UNIVERSAL APPROXIMATION BY TRANSLATES OF FUNDAMENTAL SOLUTIONS OF ELLIPTIC EQUATIONS

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Abstract. In the present work, we investigate the approximability of solutions of elliptic partial differential equations in a bounded domain Ω by universal series of translates of fundamental solutions of the underlying partial differential operator. The singularities of the fundamental solutions lie on a prescribed surface outside of Ω, known as the pseudo–boundary. The domains under consideration satisfy a rather mild boundary regularity requirement, namely, the Segment Condition. We study approximations with respect to the norms of the spaces \(C^\ell(\Omega)\) and we establish the existence of universal series. Analogous results are obtainable with respect to the norms of Hölder spaces \(C^{\ell,\nu}(\Omega)\) and Sobolev spaces \(W^{k,p}(\Omega)\). The sequence \(a = \{a_n\}_{n \in \mathbb{N}}\) of coefficients of the universal series may be chosen in \(l_1(\mathbb{N})\) but it can not be chosen in \(l^{1/p}(\mathbb{N})\).

1. Introduction

Universality has its roots in classical analysis, dating back almost a century, and continues to be an interesting area of research because of connections with such modern areas as operator theory and dynamical systems. The first universal series, a power series on \([-1, 1]\), was obtained by Fekete before 1914 (see [Pál15]); the main idea is that whatever can be approximated uniformly by polynomials can also be approximated uniformly by the partial sums of a power series. In 1945 Menchoff [Men45] showed the existence of a plethora of trigonometric series \(\sum_{n=-\infty}^{+\infty} a_n e^{int}\) with the property that for every complex measurable 2\(\pi\)–periodic function \(f\) there exists a sequence \(\{\lambda_n\}_{n \in \mathbb{N}} \subset \mathbb{N}\) such that \(\sum_{j=-\infty}^{+\infty} a_j e^{ij\lambda_n} \to f(t)\) almost everywhere, as \(n\) tends to infinity. Moreover, one can obtain \(a_n = 0\) for \(n < 0\) and the sequence \(a = \{a_n\}_{n \in \mathbb{N}}\) can be chosen to belong to \(c_0\) or to \(l^p(\mathbb{N})\), for \(p > 2\) (see [KN00]) or to \(l^{1/p}(\mathbb{N})\) if we transfer to this case one of the arguments of the present paper. In 1951 Seleznev [Sel51] showed the existence of power series with zero radius of convergence and universal approximation properties in \(\mathbb{C} \setminus \{0\}\). In the early 70’s W. Luh [Luh70] and Chui–Parnes [CP71] established, independently, the existence of universal Taylor series with strictly positive radius of convergence. The universal approximation is valid outside the closure of the domain of definition of the universal function. This result has been strengthened in [Nes96], where it was shown that the universal approximation is also valid on the boundary. Further, in [Nes03], it was established that the universal approximation is also valid for derivatives of any order.

Recently much effort has been done to investigate properties of universal Taylor series and to obtain other generic universalities [Kah00]. Further, in [NP05, BGENP], an abstract framework has been given which allows for unified simple proofs for almost all known results on the subject. This

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covers the theorem of Seleznev, universal Taylor series, universal Dirichlet series, universal Faber series, universal Laurent series, universal trigonometric series in the sense of Menchoff, universal expansions of harmonic functions, universal expansions of $C^\infty$--functions, etc. In addition to the above, the abstract framework simplifies the situation and leads to new universalities. One such new application is the content of the present paper which relates to fundamental solutions of elliptic operators.

The existence of a universal series according to the abstract theory ([NP05, BGENP]) is equivalent to a double approximation. In all previous cases this double approximation is easily verified thanks to well--known theorems of approximation, for example the theorems of Weierstrass, Runge, J. L. Walsh, Laurent 'ev, Keldysh and Mergelyan. However, in the present application, the approximation theorem that we need is relatively new ([Smy06a]) and its proof is also provided in the Appendix of the present paper. Our main result is of the following type:

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ satisfying some boundary regularity condition. Suppose that $\Omega'$ embraces $\Omega$ and let $\{y_j\}_{j \in \mathbb{N}}$ be a dense subset of $\partial \Omega'$, which is assumed not to contain any isolated points. Then there exists a sequence of real numbers $a = \{a_j\}_{j \in \mathbb{N}}$ such that the sequence $\sum_{m=0}^\infty a_j \varphi_{y_j}$, $m \in \mathbb{N}$, is dense in various spaces of solutions of the operator $L$ which extend to $\Omega$.

Here $L$ is an elliptic partial differential operator of order 2 with constant coefficients, $\varphi$ is a fundamental solution of $L$ and $\varphi_y$ denotes the translation of $\varphi$ by $y$ (i.e., $\varphi_y(x) = \varphi(x-y)$). This result may be generalized to elliptic operators with constant coefficients of order $2m$, in which case $m$ different fundamental solutions, of operators which are suitable factors of $L$, are required. Further, the sequence $a$ may be chosen in $\cap_{p>1} L^p(\mathbb{N})$ but it cannot be chosen in $L^1(\mathbb{N})$. The result is generic and we also have algebraic genericity. The proof combines the abstract theory of universal series ([NP05, BGENP]) with the method of fundamental solutions ([Bog85, KA63, FK98, Smy06a]). For related recent results we refer the reader to the papers by Gauthier and Tarkhanov [GT05] (on density results by linear combinations of translates of the Riemann zeta function) and by Stefanopoulos [Ste] (on density results by linear combinations of translates of the fundamental solution of Cauchy–Riemann equations).

The paper is organized as follows. In Section 2 we provide a description of the method of fundamental solutions and preliminary definitions and results of the abstract theory of universal series. Section 3 contains the main result on universal series of fundamental solutions. Finally, in order to avoid overloading the main text, a proof of an approximation theorem is given in the Appendix.

2. Preliminaries

2.1. The method of fundamental solutions: A Trefftz method. Let $L = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ be an elliptic partial differential operator with constant coefficients of order $m$. In Trefftz methods, the solution of the boundary value problem

$$Lu = 0 \quad \text{in } \Omega,$$

$$Bu = f \quad \text{on } \partial \Omega,$$

where $\Omega$ is an open domain in $\mathbb{R}^n$ and $Bu = f$ is the boundary condition, is approximated by linear combinations of particular solutions of (2.1a), provided that such linear combinations are dense in the set of all solutions of this equation in $\Omega$. Erich Trefftz presented this approach in 1926 [Tre26] as a counterpart of Ritz's method. A typical Trefftz method is the approximation of harmonic functions by harmonic polynomials. In his celebrated work [Mer52], Mergelyan showed in 1952 that if $\Omega$ is a bounded simply connected domain in $\mathbb{C}$, then holomorphic functions in the interior of $\overline{\Omega}$ which are
continuous in $\overline{\Omega}$ can be approximated, in the sense of the uniform norm, by polynomials, provided that $C \setminus \overline{\Omega}$ is connected, whereas in the case where $C \setminus \overline{\Omega}$ has finitely many components they can be approximated by rational functions (see Rudin [Rud87]).

