Abstract. In the present work, we investigate the approximability of solutions of elliptic partial differential equations in a bounded domain $\Omega$ by solutions of the same equations in a larger domain. We construct an abstract framework which allows us to deal with such density questions, simultaneously for various norms. More specifically, we study approximations with respect to the norms of semilocal Banach spaces of distributions. These spaces are required to satisfy certain postulates. We establish density results for elliptic operators with constant coefficients which unify and extend previous results. In our density results, $\Omega$ may possess holes and it is required to satisfy the segment condition. We observe that analogous density results do not hold in spaces where the infinitely smooth functions are not dense. Finally, we provide applications related to the method of fundamental solutions.

1. Introduction

Let $U$ be an open subset of $\mathbb{R}^n$, $\Omega$ be an open subset of $U$ with compact closure in $U$, and suppose that $L = \sum_{|\alpha| \leq m} a_\alpha(x)D^\alpha$ is an elliptic operator on $U$. Let $\mathcal{X}$ be the set of solutions of $Lu = 0$ in $U$ and $\mathcal{Y}$ be the set of solutions of $Lu = 0$ in $\Omega$. Let also $A(\overline{\Omega})$ be a Banach space of functions (or more generally of distributions) on $\overline{\Omega}$. Then we may pose the following:

Main question. Is it possible to approximate the elements of $\mathcal{Y} \cap A(\overline{\Omega})$, in the norm of the space $A(\overline{\Omega})$, by elements of $\mathcal{X}$?

F. Browder has provided a positive answer to this question in the case $A(\overline{\Omega}) = C(\overline{\Omega})$. More specifically, he has proved ([Bro62, Theorem 2]) that if $U \sim \Omega$ has no compact components, then the solutions of $Lu = 0$ in $U$, are dense in $\mathcal{X} = \{ u \in C^m(\Omega) : Lu = 0 \text{ in } \Omega \} \cap C(\overline{\Omega})$, in the sense of the uniform norm. In Browder’s result, $\Omega$ has to satisfy the cone condition, $\mathcal{L}$ and $\mathcal{L}^*$ (where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$) are required to have $C^1-$coefficients and $\mathcal{L}^*$ is required to satisfy the Condition of uniqueness for the Cauchy problem in the small in $U$:

(U) $s$. If $u \in C^m(V)$, where $V$ is an open connected subset of $\Omega$ with $\mathcal{L}^*u = 0$ in $V$, and if $u$ vanishes in a nonempty open subset of $V$, then $u$ vanishes everywhere in $V$.

Weinstock [Wei73] extended Browder’s theorem by showing that the solutions of $Lu = 0$ in $\Omega$, which are also elements of $C^k(\overline{\Omega})$, for $0 \leq k < m$, can be approximated by solutions of $Lu = 0$ in a neighborhood of $\overline{\Omega}$, where $m$ is the order of $\mathcal{L}$. In Weinstock’s work, $\mathcal{L}$ is assumed to be an elliptic
operator with constant coefficients and the domain \( \Omega \) is required to satisfy a weaker condition, the segment condition.

Answers to the main question have been obtained for linear combinations of translates of a fundamental solution \( \epsilon \) of \( L \) in the special case in which the singularities lie on a surface \( \partial \Omega' \) embracing \( \Omega \), i.e., \( \overline{\Omega} \subseteq \Omega' \) and \( \Omega' \setminus \overline{\Omega} \) does not have any compact components ([Bog85, KA63, Smy06, Smy07]). In particular, it is shown that the solutions of specific elliptic equations and systems, including the Laplace and \( m \)-harmonic equations, and the Cauchy–Navier system, can be approximated by linear combinations of translates of suitable fundamental solutions of the corresponding operators, with singularities lying on a prescribed embracing surface ([Smy06, Smy07]).

In this section we define the semilocal Banach spaces with respect to which we shall obtain our density results. We also give examples of semilocal Banach spaces. Section 4 contains the main density question. More specifically we show (Theorem 2) that every solution of \( L u = 0 \) in \( \Omega \), which lies in \( A(\overline{\Omega}) \), can be approximated by solutions of \( L u = 0 \) in \( \Omega' \), where \( \Omega' \) is open, \( \overline{\Omega} \subseteq \Omega' \) and each component of \( \mathbb{R}^n \setminus \overline{\Omega} \) contains a component of \( \mathbb{R}^n \setminus \overline{\Omega}' \). In fact, Theorem 2 is a corollary of a Runge–type theorem (Theorem 3) which allows us to approximate the solutions of \( L u = 0 \) in \( \Omega \), which belong to \( A(\overline{\Omega}) \), by solutions of \( L u = 0 \) in \( \mathbb{R}^n \setminus F \), where \( F \) is a finite set of poles; one pole in each hole of \( \Omega \). All our density results hold with respect to the norm of every Banach space which satisfies the required postulates.

We observe that analogous density results do not hold in the spaces \( \mathcal{W}^{k,\infty}(\Omega) \), \( \mathcal{M}(\overline{\Omega}) \) of signed Borel measures, and \( \text{Lip}(k,\sigma,\Omega) \) of Hölder continuous functions. Each of these fails to satisfy the fourth postulate. According to this postulate the set \( \mathcal{D} \) of test functions in \( \mathbb{R}^n \) should be dense in \( \mathcal{A} \).

We finally provide density results with linear combinations of translates of fundamental solutions in the case where the singularities of the fundamental solutions lie on the boundary of a domain \( \Omega' \) embracing \( \Omega \). In such case, fundamental solutions of suitable factors of the operator \( L \) are included in the linear combinations. Such density results establish the applicability of the method of fundamental solutions; a numerical method in the solution of elliptic boundary value problems.

The paper is organized as follows. In Section 2 we provide certain definitions and preliminary facts. In Section 3 we define the semilocal Banach spaces with respect to which we shall obtain our density results. We also give examples of semilocal Banach spaces. Section 4 contains the main density results.

\footnote{In this work \( \mathbb{N} \) is the set of non-negative integers.}
results. We also prove variations of these results which establish the applicability of the method of fundamental solutions for certain elliptic boundary value problems.

