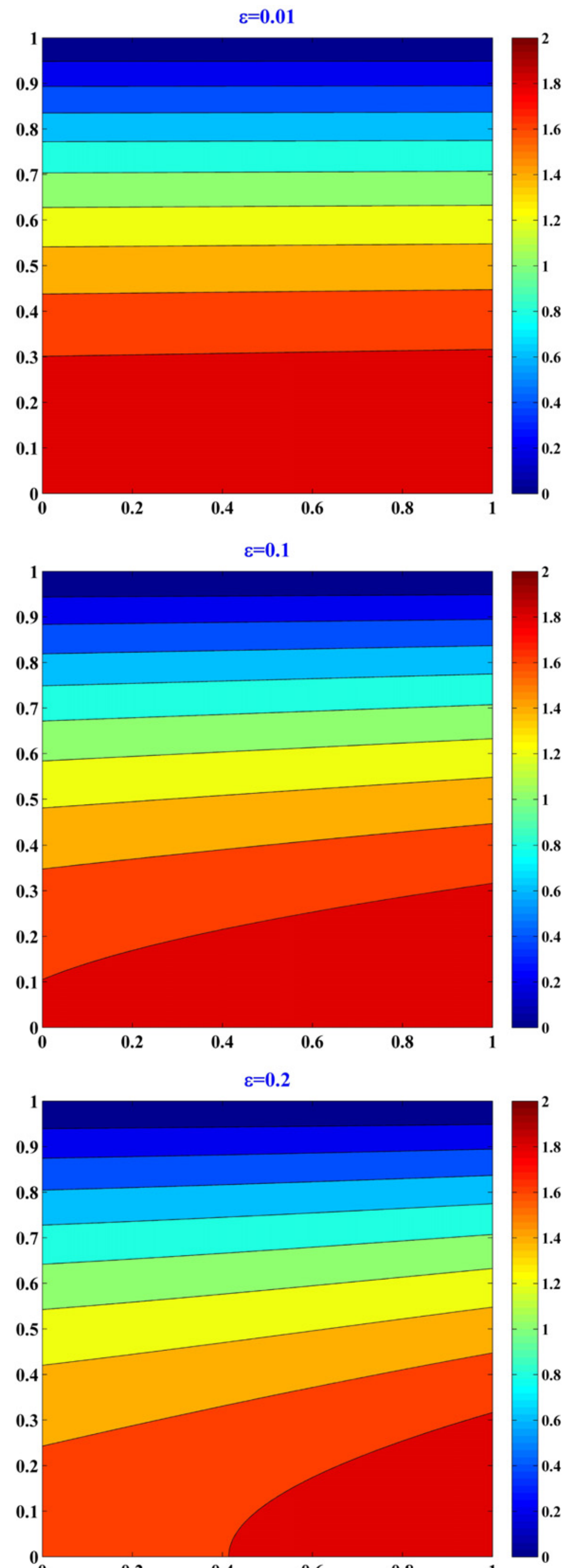

Fig. 3. Contours of u_z for $\delta = 0.1, \alpha = 0.1, Re = 0$: (top) $\varepsilon = 0.01$; (middle) $\varepsilon = 0.1$; (bottom) $\varepsilon = 0.2$.

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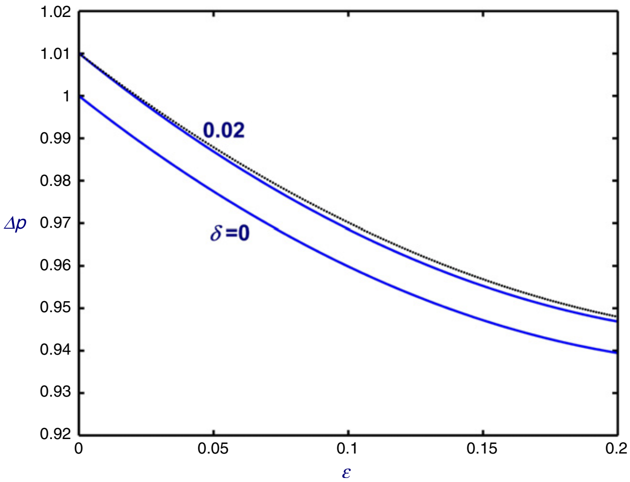

Fig. 4. Mean pressure drop in axisymmetric Poiseuille flow with $\alpha = 0.1$, $Re = 0$ and $\delta = 0$ and 0.02. The dotted line has been obtained with $\delta = 0.02$ and $Re = 1$.

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