In the method of fundamental solutions (MFS), the particular solutions of the partial differential equation under consideration are the fundamental solutions $\phi(x, y)$ of the corresponding partial differential operator. They satisfy

$$L_x \phi(\cdot, y) = \delta_y \quad \text{for every} \quad y \in \mathbb{R}^n, \quad (2.2)$$

where the notation $L_x \phi$ signifies that $\phi$ is differentiated with respect to $x$ and $\delta_y$ is the Dirac measure with unit mass at $y$, in the sense of distributions, i.e.,

$$\int_{\mathbb{R}^n} \phi(x, y) L^*_x \psi(x) \, dx = \psi(y),$$

for every $\psi \in C^0_0(\mathbb{R}^n)$, where $L^*_x = \sum_{\vert \alpha \vert \leq m} (-1)^{\vert \alpha \vert} D^\alpha (a_{\alpha} u)$. The operator $L^*$ is known as the adjoint of $L$. In particular, if $L$ is elliptic with constant coefficients, then $\phi(\cdot, y)$ is real analytic in $\mathbb{R}^n \setminus \{y\}$ and satisfies, in the classical sense, $L_x \phi(x, y) = 0$ for every $x \in \mathbb{R}^n \setminus \{y\}$. The point $y$ is known as the singularity of $\phi$. The fundamental solutions produce solutions of the corresponding inhomogeneous equation by convolution: If $u(x) = \int_{\mathbb{R}^n} \phi(x, y) f(y) \, dy$, then $L u = f$ in the sense of distributions, i.e.,

$$\int_{\mathbb{R}^n} u L^*_x \psi \, dx = \int_{\mathbb{R}^n} f \psi \, dx \quad \text{for every} \quad \psi \in C^0_0(\mathbb{R}^n).$$


Felix Browder showed¹ in 1962 that linear combinations of fundamental solutions of elliptic operators with singularities in an arbitrary open set outside a connected domain $\Omega$ without holes are dense, in the sense of the uniform norm, in the space

$$\mathcal{X} = \{ u \in C^\infty(\Omega) : L u = 0 \text{ in } \Omega \} \cap C(\overline{\Omega}).$$

Browder’s proof relies on a duality argument [Bro62]. Browder’s result extends to a partial differential operator $L$, with the property that its adjoint $L^*$ satisfies the Condition of uniqueness for the Cauchy problem in the small in $\Omega$:

(U), If $u \in C^\infty(V)$, where $V$ is an open connected subset of $\Omega$ with $L^* u = 0$ and if $u$ vanishes in a nonempty open subset of $V$, then $u$ vanishes everywhere in $V$.

Weinstock [Wei73] showed that the solutions of $L u = 0$ in $\Omega$, which are also elements of $C^\infty(\overline{\Omega})$, can be approximated by solutions of $L u = 0$ in a neighborhood of $\overline{\Omega}$, when $0 \leq \ell < m$, where $m$ is the order of $L$. In Weinstock’s work, $L$ is assumed to be an elliptic operator with constant coefficients and the domain $\Omega$ is required to satisfy a weaker condition, the Segment Condition:

**Definition 1. (The Segment Condition)** Let $\Omega$ be an open subset of $\mathbb{R}^n$. We say that $\Omega$ satisfies the Segment Condition if every $x \in \partial \Omega$ has a neighborhood $U_x$ and a nonzero vector $\xi_x$ such that, if $y \in U_x \cap \overline{\Omega}$ then $y + t \xi_x \in \Omega$ for every $t \in (0, 1)$.

¹Browder’s Theorem. Let $L$ be a linear operator with coefficients in $C^\infty(G)$, where $G$ is a domain in $\mathbb{R}^n$ without holes. Assume that both $L$ and $L^*$ satisfy condition (U), and there exists a bi-regular fundamental solution $e$ of $L$ satisfying $L_x e(\cdot, y) = \delta_y$ and $L^*_x e(\cdot, y) = \delta_y$, for every $x, y \in G$. Let $\Omega$ be an open subset of $G$, satisfying the Cone Condition, such that $\overline{\Omega} \subset G$ and $G \setminus \overline{\Omega}$ does not contain any closed connected components. Let $V$ be an open subset of $G$, such that $\overline{\Omega} \cap \overline{V} = \emptyset$. Then every solution $u$ of $L u = 0$ in $\Omega$, which lies in $C^\infty(\Omega) \cap C(\overline{\Omega})$ can be approximated, with respect to the uniform norm, by finite linear combinations of functions of the form $e(\cdot, y)$, where $y \in V$. 

A detailed survey on the extensions of Browder’s work and approximations of solutions of elliptic equations, by solutions of the same equations in larger domains can be found in [Tar95].

The MFS was introduced by Kupradze and Aleksidze [KA63] in 1963 as the method of generalized Fourier series (метод обобщённых рядов Фурье). In a typical application to the Dirichlet problem for Laplace’s equation (see [FK98, Smy06b, SK04])

\[
\begin{align*}
\Delta u &= 0 \quad \text{in } \Omega, \\
u &= f \quad \text{on } \partial \Omega,
\end{align*}
\]

the function \( \varphi(x, y) = e_1(x-y) \), where

\[
e_1(x) = \begin{cases} 
\frac{-\log |x|}{2\pi}, & \text{if } n = 2, \\
\frac{|x|^{2-n}}{(2-n)\omega_{n-1}}, & \text{if } n > 2,
\end{cases}
\]

is a fundamental solution of the Laplacian (more precisely of \(-\Delta\) which is an elliptic operator), where \( \omega_{n-1} \) is the area of the surface of the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \) and \( |\cdot| \) is the Euclidean norm in \( \mathbb{R}^n \). In the MFS, the solution of (2.3) is approximated by a finite linear combination of the form

\[
u_N(x; c) = \sum_{j=1}^{N} c_j \varphi(x, y_j) = \sum_{j=1}^{N} c_j e_1(x-y_j),
\]

where \( c = (c_j)_{j=1}^{N} \subset \mathbb{R}^N \) and \( \{y_j\}_{j=1}^{N} \), the singularities of the fundamental solutions, are located on a pseudo–boundary, i.e., a prescribed boundary \( \partial \Omega' \) of a domain \( \Omega' \) embracing \( \Omega \) (see Figure 1).