2. Preliminaries

Test functions and distributions. If $\Omega$ is an open domain in $\mathbb{R}^n$, then $\mathcal{D}(\Omega)$ is the set of infinitely differentiable functions on $\Omega$ with compact support and $\mathcal{D}'(\Omega)$ the set of distributions in $\mathbb{R}^n$. These spaces are locally convex topological vector spaces, when endowed with their usual topologies, and $\mathcal{D}'(\Omega)$ is the dual of $\mathcal{D}(\Omega)$. For simplicity we shall denote $\mathcal{D}(\mathbb{R}^n)$ by $\mathcal{D}$, and $\mathcal{D}'(\mathbb{R}^n)$ by $\mathcal{D}'$. If $\varphi \in \mathcal{D}(\Omega)$ and $\nu \in \mathcal{D}'(\Omega)$, then $\langle \varphi, \nu \rangle$ denotes the pairing between $\mathcal{D}(\Omega)$ and $\mathcal{D}'(\Omega)$ (For details see [Rud91].)

If $w$ is a function, we set $\tau_x w(y) = w(y-x)$ and $\check{w}(x) = w(-x)$. If $u$ and $v$ are measurable functions in $\mathbb{R}^n$, then their convolution $u \ast v$ is defined by

$$ (u \ast v)(x) = \int_{\mathbb{R}^n} u(y) v(x-y) \, dy, $$

provided that the integral on the right–hand side of the above exists for almost every $x \in \mathbb{R}^n$. Note that $\int_{\mathbb{R}^n} u(y) v(x-y) \, dy = \int_{\mathbb{R}^n} u(y) \tau_x \delta(y) \, dy$, which makes it natural to define

$$ (v \ast \varphi)(x) = \langle \tau_x \varphi, v \rangle, \quad v \in \mathcal{D}', \varphi \in \mathcal{D}. $$

Accordingly, if $v \in \mathcal{D}'$, then $\tau_x v$ is defined by $\langle \varphi, \tau_x v \rangle = \langle \tau_{-x} \varphi, v \rangle$, where $\varphi \in \mathcal{D}$. It is readily seen that $v \ast \varphi$ is an infinitely differentiable function and $D^\alpha (v \ast \varphi) = (D^\alpha v) \ast \varphi = v \ast (D^\alpha \varphi)$, for every multi–index $\alpha$. Also, $(v \ast \varphi) \ast \psi = v \ast (\varphi \ast \psi)$, for every $\varphi, \psi \in \mathcal{D}$ and $v \in \mathcal{D}'$.

If $\mu, \nu \in \mathcal{D}'$, and one of them (say $\nu$) has compact support, then their convolution $\mu \ast \nu$ is also defined as another distribution, and more specifically, in a natural way which extends the definition of the convolution of integrable functions.

Properties of the convolution of distributions. Let $\varphi, \psi \in \mathcal{D}$, $\mu, \nu \in \mathcal{D}'$. Furthermore, assume that $\nu$ has compact support. Then ([Rud91, Chapter 6])

(i) $D^\alpha (\mu \ast \nu) = (D^\alpha \mu) \ast \nu = \mu \ast (D^\alpha \nu)$, for every multi–index $\alpha$.

(ii) $\langle \varphi, \mu \ast \nu \rangle = \langle (\varphi \ast \nu) \ast \mu \rangle$.

(iii) If $\delta$ is the Dirac measure with unit mass at the origin, then $\delta \ast \nu = \nu$.

Also, the convolution of distributions enjoys commutativity and associativity.

Fundamental solutions. Let $\mathcal{L} = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ be a partial differential operator in $\mathbb{R}^n$ with constant coefficients. A fundamental solutions of $\mathcal{L}$ is a function $\epsilon : \mathbb{R}^n \setminus \{0\}$ satisfying $\mathcal{L} \epsilon = \delta$, where $\delta$ is the Dirac measure with unit mass at the origin, in the sense of distributions, i.e.,

$$ \langle \psi, \mathcal{L} \epsilon \rangle = \int_{\mathbb{R}^n} e(x) \mathcal{L}^* \psi(x) \, dx = \psi(0) = \langle \psi, \delta \rangle, $$

for every $\psi \in \mathcal{D}$, where $\mathcal{L}^* u = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} a_\alpha D^\alpha u$. The operator $\mathcal{L}^*$ is known as the adjoint of $\mathcal{L}$. It is readily shown that if $\epsilon$ is a fundamental solution of $\mathcal{L}$, then $\check{\epsilon}$, where $\check{\epsilon}(x) = \epsilon(-x)$, is a fundamental solution of $\mathcal{L}^*$. Also, if $\mathcal{L}$ is elliptic, then $\epsilon$ is real analytic in $\mathbb{R}^n \setminus \{0\}$ and satisfies, in the classical sense, $\mathcal{L} \epsilon(x) = 0$ for every $x \in \mathbb{R}^n \setminus \{0\}$. The fundamental solutions produce solutions of the corresponding inhomogeneous equation by convolution. If a distribution $\nu \in \mathcal{D}'(\mathbb{R}^n)$ has
compact support and \( e \) is a fundamental solution of \( \mathcal{L} \), then the convolution of \( e \) and \( v \) is a distribution defined as

\[
\langle \psi, e * v \rangle = \langle \delta * \psi, e \rangle = \int_{\mathbb{R}^n} e(x) (\delta * \psi)(x) \, dx,
\]

for \( \psi \in \mathcal{D}(\mathbb{R}^n) \). It is readily proved that \( \mathcal{L}(e * v) = (\mathcal{L}e) * v = \delta * v = v \), in the sense of distributions. Malgrange [Mal56] and Ehrenpreis [Ehr56] independently established in 1955–56 the existence of fundamental solutions for partial differential operators with constant coefficients.

The fundamental solutions of elliptic operators have the following property ([Wei73]):

**Lemma 1.** If \( \mathcal{L} \) is an elliptic operator with constant coefficients of order \( m \) in \( \mathbb{R}^n \) and \( e = e(x) \) is a fundamental solution of \( \mathcal{L} \), then \( D^a u \in L^1_{\text{loc}}(\mathbb{R}^n) \), for every \( |a| < m \). \( \square \)

**The Segment Condition.** The domains in our density results satisfy a rather weak boundary regularity requirement, namely the segment condition:

**Definition 1.** (The Segment Condition) An open subset \( \Omega \) of \( \mathbb{R}^n \) satisfies the segment condition, if for every \( x \in \partial \Omega \) there exists a neighborhood \( U_x \) and a nonzero vector \( \xi_x \in \mathbb{R}^n \), such that, if \( y \in U_x \cap \Omega \), then \( y + t \xi_x \in \Omega \) for every \( t \in (0, 1) \).

Note that the segment condition is weaker than the cone condition and allows the boundaries to have corners and cusps. Note also that \( \Omega \) satisfies the segment condition if and only if \( \mathbb{R}^n \setminus \overline{\Omega} \) does. Also, the boundaries of domains satisfying this condition are \( (n-1) \)-dimensional and their measure is zero. However, if a domain satisfies the segment condition it cannot lie on both sides of any part of its boundary ([AF03]). It is not hard to prove that, if a domain satisfies the segment condition, then every connected component of its complement has a nonempty interior. In fact, domains satisfying the segment condition coincide with the interior of their closure. Finally, bounded domains satisfying the segment condition can have only finitely many holes (i.e., their complement can have finitely many connected components).

### 3. Semilocal Spaces

**Semilocal Banach spaces.** For simplicity we shall denote \( \mathcal{D}(\mathbb{R}^n) \) by \( \mathcal{D} \), and \( \mathcal{D}'(\mathbb{R}^n) \) by \( \mathcal{D}' \). For further details see [Rud91]. Let \( A \) be a subspace of \( \mathcal{D}' \) which is a Hausdorff space and a locally convex topological vector space. We say that \( A \) is semilocal ([Hör83b, Tar95]) if the following two postulates hold:

- **[P1]** The inclusion \( A \hookrightarrow \mathcal{D}' \) is continuous.
- **[P2]** If \( \varphi \in \mathcal{D} \) and \( u \in A \), then \( \varphi u \in A \), and the mapping \( T : A \to A \), with \( Tu = qu \), is continuous.

In this work we shall consider semilocal spaces which are also Banach spaces with norm \( |\cdot|_A \) and satisfy the following additional postulates:

- **[P3]** \( \mathcal{D} \subset A \) and the inclusion \( \mathcal{D} \hookrightarrow A \) is continuous.
- **[P4]** \( \mathcal{D} \) is dense in \( A \).

\(^\text{3}\)While the first, second and third postulates are satisfied by most of the function spaces used in the theory of PDEs, the fourth one is not satisfied by the Sobolev spaces \( W^{k,\infty}(\mathbb{R}^n) \), the space of Borel measures \( \mathfrak{M}(\mathbb{R}^n) \) and \( \text{Lip}(k, \sigma, \overline{\Omega}) \) of Lipschitz functions. It is however satisfied by the Sobolev spaces \( W^{k,p}(\mathbb{R}^n) \), \( p \in [1, \infty) \), the spaces of \( C^1 \)-functions and the spaces \( \text{lip}(k, \sigma, \overline{\Omega}) \) of uniformly Lipschitz continuous functions. However, if \( A \) satisfies all the postulates except of the fourth, and \( B \) is the closure of \( \mathcal{D} \) in \( A \), then \( B \) satisfies all the postulates.
According to the third postulate, if \( \{q_\ell\}_{\ell \in \mathbb{N}} \subset D \) and \( q_\ell \to 0 \), in the topology of \( D \), then \( |q_\ell|_A \to 0 \). The second postulate implies that whenever \( v \) is a continuous linear functional on \( A \) (i.e., \( v \in A' \)) and \( \varphi \in D \), then \( \varphi v \in A' \), where \( \varphi v(u) = v(\varphi u) \). It is readily seen that the third postulate implies that the inclusion \( A' \hookrightarrow D' \) is continuous, and therefore \( A' \) is also semilocal. The third and fourth postulates imply that \( A \) coincides (up to an isometry) with the completion of \( D \) with respect to \( | \cdot |_A \).

**Notation.** In order to avoid confusions, we shall be denoting by \( v(u) \) the pairing between \( A \) and \( A' \), while \( \langle \varphi, v \rangle \) is the pairing between \( D \) and \( D' \).

The third postulate also implies that the elements of \( A' \), when restricted to \( D \), define distributions, and due to the fourth postulate, different elements of \( A' \) define different distributions (because \( D \) is dense in \( A \)). We may thus identify the elements of \( A' \) with distributions. On the other hand, a distribution \( v \) defines an element of \( A' \) if and only if \( \sup_{|\varphi|_A = 1} |\langle \varphi, v \rangle| < \infty \).

The following postulate guarantees the homogeneity of \( A \):

\[ [P_5] \text{ If } u \in A, \text{ then } \tau_x u \in A \text{ and } |\tau_x u|_A = |u|_A \text{ for all } x \in \mathbb{R}^n. \]

The following technical fact is a consequence of the fifth postulate:

**Proposition 1.** If the Banach space \( A \) satisfies the postulates \([P_1 - P_5]\), \( v \in A' \), \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \) and the distribution \( \mu = f * v \) has compact support, then \( \mu \) defines a continuous linear functional on \( A \).

**Proof.** Since \( v \) has compact support, then \( f * v \) defines also a distribution, and for every \( \varphi \in D \), we clearly have that \( v * \varphi \in D \), and

\[ \langle \varphi, f \ast v \rangle = \langle (v \ast \varphi)c, f \rangle. \]

Note that \( (v \ast \varphi) \in D \). Also,

\[ \langle v \ast \varphi(x), \varphi(x) \rangle = \langle v \ast \varphi(x), \varphi(x) \rangle = \langle \varphi, \tau_x v \rangle = \langle \tau_{-x} \varphi, v \rangle = \langle \tau_{-x} \varphi, v \rangle. \]

Let \( \psi \in D \) be such that \( \psi \equiv 1 \) in a neighborhood of the support of \( f \ast v \). Then \( \langle \varphi, f \ast v \rangle = \langle \psi \varphi, f \ast v \rangle \), for every \( \varphi \in D \). Therefore,

\[ \langle \varphi, f \ast v \rangle = \langle \psi \varphi, f \ast v \rangle = \int_{\mathbb{R}^n} (v \ast (\psi \varphi))(x) f(x) \, dx = \int_{\mathbb{R}^n} v(\tau_{-x} (\psi \varphi))(x) f(x) \, dx. \]