**Definition 2. (The Embracing Pseudo-boundary)** Let \( \Omega, \Omega' \) be open connected subsets of \( \mathbb{R}^n \). We say that \( \Omega' \) embraces \( \Omega \) if \( \overline{\Omega} \subset \overline{\Omega'} \), and for every connected component \( V \) of \( \mathbb{R}^n \setminus \overline{\Omega} \), there is an open connected component \( V' \) of \( \mathbb{R}^n \setminus \overline{\Omega'} \) such that \( \overline{V'} \subset V \).

**Figure 1.** In each figure, the grey region is the domain \( \Omega \); the broken lines correspond to the embracing pseudo–boundary \( \partial \Omega' \).

Comprehensive reference lists of applications of the MFS can be found in [CGML04, DEW00, FKM03, GC99] whereas a rigorous mathematical foundation of the MFS for the numerical solution of a variety of boundary value problems in mathematical physics can be found in [Ale91].
The question whether linear combinations of fundamental solutions with singularities lying on a prescribed pseudo–boundary are dense in the set of all solutions of the corresponding equation has been studied by Kupradze and Aleksidze [KA63], Bogomolny [Bog85] and Smyrlis [Smy06a]. In particular, density results with respect to the supremum norm appear in [KA63] and [Bog85], while density results with respect to the norms of the Hölder spaces $C^{\ell,\nu}(\Omega)$ and the norms of Sobolev spaces $W^{k,p}(\Omega)$ appear in [Smy06a]. A typical such result is the following:

**Theorem 1.** Let $\Omega$, $\Omega'$ be open domains in $\mathbb{R}^n$ with $\Omega$ bounded and satisfying the Segment Condition and $\Omega'$ embracing $\Omega$ and let $\ell$ be a nonnegative integer. Then the space $X$ of finite linear combinations of the form $\sum_{j=1}^{N} c_j e_1(x - y_j)$, where $e_1$ is given by (2.4) and $(y_j)_{j=1}^N \subset \partial \Omega'$, is dense in

$$\mathcal{Y}_\ell = \left\{ u \in C^2(\Omega) : \Delta u = 0 \text{ in } \Omega \right\} \cap C^\ell(\overline{\Omega}),$$  \hspace{1cm} (2.6)

with respect to the norm of the space $C^\ell(\overline{\Omega})$ if $n \geq 3$.

**Proof.** See Appendix.

**Remarks 2.1.** Theorem 1 can be extended in several ways. Analogues of this Theorem exist for more general elliptic equations and elliptic systems and also for different norms. (See [Smy06a].) In particular,

(i) Theorem 1 also holds for $n = 2$ in which case the constant function should be included in the linear combinations or the fundamental solution of the Laplacian, $e_1(x) = -\frac{1}{2\pi} \log |x|$ by a rescaled one

$$e_1^R(x) = -\frac{1}{2\pi} \log \frac{|x|}{R},$$  \hspace{1cm} (2.7)

with $R > \text{diameter}(\Omega')$. (See [Smy06a].)

(ii) Theorem 1 also holds for the solutions of the modified Helmholtz equation $\Delta u - \kappa^2 u = 0$. Modified Helmholtz operator and has as fundamental solution the function

$$e_1(x, x^2) = \begin{cases} -\frac{K_0(|x|)}{2\pi} & \text{if } n = 2, \\ -\frac{e^{-\kappa|x|}}{4\pi|x|} & \text{if } n = 3, \end{cases}$$

where $K_0(r)$ is the modified Bessel function of the second kind.

(iii) In the case of the Biharmonic equation $\Delta^2 u = 0$, the solutions can be approximated by linear combinations of the form

$$u_N(x, c, d) = \sum_{j=1}^{N} c_j e_1(x - y_j) + \sum_{j=1}^{N} d_j e_2(x - y_j),$$  \hspace{1cm} (2.8)

where $e_2$ is the (standard) fundamental solution of the biharmonic operator, in the case $n \geq 3$. If $n = 2$ one should include the function 1 and $|x|^2$ in the linear combination or rescale $e_1$ as in the case of the Laplacian. Analogous density results hold for the solutions of the poly–harmonic equation $\Delta^2 u = 0$, in which case the right hand side of (2.8) contains translates of the fundamental solutions of the operators $\Delta^j$, $j = 1, \ldots, k$.

(iv) Theorem 1 holds for the norms of the Sobolev spaces $W^{k,p}(\Omega)$, $k \in \mathbb{Z}$, $p \in [1, \infty]$ and the norms of the Hölder spaces $C^{\ell,\nu}(\Omega)$, $\ell \in \mathbb{N}$, $\nu \in (0, 1)$. Further, if $(p_{\ell})_{\ell \in \mathbb{N}}$ is a sequence of positive reals tending to infinity, then every Sobolev space $W^{k,p}(\Omega)$, $k \in \mathbb{Z}$, $p \in [1, \infty)$, with $\Omega$ bounded, contains a space $W^{n,p_n}(\Omega)$, for sufficiently large $n$. Baire’s Theorem will yield the existence of a series which is universal simultaneously for all Sobolev spaces $W^{k,p}(\Omega)$, $k \in \mathbb{Z}$, $p \in [1, \infty)$.

(v) If the pseudo–boundary $\partial \Omega'$ is an analytic surface, then in the density results the singularities are only required to lie on an open subset of $\partial \Omega'$. In particular, if $n = 2$ and $\partial \Omega'$ is analytic and connected, then for any infinite set of singularities $\{y_j\}_{j \in \mathbb{N}} \subset \partial \Omega'$ the harmonic functions in $\Omega$ can be approximated by linear combinations of the functions $\{e_1^R(x - y_j)\}_{j \in \mathbb{N}}$ given by (2.7), with $R > \text{diameter}(\Omega')$.  

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2.2. Abstract formulation of the theory of universal series. Let $X$ be a Banach space on $\mathbb{R}$ or $\mathbb{C}$ (from now on $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and $\{x_k\}_{k \in \mathbb{N}} \subset X$.

**Definition 3.** A sequence $a = \{a_k\}_{k \in \mathbb{N}} \in \mathbb{K}^\mathbb{N}$ belongs to the class $\mathcal{U}$ if the sequence of partial sums $\sum_{j=0}^k a_jx_j, k \in \mathbb{N}$, is dense in $X$. $\mathcal{U}$ is the class of unrestricted universal series.