The last integral is well defined since \( w(x) = v(\tau_{-x} (\psi \varphi)) \) is supported in

\[ K = \text{supp } \psi - \text{supp } v = \{x - y : x \in \text{supp } \psi \text{ and } y \in \text{supp } v\}, \]

which is compact and \( w \in D \). Thus,

\[ |\langle \varphi, f \ast v \rangle| = \left| \int_K v(\tau_{-x} (\psi \varphi)) f(x) \, dx \right| \leq \left( \int_K |f(x)| \, dx \right) \cdot \sup_{x \in K} |v(\tau_{-x} (\psi \varphi))| \]

Also,

\[ \sup_{x \in K} |v(\tau_{-x} (\psi \varphi))| \leq |v|_{A'} \cdot |\psi \varphi|_A \leq c \cdot |v|_{A'} \cdot |\varphi|_A, \]

where \( c = \sup_{|\varphi|_A = 1} |\psi \varphi|_A \). Clearly, \( c < \infty \), due to the second postulate. Altogether, for every \( \varphi \in D \), we have \( |\langle \varphi, f \ast v \rangle| \leq C |\varphi|_A \), where \( C = c \cdot |v|_{A'} \cdot \int_K |f(x)| \, dx \), and thus \( f \ast v \) defines an element of \( A' \). This concludes the proof of the proposition.

**Remark 1.** Proposition 1 holds even if the fifth postulate is replaced by the following weaker requirement

\[ [P'_5] \text{ If } u \in A \text{ and } \xi \in \mathbb{R}^n, \text{ then } \tau_x u \in A \text{ and for every } K \subset \mathbb{R}^n \text{ compact, there exists a } c_K > 0, \text{ such that } |\tau_x u|_A \leq c_K |u|_A, \text{ for all } \xi \in K \text{ and } u \in A. \]

Also, if the locally integrable function $f$, in Proposition 1, is replaced by an element of $\mathfrak{M}_{\text{loc}}(\mathbb{R}^n)$, where

$$\mathfrak{M}_{\text{loc}}(\mathbb{R}^n) = \{\mu \in \mathcal{D}' : \zeta \mu \text{ is a signed Borel measure on } \mathbb{R}^n \text{ for every } \zeta \in \mathcal{D}\},$$

then Proposition 1 still holds.

Combination of Lemma 1 and Proposition 1 yields the following result:

**Proposition 2.** Let $\mathcal{A}$ be a semilocal Banach space in $\mathbb{R}^n$ which satisfies postulates $[P_1 - P_5]$ and $\nu \in \mathcal{A}'$. Let also $\mathcal{L} = \sum_{|\alpha| \leq m} a_{\alpha} D^\alpha$ be an elliptic operator of order $m$ with constant coefficients in $\mathbb{R}^n$ and $\epsilon = e(x)$ be a fundamental solution of $\mathcal{L}$, and $\psi \in \mathcal{D}$. If the distributions $\nu$ and $e \ast \nu$ have compact supports, then $\mathcal{L}(\psi(e \ast \nu))$ defines an elements of $\mathcal{A}'$.

**Proof.** The distribution $\mathcal{L}(\psi(e \ast \nu))$ can be expressed as

$$\mathcal{L}(\psi(e \ast \nu)) = \psi \mathcal{L}(e \ast \nu) + \sum_{|\beta| < m} \mathcal{L}_\beta \psi(D^\beta e \ast \nu) = \psi \nu + \sum_{|\beta| < m} \mathcal{L}_\beta \psi(D^\beta e \ast \nu), \quad (3.1)$$

where $\mathcal{L}_\beta$ is a partial differential operator with constant coefficients, for every $|\beta| < m$. If $|\beta| < m$, then $D^\beta \in L^1_{\text{loc}}(\mathbb{R}^n)$, as a consequence of Lemma 1. Also, $D^\beta e \ast \nu$ has compact support, for every $|\beta| < m$; in fact $\text{supp} D^\beta e \ast \nu = \text{supp} D^\beta (e \ast \nu) \subset \text{supp} e \ast \nu$. Proposition 1 implies that $D^\beta e \ast \nu$ defines an element of $\mathcal{A}'$, for every $|\beta| < m$. Each term on the right-hand side of (3.1) defines an element of $\mathcal{A}'$, as a consequence of the second postulate, and so does their sum. □

If $\mathcal{A}$ satisfies the postulates $[P_1 - P_3]$, then $\mathcal{A}'$ does not necessarily satisfy the fourth postulate. For example, if $\mathcal{A}$ is the set of continuous functions in $\mathbb{R}^n$ which vanish at infinity, equipped with the supremum norm, then $\mathcal{A}'$ is $\mathfrak{M}(\mathbb{R}^n)$, the set of signed Borel measures on $\mathbb{R}^n$, equipped with the total variation norm. Every test function in $\mathbb{R}^n$ defines a signed Borel measure. (If $\phi \in \mathcal{D}$, then $\phi dx \in \mathfrak{M}(\mathbb{R}^n)$.) On the other hand, the closure of $\mathcal{D}$ in $\mathfrak{M}(\mathbb{R}^n)$ is just $L^1(\mathbb{R}^n)$, which is a proper subset of $\mathfrak{M}(\mathbb{R}^n)$. We have the following result:

**Proposition 3.** If the space $\mathcal{A} \subset \mathcal{D}'$ satisfies the postulates $[P_1 - P_3]$ and $[P_5]$, then so does $\mathcal{A}'$. If $\mathcal{A}$ satisfies the postulate $[P_4]$ as well, then $\mathcal{D}$ is weak" dense in $\mathcal{A}'$.