Remarks on universal series. The properties that follow are straightforward implications of the definition of universal series:

(i) If $a = \{a_k\}_{k \in \mathbb{N}}$ is a universal series, then for every $x \in X$, there exists an increasing sequence $\{\lambda_k\}_{k \in \mathbb{N}} \subset \mathbb{N}$, such that

$$\lim_{k \to \infty} \sum_{j=0}^{\lambda_k} a_j x_j = x.$$  

(ii) If $p$ is a positive integer and $a = \{a_k\}_{k \in \mathbb{N}}$ belongs to $\mathcal{U}$, then so does $a^p$ with

$$a_k^p = \begin{cases} 0 & \text{if } k < p, \\ a_k & \text{if } k \geq p. \end{cases}$$

(iii) If $\mathcal{U} \neq \emptyset$, then $X$ is separable.

2.3. Restricted universal series. Of interest is whether universal series exist in specific subspaces of $\mathbb{K}^\mathbb{N}$. Let $A$ be a linear subspace of $\mathbb{K}^\mathbb{N}$ which is a Fréchet space on $\mathbb{K}$. In particular, we require that the space $A$ satisfies the following postulates:

**Postulates on the topology of $A$.**

$A_1$ The projections $\pi_k : A \to \mathbb{K}$, where $\pi_k(\{a_j\}_{j \in \mathbb{N}}) = a_k$, are continuous, for all $k \in \mathbb{N}$.

$A_2$ Let $G = \{\{a_k\}_{k \in \mathbb{N}} \in A : a_k \neq 0 \text{ holds only for finitely many } k \in \mathbb{N}\}$. Then $G \subset A$.

$A_3$ Let $\{e^j\}_{j \in \mathbb{N}}$ be the canonical basis of $\mathbb{K}^\mathbb{N}$, i.e., $e^j = (\delta_{jk})_{k \in \mathbb{N}}$. Then for every $a = \{a_k\}_{k \in \mathbb{N}} \in A$

$$\lim_{k \to \infty} \sum_{j=0}^k a_j e^j = a,$$

with respect to the distance of $A$.

Set $\mathcal{U}_A = \mathcal{U} \cap A$ – The class of restricted universal series.

**Remark 2.2.** More generally, postulate $A_3$ may be replaced by $\overline{G} = A$. Then $\mathcal{U}_A$ is defined differently and it does not always coincide with $\mathcal{U} \cap A$ (see [BGENP, NP05]).

The main result of the abstract theory of universal series. The result that follows will allow us to obtain universal series of translates of fundamental solutions.

**Theorem 2.** Assume that $A$ is a Fréchet space satisfying the postulates $A_1$, $A_2$ and $A_3$ and let $d(\cdot, \cdot)$ be the distance in $A$ and $\| \cdot \|_X$ the norm of $X$. Then the following are equivalent:

(i) $\mathcal{U}_A \neq \emptyset$.

(ii) For every $x \in X$ and $\varepsilon > 0$, there exist $k \in \mathbb{N}$ and $\gamma_0, \ldots, \gamma_k \in K$ such that

$$\| \gamma_0 x_0 + \cdots + \gamma_k x_k - x \|_X < \varepsilon \quad \text{and} \quad d(\gamma_0 e^0 + \cdots + \gamma_k e^k, 0) < \varepsilon.$$

(iii) $\mathcal{U}_A$ is a dense $G_\delta$ in $A$.

(iv) $\mathcal{U}_A \cup \{0\}$ contains a dense linear subspace of $A$.

**Proof.** See Nestoridis & Papadimitropoulos [NP05] and Bayart, Grosse–Erdmann, Nestoridis & Papadimitropoulos [BGENP].
Remark 2.3. Let \( \mu \) be an infinite subset of \( \mathbb{N} \). We define \( \mathcal{U}^\mu \cap A \) to be the set of sequences

\[
a = \{a_j\}_{j \in \mathbb{N}} \in \mathbb{K}^\mathbb{N} \cap A,
\]

with the property that for every \( x \in X \) there exists a sequence \( \{\lambda_n\}_{n \in \mathbb{N}} \subset \mu \) so that

\[
\lim_{n \to \infty} \sum_{j=0}^{\lambda_n} a_j x_j = x.
\]

Then the conditions (i) – (iv) of Theorem 2 are equivalent to each of the following conditions:

(i) \( \mathcal{U}^\mu \cap A \) is a \( G_\delta \) and dense in \( A \).

(ii) \( \left( \mathcal{U}^\mu \cap A \right) \cup \{0\} \) contains a vector space dense in \( A \).

For a proof see [BGENP].

3. Universal series of fundamental solutions

3.1. A typical application. Let \( \Omega \) be an open bounded domain in \( \mathbb{R}^n \), \( n \geq 3 \), satisfying the segment condition and assume that \( \Omega' \) embraces \( \Omega \). Let \( D = \{y_j\}_{j \in \mathbb{N}} \subset \partial \Omega' \) be countable and dense in \( \partial \Omega' \). We assume that \( \partial \Omega' \) does not contain any isolated points. The corresponding function space shall be

\[
X = \left\{ u \in C^2(\Omega) : \Delta u = 0 \text{ in } \Omega \right\} \cap C^1(\overline{\Omega}),
\]

which is a Banach space with respect to the norm of \( C^1(\overline{\Omega}) \).

Let \( e_1 \) be the fundamental solution of \( -\Delta \) in \( \mathbb{R}^3 \) and \( \chi_j(x) = e_1(x - y_j) \), where \( y_j \in D \).

Question. Is it possible to find a sequence \( a = \{a_k\}_{k \in \mathbb{N}} \in \mathbb{K}^\mathbb{N} \) such that the sequence of partial sums \( \left\{ \sum_{j=0}^{\lambda} a_j \chi_j \right\}_{k \in \mathbb{N}} \) is dense in \( X \)?

The set of such sequences \( a \) is denoted by \( \mathcal{U} \).

Positive and negative answers. In particular, we are interested in finding sequences \( a = \{a_k\}_{k \in \mathbb{N}} \in \mathcal{U} \) belonging to specific subspaces of \( \mathbb{K}^\mathbb{N} \):

- \( a \in c_0 = \left\{ \{a_k\}_{k \in \mathbb{N}} : \lim_{k \to \infty} a_k = 0 \right\} \).
- \( a \in \ell^p(\mathbb{N}) \), \( p \in [1, \infty) \). (Note that \( \ell^\infty(\mathbb{N}) \) is non-separable.)
- \( a \in \bigcap_{p>1} \ell^p(\mathbb{N}) \).

We have the following result:

Theorem 3. Let \( \{\chi_j\}_{j \in \mathbb{N}} \subset X \) be as defined above and \( \mathcal{U} \subset \mathbb{K}^\mathbb{N} \) be the class of unrestricted universal series corresponding to \( \{\chi_j\}_{j \in \mathbb{N}} \). Then

(i) \( \mathcal{U} \neq \emptyset \).

(ii) \( \mathcal{U}_{c_0} = \mathcal{U} \cap c_0 \neq \emptyset \).

(iii) \( \mathcal{U}_b = \mathcal{U} \cap \ell^p(\mathbb{N}) \neq \emptyset \) for every \( p \in (1, \infty) \).

(iv) \( \mathcal{U}_{\ell^p} = \mathcal{U} \cap \left( \bigcap_{p>1} \ell^p(\mathbb{N}) \right) \neq \emptyset \).

(v) \( \mathcal{U}_1 = \mathcal{U} \cap \ell^1(\mathbb{N}) = \emptyset \).

Furthermore, the sets \( \mathcal{U} \), \( \mathcal{U}_{c_0}, \mathcal{U}_{\ell^p(\mathbb{N})} \) with \( p \in (1, \infty) \) and \( \mathcal{U}_{\ell^p} \) are dense \( G_\delta \) in \( \mathbb{K}^\mathbb{N}, c_0, \ell^p(\mathbb{N}) \) and \( \bigcap_{p>1} \ell^p(\mathbb{N}) \), respectively, and they contain a dense vector subspace except zero.
Proof. Since
\[ \bigcap_{p>1} P(N) \subset P(N) \subset c_0(N) \subset K^N, \]
for every \( q \in (1, \infty) \), and since \( P(N) \cap_{p>1} P(N), c_0(N) \) and \( K^N \) satisfy the postulates for \( A \), it suffices to show that
\[ \mathcal{U} \cap \left( \bigcap_{p>1} P(N) \right) \neq \emptyset. \]
The space \( \bigcap_{p>1} P(N) \) is a Fréchet space with distance
\[ d(a, b) = \sum_{j=1}^{\infty} \frac{1}{2^j} \cdot \frac{\|a - b\|_{1+2^{-j}}}{1 + \|a - b\|_{1+2^{-j}}}, \]
where \( \| \cdot \|_p \) is the norm of \( P(N) \). Let \( \varepsilon > 0 \) and \( u \in X \). Let \( N \in \mathbb{N} \) be such that \( \sum_{j=N+1}^{\infty} 2^{-j} < \varepsilon/2 \) and \( M \) be a sufficiently large positive integer to be defined later. By Theorem 1 there exist \( k \in \mathbb{N} \) and \( \gamma_0, \ldots, \gamma_k \in K \) such that
\[ \left| \sum_{j=0}^{k} \chi_j x_j - u \right|_t < \varepsilon, \]
where \( | \cdot |_t \) is the norm of the space \( C^t(\mathcal{I}) \). Since \( \partial \Omega \) does not contain any isolated points, for every \( j = 0, \ldots, k \) we can find distinct \( \chi_{j,0}, \ldots, \chi_{j,m_j} \), close to \( \chi_j \), such that
\[ \left| \frac{1}{M} \sum_{j=0}^{k} \sum_{i=1}^{M} \chi_{j,i} x_j - u \right|_t < \varepsilon. \tag{3.9} \]
Clearly
\[ \sum_{j=0}^{k} \sum_{i=1}^{M} \frac{1}{M^2} |\gamma_j^i|^p = \frac{1}{M^{p-1}} \sum_{j=0}^{k} |\gamma_j|^p \to 0, \]
as \( M \) tends to infinity, where \( p = 1 + 2^{-N} \). Therefore, we can choose sufficiently large \( M \) so that
\[ \|\gamma_j\|_{1+2^{-j}} < \frac{\varepsilon}{2N} \quad \text{for} \quad j = 1, \ldots, N, \]
where \( \gamma_j \) is the finite sequence of \( \gamma_j^i/M \) corresponding to the coefficients \( \chi_{j,i} \)’s. If follows that \( d(\gamma, 0) < \varepsilon \) and in combination with (3.9) it implies that \( \mathcal{U}_{\gamma, c_0(N)} \neq \emptyset. \)
\[ \mathcal{U}_{\gamma} = \emptyset: \]
Let \( u \in X \) and \( d = \text{dist}(\overline{\Omega}, \partial \Omega) > 0 \). If there are \( \gamma_0, \ldots, \gamma_k \) such that
\[ |\gamma_0| + \cdots + |\gamma_k| < \varepsilon \quad \text{and} \quad \left| \sum_{j=0}^{k} \gamma_j x_j - u \right|_t < \varepsilon, \]
then for \( x \in \overline{\Omega} \) we shall have
\[ |u(x)| - \varepsilon \leq \left| \sum_{j=0}^{k} \gamma_j x_j(x) \right| \leq \frac{1}{(n-2)a_{n-1}} \sum_{j=0}^{k} |\gamma_j| \sup_{x \in \overline{\Omega}} \frac{1}{|x - y|^{n-2}} \]
\[ \leq \frac{1}{(n-2)a_{n-1}} \cdot \varepsilon \frac{1}{d^{n-1}} = c \varepsilon. \]
Thus \( |u(x)| \leq (1 + c) \varepsilon \) for every \( x \) in \( \overline{\Omega} \) which leads to contradiction since it requires from \( u \) to be arbitrarily small. \( \square \)
3.2. Universal series of fundamental solutions for domains not satisfying the segment condition.

In such case it is still possible to find a universal series with singularities on a prescribed pseudo–boundary with coefficients in \( \bigcap_{n=1}^{\infty} P(\mathbb{N}) \) approximating all functions which are harmonic in a neighborhood of \( \overline{\Omega} \) with respect to the norm of \( C^2(\overline{\Omega}) \). In particular, if \( u \) is harmonic in \( V \), where \( \overline{\Omega} \subset V \), we can find an open set \( W \) satisfying the segment condition and \( \overline{\Omega} \subset W \subset \overline{W} \subset V \) and having \( \partial\Omega' \) as a pseudo–boundary. Theorem 3 can now be applied to \( W \).