**Proof.** We have already explained why the first and second postulates are satisfied by $\mathcal{A}'$. The satisfaction of the fifth postulate follows from the fact that $\tau_\xi \nu(u) = \nu(\tau_\xi u)$, for every $\xi \in \mathbb{R}^n$, $u \in \mathcal{A}$ and $\nu \in \mathcal{A}'$. We next show that $\mathcal{A}'$ satisfies the third postulate. First we observe that, by virtue of Proposition 1, $\phi * (\psi \nu) \in \mathcal{D} \cap \mathcal{A}'$, for every $\phi, \psi \in \mathcal{D}$ and $\nu \in \mathcal{A}'$, and thus we can find a $\omega \in \mathcal{D} \cap \mathcal{A}' \neq \{0\}$. (Note that if $\zeta_\epsilon$ is an approximate identity (i.e., $\zeta_\epsilon(x) = e^{-\epsilon x}(e^{-x} x)$, where $\zeta \in \mathcal{D}(B_1)$, $\zeta \geq 0$ and $\int \zeta dx = 1$), then for every $\nu \in \mathcal{D}'$, $\zeta_\epsilon \ast \nu \to \nu$ in the sense of distributions, as $\epsilon \to 0$, and thus if $\nu \neq 0$, then $\zeta_\epsilon \ast \nu \neq 0$, if $\epsilon$ is sufficiently small.) We may assume that $\phi$ is positive in some ball $B(x_0, \epsilon) \subset \mathbb{R}^n$. Let $\zeta \in \mathcal{D}(B(x_0, \epsilon))$, which is positive in $B(x_0, \epsilon/2)$. Here, $B(x, r)$ denotes the open ball of radius $r$ centered at $x$. Then $\zeta \omega \in \mathcal{A}'$, due to the second postulate, and $\tau_\xi \psi \in \mathcal{A}'$, for every $\xi \in \mathbb{R}^n$, since $\mathcal{A}'$ satisfies the fifth postulate. Also, $\zeta \omega$ is a non-negative test function, which is supported in $B(x_0, \epsilon)$ and it positive in $B(x_0, \epsilon/2)$. If $\psi$ is an arbitrary test function and $K = \text{supp} \psi$, then we can find $\xi_1, \ldots, \xi_j \in \mathbb{R}^n$, such that $\psi = \sum_{i=1}^j \tau_{\xi_i}(\zeta_\omega)$ is positive in a neighborhood of $K$. Clearly, $\phi/\psi \in \mathcal{D}$, and since $\psi \in \mathcal{A}'$, then $\phi = \psi \cdot (\phi/\psi) \in \mathcal{A}'$. Finally, if $\mathcal{A}$ satisfies postulates $[P_1 - P_5]$, then it is not hard to show that if $\nu \in \mathcal{A}'$ and $\xi_\epsilon$ is an approximate identity, then $\zeta_\epsilon \ast \nu \in \mathcal{A}'$, due to Proposition 1, and $\zeta_\epsilon \ast \nu$ tends to $\nu$ in the weak" sense of $\mathcal{A}'$, as $\epsilon$ tends to zero. □
Definition of $A(\Omega)$. Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $A$ be a Banach space satisfying the postulates $[P_1 - P_5]$. We set

$$A_0(\Omega) = \{ u \in A : \text{supp } u \subset \Omega \}.$$ 

Clearly, $A_0(\Omega)$ is a closed subspace of $A$ and thus, a Banach space. $A'_0(\Omega)$ is defined accordingly,

$$A'_0(\Omega) = \{ u \in A' : \text{supp } u \subset \Omega \}.$$ 

Since $\mathcal{D} \subset A'$, then $\mathcal{D}(\Omega) \subset A'_0(\Omega)$.

**Proposition 4.** If $\Omega$ is bounded and satisfies the segment condition and $A \subset \mathcal{D}'$ satisfies postulates $[P_1 - P_5]$, then $\mathcal{D}(\Omega)$ is weak$^*$ dense in $A'_0(\Omega)$.

**Proof.** In order to show this, let $\{U_j\}_{j=0}^I$ be an open cover of $\Omega$, and $\{\xi_j\}_{j=1}^I \subset \mathbb{R}^n$, be non-zero vectors such that $\{U_j\}_{j=1}^I$ is a cover of $\partial \Omega$ and for every $\epsilon \in (0,1)$, $j = 1, \ldots, I$, and $x \in \Omega \cap U_j$, we have that $x + t\xi_j \in \Omega$. Let $\{\psi_j\}_{j=0}^I \subset \mathcal{D}$ be an infinitely differentiable partition of unity subordinate to $\{U_j\}_{j=0}^I$, and $\delta_j$ be an approximate identity. It is not hard to show that, for suitable $\delta_k, \epsilon_k \searrow 0$, we have

$$\sum_{j=0}^I \delta_k \ast (\tau_j \epsilon_j (\psi_j \mu)) \in \mathcal{D}(\Omega) \quad \text{and} \quad \sum_{j=0}^I \delta_k \ast (\tau_j \epsilon_j (\psi_j \mu)) \rightarrow \nu,$$

in the weak$^*$ sense of $A'_0(\Omega)$. \hfill \Box

Next, we define, for $\varphi \in \mathcal{D}$, the seminorm

$$|\varphi|_{A(\Omega)} = \sup_{\nu \in A'_0(\Omega), \sum_{j=0}^I \delta_k \ast (\tau_j \epsilon_j (\psi_j \mu)) \in \mathcal{D}(\Omega), \sum_{j=0}^I \delta_k \ast (\tau_j \epsilon_j (\psi_j \mu)) \rightarrow \nu} |\nu(\varphi)|.$$ 

Note that $|\varphi|_{A(\Omega)} \leq |\varphi|_A$. If we set $\mathcal{N} = \{ \varphi \in \mathcal{D} : |\varphi|_{A(\Omega)} = 0 \}$, then $| \cdot |_{A(\Omega)}$ defines a norm on $\mathcal{D} / \mathcal{N}$. We denote by $A(\Omega)$ the completion of $\mathcal{D} / \mathcal{N}$, with respect to the norm $| \cdot |_{A(\Omega)}$. Equivalently, $A(\Omega)$ is the closure of $\mathcal{D} / \mathcal{N}$ (or of $A(\Omega) = \{ \varphi_{\Omega} : \varphi \in \mathcal{D} \}$) in the dual of $A'_0(\Omega)$.

The following result allows us to identify the dual of $A(\Omega)$ with $A'_0$.

**Proposition 5.** Let $\nu \in A'_0$. Then the functional $\nu$ is well-defined on $\mathcal{D} / \mathcal{N}$ and extends to a continuous linear functional on $A(\Omega)$. In particular

$$|\nu|_{A(\Omega)} = \sup_{\varphi \in A(\Omega)} |\varphi|.$$ 

Conversely, if $\mu \in (A(\Omega))'$, then there exists a unique $\nu \in A'_0$, such that $\mu(u) = \lim_{\ell \to \infty} \nu(\varphi_{\ell})$, for every sequence $\{ \varphi_{\ell} \}_{\ell \in \mathbb{N}} \subset \mathcal{D}$, with the property that $\{ \varphi_{\ell} \}_{\ell \in \mathbb{N}}$ converges to $u$ in the norm of $A(\Omega)$.