**Remark 3.1.** In the case of analytic connected pseudo–boundary \( \partial\Omega' \subset \mathbb{R}^2 \), it suffices to consider an infinite countable set \( D = \{ y_j \}_{j \in \mathbb{N}} \subset \partial\Omega' \) with an accumulation point. In such case, we have unrestricted universal series.

**Remark 3.2.** Our results combined with [NP05] or [BGENP] provide the existence of universal series realizing universal approximations by the Cesàro averages \( \sigma^\alpha \) of the partial sums of the universal series. The result is generic and we have algebraic genericity also. Furthermore, we may also have an extra supremum for \( \alpha \in K \), where \( K \) is any compact subset of \((-1, \infty)\).

3.3. Approximation of solutions of the biharmonic equation. The (standard) fundamental solution of the biharmonic operator \( \mathcal{L} = \Delta^2 \) is given by

\[
e_2(x) = \begin{cases} 
\frac{1}{8\pi} |x|^2 (\log |x| - 1) & \text{if } n = 2, \\
\frac{1}{4 \omega_3} \log |x| & \text{if } n = 4, \\
\frac{|x|^{4-n}}{2(4-n)(2-n) \omega_{n-1}} & \text{if } n \neq 2, 4,
\end{cases}
\]

As mentioned earlier (see (iii) in Remarks 2.1), the solutions of the biharmonic equation \( \Delta^2 u = 0 \) can be approximated by linear combinations of the form

\[
u_N(x; c, d) = \sum_{j=1}^{N} c_j e_1(x - y_j) + \sum_{j=1}^{N} d_j e_2(x - y_j),
\]

for \( n \geq 3 \), where \( e_1 \) is given by (2.4).

**Construction of a universal series for the solutions of the biharmonic equation.** Let \( D = \{ y_j \}_{j \in \mathbb{N}} \) be a dense subset of the pseudo–boundary \( \partial\Omega' \) and \( \{ x_j \}_{j \in \mathbb{N}} \) defined as

\[
\lambda_{2j-1}(x) = e_1(x - y_j), \quad \lambda_{2j}(x) = e_2(x - y_j),
\]

where \( j \in \mathbb{N} \). Let \( \mu \subset \mathbb{N} \) be the set even numbers, then by Remark 2.3 there exists a universal series \( a \in \mathcal{Y}^\mu \). In particular, if we set

\[
c_j = a_{2j-1} \quad \text{and} \quad d_j = a_{2j},
\]

then we obtain that the partial sums of the form

\[
\sum_{j=1}^{2N} a_j \lambda_j(x) = \sum_{k=1}^{N} c_k e_1(x - y_k) + \sum_{k=1}^{N} d_k e_2(x - y_k),
\]

are dense in the set of the solutions of the biharmonic equation in \( \Omega \).

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Appendix

Proof of Theorem 1. Both sets $X$ and $Y_I$ are linear subspaces of $C^f(\bar{\Omega})$. If $\nu \in \left(C^f(\bar{\Omega})\right)'$, then there exist $\{v_{\alpha}\}_{\alpha \in \ell} \subset \mathcal{M}(\bar{\Omega})$, such that

$$v(u) = \langle u, \nu \rangle = \sum_{\alpha \in \ell} D^n u \, dv_{\alpha} \quad \text{for every} \quad u \in C^f(\bar{\Omega}). \tag{A.1}$$

This representation can be achieved by the isometric imbedding $\mathcal{P} : C^f(\bar{\Omega}) \to C^{f(\bar{\Omega})}$, where $\bar{\Omega}^f$ is the union of mutually disjoint copies $\{\Omega_{\alpha}\}_{\alpha}$ of $\bar{\Omega}$ and $\mathcal{P} u = (D^n u)_{\alpha}$. The dual of $C^{f(\bar{\Omega})}$ is representable by a sum of signed Borel measures $\{v_{\alpha}\}_{\alpha \in \ell}$, with supp $v_{\alpha} \subset \bar{\Omega}$, and since $C = \mathcal{P}[C^f(\bar{\Omega})]$ is a closed subspace of $C^{f(\bar{\Omega})}$, every bounded linear functional on $C$ can be extended to a bounded linear functional $\nu$ on $C^{f(\bar{\Omega})}$, due to the Hahn–Banach theorem, and thus $\nu$ can be expressed in the form (A.1). From the Hahn–Banach theorem, it suffices to show that $X^+ \subset Y^+_I$, i.e.,

\[
\text{if } \nu \in \left(C^f(\bar{\Omega})\right)' \text{ and } \langle u, \nu \rangle = 0 \text{ for every } u \in X \text{ then } \langle u, \nu \rangle = 0. \text{ for every } u \in Y_I.
\]

Let $\nu \in \left(C^f(\bar{\Omega})\right)'$ be such that $\langle u, \nu \rangle = 0$, for every $u \in X$. In particular, if $x \in \partial Y_I$, then the function $u(y) = e_1(y-x) = \tau_{-x} e_1(y)$ belongs to $X$ and

$$0 = \langle u, \nu \rangle = \langle \tau_{-x} e_1, \nu \rangle = \nu(\tau_{-x} e_1) = \nu(e_1 (x)).$$

Thus the convolution $\delta = e_1 * \nu$ vanishes on $\partial Y_I$. Note that $\delta$ defines a distribution in $\mathbb{R}^n$, as a convolution of two distributions, one of which (namely $\nu$) is of compact support (i.e., supp $\nu \subset \bar{\Omega}$). Meanwhile, $\delta$ is real analytic, and in fact, harmonic function in $\mathbb{R}^n \setminus \bar{\Omega}$. Also $-\Delta \delta = \nu$ in the sense of distributions in $\mathbb{R}^n$. Let $U$ be the unbounded connected component of $\mathbb{R}^n \setminus \bar{\Omega}$. Since $\bar{\Omega}$ embraces $\Omega$, there is a connected component $U'$ of $\mathbb{R}^n \setminus \bar{\Omega}$ such that $\bar{U} \subset U$. Clearly, $\partial U' \subset \partial Y_I$, and therefore $\delta$ vanishes on $\partial U'$. If $U'$ is bounded, then $\delta$ vanishes in $U'$, from the maximum principle. Consequently, $\delta$ vanishes in the whole of $U$, being a real analytic function. If $U'$ is unbounded, then we have

$$\delta(x) = \langle \tau_{-x} e_1, \nu \rangle = \sum_{|\alpha| \leq \ell} \int_{\bar{\Omega}} D^n e_1(y-x) \, dv_{\alpha}(y),$$

and thus

$$|\delta(x)| \leq \left( \sum_{|\alpha| \leq \ell} \|v_{\alpha}\| \right) \left( \sup_{x \in \bar{\Omega}} |D^n e_1(y-x)| \right). \tag{A.2}$$