**Proof.** If $\nu \in A'_0(\Omega)$, $u \in A(\Omega)$ and $\{\omega_\ell\}_{\ell \in \mathbb{N}} \subset \mathcal{D} / \mathcal{N}$ is a $| \cdot |_{A(\Omega)}$–Cauchy sequence converging to $u$, then $\nu(\omega_\ell)$ is well-defined, and the sequence $\{\nu(\omega_\ell)\}_{\ell \in \mathbb{N}}$ is convergent. In fact, the limit $\lim_{\ell \to \infty} \nu(\omega_\ell)$ is independent of the choice of the Cauchy sequence, and we denote it by $\nu(u)$. It is readily seen that $\nu(u)$ extends the definition of $\nu(\varphi)$, $\varphi \in \mathcal{D}$. Also, $|\nu(\varphi)| \leq |\nu|_A |\varphi|_A$, since if $\nu \in A'_0(\Omega) \setminus \{0\}$, then

$$|\nu(\varphi)| = |\nu|_A \left| \frac{1}{|\nu|_A} \nu(\varphi) \right| \leq |\nu|_A \sup_{\mu \in A'_0(\Omega)} |\mu(\varphi)| = |\nu|_A |\varphi|_{A(\Omega)}.$$
Therefore, the elements of \( A_0'(\Omega) \) define continuous linear functionals on \( A(\Omega) \) and in particular, 
\[
|v|_{(A(\Omega))'} \leq |v|_{A'}, \\
\text{for every } v \in A_0'(\Omega).
\]
On the other hand if \( v \in A_0'(\Omega) \) and \( \varepsilon > 0 \), there exists a \( \varphi \in \mathcal{D} \), with \( |\varphi|_{A'} = 1 \) and
\[
|v|_{A'} - \varepsilon < |v(\varphi)| \leq |v|_{(A(\Omega))'},
\]
thus \( |v|_{A'} \leq |v|_{(A(\Omega))'} \) and altogether \( |v|_{A'} = |v|_{(A(\Omega))'} \). Finally, if \( \mu \in (A(\Omega))' \), then there is a \( c > 0 \), such that for every \( \varphi \in \mathcal{D} \),
\[
|\mu(\varphi)| \leq c|\varphi|_{A(\Omega)} \leq c|\varphi|_{A'}.
\]
Therefore, there exists a unique \( \nu \in A' \), which satisfies \( \nu(\varphi) = \mu(\varphi) \), for every \( \varphi \in \mathcal{D} \). Also, if \( \psi \in \mathcal{D}(\mathbb{R}^n \setminus \Omega) \), then \( \lambda(\psi) = 0 \), for every \( \lambda \in A'_0(\Omega) \), and thus \( |\psi|_{A(\Omega)} = 0 \), which implies that \( \nu(\psi) = \mu(\psi) = 0 \). Consequently, \( \text{supp} \nu \subset \Omega \) and therefore \( \nu \) defines an elements of \( A'_0(\Omega) \).

The elements of \( A(\Omega) \). Assume that \( \Omega \) is an open bounded domain in \( \mathbb{R}^n \) satisfying the segment condition and \( A \subset \mathcal{S}' \) is a Banach space which satisfies postulates \( [P_1 - P_5] \). If \( \varphi \in \mathcal{D} \), then \( \varphi \) defines an element of \( A(\Omega) \), and if \( \varphi, \psi \in \mathcal{D} \) coincide in \( \Omega \), they define the same element of \( A(\Omega) \), whereas if \( \varphi, \psi \) differ in \( \Omega \), then they define different elements of \( A(\Omega) \). In fact, Proposition 3 implies that \( \mathcal{D} \subset A' \), and thus \( (A(\Omega)) \subset A'(\Omega) \), which allows us to find a \( \zeta \in \mathcal{D}(\Omega) \), such that \( \zeta(\varphi) - \zeta(\psi) = \int_{\Omega} \zeta(\varphi - \psi) \neq 0 \). Therefore, the set \( \mathcal{D}_{\mid \Omega} = \{ \varphi \mid \Omega : \varphi \in \mathcal{D} \} \) can be viewed as a subset of \( A(\Omega) \), and in fact a dense one. If \( u \) is an element of \( A(\Omega) \), then \( u \) represents a limit of a \( A(\Omega) \)-Cauchy sequence in \( \mathcal{D}_{\mid \Omega} \). Furthermore, if \( u \in A \), then \( u \) defines an element of \( A(\Omega) \), since \( u \) is a limit of a \( A \)-Cauchy sequence \( \{ \varphi_i \} \in \mathcal{D} \subset A \), and thus a limit of a \( A(\Omega) \)-Cauchy sequence.

Also, every element \( u \) of \( A(\Omega) \) defines a unique distribution \( T_u \) in \( \Omega \), since convergence in \( A \) is stronger than the convergence in \( \mathcal{D}'(\Omega) \). We shall see that, different elements of \( A(\Omega) \) define different distributions in \( \Omega \), provided that \( \Omega \) is bounded and satisfies the segment condition. It suffices to show that, if \( T_u = T_v \Rightarrow u = v \), implies also that behavior of the elements of \( A(\Omega) \) on \( \partial \Omega \) is determined form their behavior in \( \Omega \).

It is also noteworthy that, if \( \Omega_1, \Omega_2 \) are open subsets of \( \mathbb{R}^n \) and \( \Omega_1 \subset \Omega_2 \), then the elements of \( A(\Omega_2) \) define elements of \( A(\Omega_1) \). To every element \( u \in A(\Omega_2) \) corresponds an \( A(\Omega_1) \)-Cauchy sequence \( \{ \varphi_i \} \in \mathcal{D} \subset A \), which is also \( A(\Omega_1) \)-Cauchy; therefore, \( u \) defines an element in \( A(\Omega_1) \), which is clearly unique. In fact, this observation allows us to define on \( A(\Omega_2) \) the restriction \( \mathcal{R}_{\Omega_1} u \) of \( u \) on \( \Omega_1 \), which defines a continuous imbedding in \( A(\Omega_1) \); in fact, a contractive one.

The solution spaces \( A^\text{ext}_L(\Omega) \) and \( A^\text{ext}_C(\Omega) \). Let \( L = \sum_{|\alpha| \leq m} a_\alpha D^\alpha \) be an elliptic operator. We next define
\[
A^\text{ext}_L(\Omega) = \{ u \in A(\Omega) : Lu = 0 \text{ in } \Omega \}.
\]

The set \( A^\text{ext}_L(\Omega) \) contains exactly those elements of \( A(\Omega) \) for which, the corresponding distributions in \( \mathcal{D}'(\Omega) \) satisfy the equation \( Lu = 0 \), in the sense of distributions. We also define as \( A^\text{ext}_C(\Omega) \) the set of elements in \( A(\Omega) \) which can be approximated by test functions which satisfy the equation \( Lu = 0 \).
in a neighborhood of $\overline{\Omega}$. Clearly, $A^{ext}_{L^p}(\overline{\Omega}) \subset A^{int}_{L^p}(\overline{\Omega})$. We shall see that, if $\Omega$ is bounded and satisfies the segment condition, then $A^{ext}_{L^p}(\overline{\Omega}) = A^{int}_{L^p}(\overline{\Omega})$.

**Local spaces.** Let $\Omega$ be an open subset of $\mathbb{R}^n$. We set

$$A_{loc}(\Omega) = \{ v \in \mathcal{D}' : \varphi v \in A(\overline{\Omega}), \text{ for all } \varphi \in \mathcal{D} \}.$$  

Let $\Omega_\nu$, $\nu \in \mathbb{N}$, be a sequence of open subsets of $\mathbb{R}^n$, such that $\overline{\Omega}_\nu \subset \Omega_{\nu+1}$ and $\bigcup_{\nu \in \mathbb{N}} \Omega_\nu = \Omega$. For every $\nu \in \mathbb{N}$ we define on $A_{loc}(\Omega)$ the seminorm

$$\| u \|_\nu = \| R_{\Omega_\nu} u \|_{\Omega_\nu},$$

and subsequently, the invariant metric

$$d(u, v) = \sum_{\nu=1}^{\infty} \frac{1}{2^\nu} \frac{\| u - v \|_\nu}{1 + \| u - v \|_\nu},$$

with respect to which $A_{loc}(\Omega)$ becomes a Fréchet space. Clearly, the induced topology is independent of the choice of the $\Omega_\nu$’s.

**Remark 2.** Usually, $A(\overline{\Omega})$ is defined as $A/A_0(\mathbb{R}^n \setminus \Omega)$, where

$$A_0(\overline{\Omega}) = \{ u \in A : \text{supp } u \subset \overline{\Omega} \}.$$  

The two definitions do not coincide if the boundary of $\Omega$ is not sufficiently smooth. In the case $A = W^{k,p}(\mathbb{R}^n)$, with $k \geq 1$ and $p \in [1, \infty)$, the space $A(\overline{\Omega})$, according to the latter definition, does not in general coincide with $W^{k,p}(\Omega)$ for domains satisfying the segment condition, since (according to the latter definition) the elements of $A(\overline{\Omega})$ extend to elements of $A$.

However, if $\Omega$ satisfies the segment condition, then the elements of $W^{k,p}(\Omega)$, $k \geq 1$, do not, in general, extend to elements of $W^{k,p}(\mathbb{R}^n)$.

Meanwhile, as we shall see, in our definition, $A(\overline{\Omega})$ does coincide with $W^{k,p}(\Omega)$ for domains satisfying the segment condition.

**Examples of semilocal Banach spaces.** Most of the Banach spaces which are used in Harmonic Analysis and PDEs are semilocal, i.e., they satisfy the first and second postulates. However, spaces as for example $L^\infty(\mathbb{R}^n)$, $W^{k,\infty}(\mathbb{R}^n)$ and the Hölder spaces do not satisfy the fourth postulate, i.e., $\mathcal{D}$ is not dense in $\mathcal{A}$. Nevertheless, if $u \in A(\overline{\Omega})$ can be approximated by functions which are solutions of the elliptic equation $Lu = 0$ in a neighborhood of $\overline{\Omega}$, then $u$ can also be approximated by elements of $\mathcal{D}$. Hence, in our framework, the fourth postulate is only a reasonable restriction.

**Lebesgue spaces.** If $p \in [1, \infty)$, then $A = L^p(\mathbb{R}^n)$ is a Banach space, it is a subset of $\mathcal{D}'$ and it satisfies postulates $[P_1 - P_5]$. Also, $(L^p(\mathbb{R}^n))^\prime = L^{q}(\mathbb{R}^n)$, where $1/p + 1/q = 1$, and if $\Omega \subset \mathbb{R}^n$ is open, then

$$A^\prime_0(\Omega) = \{ v \in L^q(\mathbb{R}^n) : \text{supp } v \subset \overline{\Omega} = L^q(\Omega).$$

Clearly, if $\{ \varphi_\nu \} \subset \mathcal{D}$ is $A(\overline{\Omega})$-Cauchy, then it defines a unique element of $L^p(\Omega)$, while sequences with different limits correspond to different elements of $L^p(\Omega)$. In fact, $\| \cdot \|_{A(\overline{\Omega})} = \| \cdot \|_{L^p(\Omega)}$. Conversely, each element of $L^p(\Omega)$ is a limit of a $A(\overline{\Omega})$-Cauchy sequence $\{ \varphi_\nu \} \subset \mathcal{D}$, since $\{ \varphi_\nu : \nu \in \mathcal{D} \}$ is dense in $L^p(\Omega)$. Also, $A_{loc}(\Omega)$ coincides with $L^p_{loc}(\Omega)$.