It is not hard to show that for $x$ large (and $y$ in $\bar{\Omega}$), we have $D^n e_1(y-x) = O(|x|^{2-n-|\alpha|})$, which combined with (A.2) provides that $\delta(x) = O(|x|^{2-n})$. Therefore $\lim_{x \to \infty} \delta(x) = 0$. Since $\delta$ vanishes also on $\partial U'$ and is arbitrarily small on $S_R = \{x \in \mathbb{R}^n : |x| = R\}$, for $R$ sufficiently large, then by the maximum principle, $\delta$ vanishes in the whole of $U'$. Thus $\delta$ vanishes in the whole of $U$, since $\delta$ vanishes in $U'$, a nonempty open subset of $U$. If $U$ is a bounded component of $\mathbb{R}^n \setminus \bar{\Omega}$, then according to Definition 2, there is an open component $U'$ of $\mathbb{R}^n \setminus \bar{\Omega}$ such that $\bar{U} \subset U$. In particular, $\partial U' \subset \partial Y_I$ and thus $\delta$ vanishes in $\partial U'$. Therefore, $\delta$ vanishes in the whole of $U'$, and, since $\delta$ is harmonic in $U$, it has to vanish in the whole of $U$. Consequently, $\delta$ vanishes in $\mathbb{R}^n \setminus \bar{\Omega}$, and thus supp $\delta \subset \bar{\Omega}$.

We now need the following lemma:
Lemma 1. Let \( e = e(x) \) be the fundamental solution of the Laplacian given by (2.4). Also, let \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \) satisfying the Segment Condition and \( \nu \in C^r(\overline{\Omega}) \). If \( \delta = e * \nu \) is the convolution of the distributions \( e \) and \( \nu \) and \( \text{supp} \ \delta \subset \overline{\Omega} \), then there exist \( \psi_k \in \mathfrak{C}_0^\infty(\Omega) \), such that for every \( u \in \mathcal{Y}_f \)

\[
- \lim_{k \to \infty} \int_{\Omega} u(x) \Delta \psi_k(x) \, dx = \langle u, \nu \rangle.
\]

Proof. We postpone it for a while.

Let \( u \in \mathcal{Y}_f \). Then by virtue of Lemma 1 we have that

\[
\langle u, \nu \rangle = - \lim_{k \to \infty} \int_{\Omega} u \Delta \psi_k \, dx = - \lim_{k \to \infty} \int_{\Omega} \Delta u \psi_k \, dx = 0,
\]

which concludes the proof of Theorem 1.

\[\square\]

Proof of Lemma 1. We provide a proof for the case in which the Laplacian is replaced by a linear elliptic operator with constant coefficients \( \mathcal{L} \) of order \( m \) and \( e \) is a fundamental solution of \( \mathcal{L} \).

**Fact I.** If \( \overline{\Omega} \cap \Omega \neq \emptyset \), then every bounded linear functional \( \nu \) on \( C(\overline{\Omega}) \) defines a distribution in \( \Omega \) of the form \( T = \sum_{\beta \leq p+1} \|D^\beta e\|_{L^q} \nu_\beta \), where \( \{\nu_\beta\}_{\beta \leq p+1} \subset L^p(\mathbb{R}^n) \) and \( 1 < q < n/(n-1) \).

If \( \psi \in \mathcal{D}(\Omega) \), then its restriction in \( \overline{\Omega} \) belongs to \( C(\overline{\Omega}) \), and thus \( \mu \) defines a linear functional \( T_\mu(\psi) = \int_{\Omega} \psi \, d\mu \) on \( \mathcal{D}(\mathbb{R}^n) \), which is clearly continuous. Sobolev Imbedding Theorem (see [AF03, Theorem 4.12]) implies that \( W^{1,p}(\Omega) \subset C(\overline{\Omega}) \), for every \( p > n \), provided that \( \Omega \) satisfies the Cone Condition. Thus a measure \( \mu \in \mathfrak{N}(\overline{\Omega}) \), which is an element of \( (C(\overline{\Omega}))' \), defines a continuous linear functional on \( W^{1,p}(\Omega) \). Therefore, \( \mu \) can be represented as

\[
\mu(u) = \sum_{|\beta| \leq 1} \int_{\Omega} D^\beta u \psi_\beta \, dx,
\]

for suitable \( \{\psi_\beta\}_{|\beta| \leq 1} \subset L^q(\Omega) \), with \( 1/p + 1/q = 1 \). Thus, for every \( \psi \in \mathcal{D}(\overline{\Omega}) \), we have

\[
\int_{\Omega} \psi \, d\mu = \sum_{|\beta| \leq 1} \int_{\Omega} D^\beta \psi \psi_\beta \, dx = \sum_{|\beta| \leq 1} \int_{\Omega} \psi D^\beta \psi_\beta \, dx,
\]

which provides the representation \( \mu = \sum_{|\beta| \leq 1} (-1)^{|\beta|} D^\beta \psi_\beta \in W^{-1,q}(\Omega) \).

Clearly, if \( \psi \in \mathcal{D}(\Omega) \), then its restriction in \( \overline{\Omega} \) belongs to \( C(\overline{\Omega}) \), and thus \( \nu \) defines a linear functional on \( \mathcal{D}(\Omega) \). It is possible to find \( \{\nu_\beta\}_{|\beta| \leq 1} \subset \mathfrak{N}(\overline{\Omega}) \), such that

\[
\nu(\psi) = \sum_{|\beta| \leq 1} \int_{\Omega} D^\beta \psi \, d\nu_\beta = \sum_{|\beta| \leq 1} T_{\nu_\beta}(D^\beta \psi) = \sum_{|\beta| \leq 1} (-1)^{|\beta|} D^\beta T_{\nu_\beta}(\psi),
\]

which is continuous on \( \mathcal{D}(\Omega) \) and consequently \( \nu \in \mathcal{D}'(\Omega) \). Using (A.3), we finally obtain

\[
T_\nu = \sum_{|\beta| \leq 1} (-1)^{|\beta|} D^\beta T_{\nu_\beta} = \sum_{|\beta| \leq 1} (-1)^{|\beta|} D^\beta \psi_\beta,
\]

in the sense of distributions, for suitable \( \{\nu_\beta\}_{|\beta| \leq 1} \subset L^q(\Omega) \), for some \( 1 < q < n/(n-1) \). Thus \( \nu \in W^{-1,q}(\Omega) \).

**Fact II.** The convolution \( \delta = e \ast \nu \) belongs to \( W_0^{m-\ell-1,q}(\Omega) \).

The convolution \( \delta = e \ast \nu \) belongs to \( W_0^{m-\ell-1,q}(\mathbb{R}^n) \) due to standard elliptic regularity theory. (For a proof see Theorem 7.9.17 in [Hör83] and the discussion that follows.) We already know that \( \text{supp} \ \delta \subset \overline{\Omega} \) and that \( \delta \in W^{m-\ell-1,q}(\Omega) \). If \( m \leq \ell + 1 \), there is nothing left to prove, since \( W_0^{k,q}(\Omega) = W^{k,q}(\Omega) \), when
$k \leq 0$. On the other hand, if $m-\ell-1 > 0$, then what needs to be proved is a consequence of the following result (for a proof see [AF03, Theorem 5.29]).

Let $V$ be an open subset of $\mathbb{R}^n$ satisfying the Segment Condition and $k \geq 1$. Then a function $u$ belongs to $W_0^{k,p}(V)$ if and only if the zero extension of $u$ belongs to $W^{k,p}(\mathbb{R}^n)$.

**Construction of the sequence $\{\psi_k\}_{k \in \mathbb{N}}$.**

Since $\partial \Omega$ satisfies the Segment Condition, then for every $x \in \partial \Omega$, there exist a vector $\xi_x \in \mathbb{R}^n \setminus \{0\}$ and an open neighborhood $U_x$ of $x$, such that if $y \in U_x \cap \Omega$ then $y + t \xi_x \in \Omega$ for every $t \in (0,1)$. Let $V_x$ be an open set in $\mathbb{R}^n$ satisfying

$$x \in V_x \subset U_x. \quad (A.4)$$

Since $\partial \Omega$ is compact, there is a finite collection of such neighborhoods $\{V_j\}_{j=1}^J$ covering $\partial \Omega$. Let $\{U_j\}_{j=1}^J$ be the corresponding $U_x$'s in (A.4), i.e., $\overline{V}_j \subset U_j$. The collection $\{V_j\}_{j=1}^J$ becomes an open cover of $\Omega$ with the addition of another open set $V_0$ such that $\overline{V}_0 \subset \Omega$. Let $\{\psi_j\}_{j=0}^J$ be an infinitely differentiable partition of unity corresponding to the covering $\{V_j\}_{j=0}^J$ of $\Omega$. Clearly, $\{\psi_j\}_{j=0}^J \subset W_0^{s,\delta}(\Omega)$, where $s = m-\ell-1$. Also, $\psi_j \in W^{s,\delta}(\mathbb{R}^n)$. We denote by $\tau_{jx}$ the translation operator by $\epsilon \xi_x$, where $\epsilon \in [0,1]$, i.e.,

$$\left( \tau_{jx} \circ w \right)(x) = \begin{cases} w(x + \epsilon \xi_x) & \text{if } j = 1, \ldots, J, \\ w(x) & \text{if } j = 0. \end{cases}$$

We also define $\psi_j^{\epsilon} = \tau_{jx} \circ \psi_j$ and $\psi_j \circ (\psi_j \delta)$. It is readily seen that

$$\delta = \delta(\epsilon) = \min_{i \leq j \leq \ell} \text{dist}(\partial \Omega, \tau_{jx}[\overline{\Omega} \cap V_j]) > 0. \quad (A.5)$$

Also, if we let $\delta_\epsilon = \sum_{j=0}^J \delta_{j\epsilon}$, then we have $\sup \delta_\epsilon \subset \overline{\Omega}^{\delta}$ and $\delta_\epsilon \in W_0^{s,\delta}(\Omega)$. Clearly, $\mathcal{L} \delta_\epsilon = \sum_{j=0}^J \mathcal{L} \delta_{j\epsilon}$.

We next show that $\lim_{\epsilon \searrow 0} \sum_{j=0}^J \mathcal{L} \delta_{j\epsilon} = \mathcal{L} \delta = v$, in the weak* sense of $C'(\overline{\Omega})$. We observe that

$$\mathcal{L} \delta_{j\epsilon} = \tau_{j\epsilon} \circ (\mathcal{L}(\psi_j \delta)) = \tau_{j\epsilon} \circ (\mathcal{L}(\psi_j \delta)).$$

Thus,

$$\lim_{\epsilon \searrow 0} \mathcal{L} \delta_\epsilon = \lim_{\epsilon \searrow 0} \sum_{j=0}^J \mathcal{L} \delta_{j\epsilon} = \lim_{\epsilon \searrow 0} \sum_{j=0}^J \tau_{j\epsilon} \circ (\psi_j \delta) = J \psi \delta = v,$$

in the weak* sense of $\left( C'(\overline{\Omega}) \right)$. Let $\zeta \in C_0^\infty(B_1)$, $\zeta \geq 0$ and $\int_{B_1} \zeta = 1$, where $B_\rho = \{ x \in \mathbb{R}^n : |x| < \rho \}$, and $\zeta_\eta(x) = \eta^{-n} \zeta(\eta^{-1} x)$, for $\eta > 0$. Clearly, $\zeta_\eta \in C_0^\infty(B_\eta)$ and $\int_{B_\eta} \zeta_\eta = 1$. Also, $\zeta_\eta \circ \delta_\epsilon \in C_0^\infty(\Omega)$, and for every $\eta < \delta$, we have that

$$\sup \zeta_\eta \circ \delta_\epsilon \subset B_\eta \subset \Omega,$$

and thus $\zeta_\eta \circ \delta_\epsilon \in C_0^\infty(\Omega)$. The sequence $\{\psi_k\}_{k \in \mathbb{N}}$ we are seeking can be constructed from functions of the form $\zeta_k \circ \delta_\epsilon$, where $\epsilon = 1/k$ and $\eta$ is suitably chosen in the interval $(0, \delta)$, where $\delta$ is given by (A.5). Indeed, the sequence

$$\mathcal{L} \psi_k = \mathcal{L}(\zeta_k \circ \delta_\epsilon) = \zeta_k \mathcal{L} \delta_\epsilon, \quad k \in \mathbb{N},$$

converges to $v$ in the weak* sense of $C'(\overline{\Omega})$, since for every $\varphi \in C'(\overline{\Omega})$

$$v(\varphi) = \lim_{\epsilon \searrow 0} \mathcal{L} \delta_\epsilon(\varphi) = \lim_{\epsilon \searrow 0} \mathcal{L} \zeta_k \circ \delta_\epsilon(\varphi).$$

$\square$
References


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