Note that, $\mathcal{D}$ is not dense in $L^\infty(\mathbb{R}^n)$.
Sobolev spaces. If \( \Omega \subset \mathbb{R}^n \) open, \( k \in \mathbb{N} \) and \( p \in [1, \infty) \), then the space
\[
W^{k,p}(\Omega) = \{ u \in \mathcal{D}'(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for every } |\alpha| \leq k \},
\]
with norm \( |u|_{W^{k,p}(\Omega)} = \left( \sum_{|\alpha| \leq k} |D^\alpha u|_{L^p(\Omega)}^p \right)^{1/p} \) is a Banach space. In fact, \( \mathcal{A} = W^{k,p}(\mathbb{R}^n) \) satisfies postulates \([P_1 - P_5]\). We shall see that, if \( \Omega \) satisfies the segment condition, then \( \mathcal{A}(\overline{\Omega}) \) coincides with \( W^{k,p}(\Omega) \). At first, if \( \varphi \in \mathcal{D} \), then
\[
|\varphi|_{\mathcal{A}(\overline{\Omega})} = \sup_{v \in \mathcal{A}'(\Omega) \setminus \{0\}} \|v\|_{\mathcal{A}'(\Omega)} = \sup_{\|\mu\|_{W^{k,p}(\Omega)} = 1} \|\varphi\|_{W^{k,p}(\Omega)}
\]
since every \( v \in \left( W^{k,p}(\mathbb{R}^n) \right)' \), which is supported in \( \overline{\Omega} \), defines an element of \( \left( W^{k,p}(\Omega) \right)' \), and every element of \( \left( W^{k,p}(\Omega) \right)' \) is an element of \( \left( W^{k,p}(\Omega) \right)' \) supported in \( \overline{\Omega} \). Clearly, if \( u \in \mathcal{A}(\overline{\Omega}) \), then \( u \) is a \( W^{k,p}(\Omega) \)-limit of elements of \( \mathcal{D} \). Conversely, if \( u \in W^{k,p}(\Omega) \), then \( u \) can be approximated by elements of \( \mathcal{D} \), since \( \Omega \) satisfies the segment condition (see [AF03, Theorem 3.22]). Therefore, the elements of \( \mathcal{A}(\overline{\Omega}) \) can be identified, in an isometric isomorphism, with the elements of \( W^{k,p}(\Omega) \). Finally, it is clear that \( \mathcal{A}_{\text{loc}}(\Omega) \) coincides with \( W^{k,p}_{\text{loc}}(\Omega) \).

Note that, \( \mathcal{D} \) is not dense in \( W^{k,\infty}(\mathbb{R}^n) \). The space \( \mathcal{A} = W^{k,p}(\mathbb{R}^n) \) satisfies postulates \([P_1 - P_5]\). We shall next see that \( \mathcal{A}'(\overline{\Omega}) \) coincides with \( W^{k,p}_{\text{loc}}(\Omega) \).

Sobolev spaces of negative norm. Let \( k \) be a positive integer and \( p \in (1, \infty) \). If \( \varphi \) is a test function, we set
\[
|\varphi|_{-k,p} = \sup_{v \in W^{-k,p}(\mathbb{R}^n)} \int_{\mathbb{R}^n} \varphi(x) v(x) \, dx,
\]
where \( 1/p + 1/q = 1 \) and \( |\cdot|_{-k,p} \) is the norm of the Sobolev space \( W^{-k,q}(\mathbb{R}^n) \). Clearly, \( |\cdot|_{-k,p} \) defines a norm on \( \mathcal{D} \), and let \( W^{-k,p}(\mathbb{R}^n) \) be the completion of \( \mathcal{D} \) with respect to this norm. The Banach space \( W^{-k,p}(\mathbb{R}^n) \) can be viewed as a subspace of \( \mathcal{D}' \), since the \( |\cdot|_{-k,p} \)-convergence is stronger than the convergence in \( \mathcal{D}' \). It can be shown that \( \mathcal{A}' \) coincides with \( W^{-k,q}(\mathbb{R}^n) \) and \( \mathcal{A}'(\overline{\Omega}) \) coincides with \( W^{-k,p}_{\text{loc}}(\Omega) \). Finally, \( \mathcal{A}(\overline{\Omega}) \) coincides with \( W^{-k,p}_{\text{loc}}(\Omega) \).

Spaces of smooth functions. If \( k \in \mathbb{N} \), then \( C^k_0(\mathbb{R}^n) \) is the space of functions with continuous partial derivatives up to order \( k \) which vanish at infinity, i.e., \( \lim_{|x| \to \infty} D^\alpha u(x) = 0 \), for every \( |\alpha| \leq k \), and it is a Banach space with respect to the norm \( |u|_k = \max_{|\alpha| \leq k} \sup_{x \in \mathbb{R}^n} |D^\alpha u(x)| \). It is readily seen that \( \mathcal{A} = C^k_0(\mathbb{R}^n) \) satisfies postulates \([P_1 - P_5]\). If \( \Omega \subset \mathbb{R}^n \) is open and bounded, then \( C^k(\overline{\Omega}) \) is the space of functions with continuous partial derivatives up to order \( k \) which extend continuously to \( \overline{\Omega} \), and it is a Banach space with respect to the norm \( |u|_{k,\overline{\Omega}} = \max_{|\alpha| \leq k, x \in \overline{\Omega}} |D^\alpha u(x)| \). According to Whitney’s Extension Theorem (see [Hör83a, Theorem 2.3.6]), every \( u \in C^k(\overline{\Omega}) \) extends to a \( \tilde{u} \in \mathcal{E}_u \subset C^k_0(\mathbb{R}^n) \) and there is a \( c > 0 \), depending only on \( \Omega \) and \( k \), such that
\[
|\tilde{u}|_k \leq c |u|_{k,\overline{\Omega}} \text{ for every } u \in C^k(\overline{\Omega})\]

The above allow us to identify \( \mathcal{A}(\overline{\Omega}) \) with \( C^k(\overline{\Omega}) \). Also, \( \mathcal{A}_{\text{loc}}(\Omega) \) coincides with \( C^k(\Omega) \), the space of \( k \) times continuously differentiable functions in \( \Omega \), endowed with the topology of convergence with respect to the norms \( |\cdot|_{k,K} \), for all compact subsets \( K \) of \( \Omega \).
Lipschitz spaces. Let \( \sigma \in (0, 1) \), \( F \subset \mathbb{R}^n \), and \( u : F \to \mathbb{R} \). We set

\[
\omega_\sigma(u, \delta, F) = \sup_{x, y \in F, |x-y|<\delta} \frac{|u(x)-u(y)|}{|x-y|^\sigma}
\]

and

\[
|u|_{0, \sigma, F} = \sup_{\delta>0} \omega_\sigma(u, \delta, F).
\]

If \( \Omega \subset \mathbb{R}^n \), then the space \( \text{Lip}(0, \sigma, \Omega) \) contains all the bounded continuous functions \( u \), for which \( |u|_{0, \sigma} < \infty \). \( \text{Lip}(0, \sigma, \Omega) \) is a Banach space with respect to the norm \( | \cdot |_{0, \sigma, \Omega} = | \cdot | + | \cdot |_{0, \sigma, \Omega} \). In general, if \( k \in \mathbb{N} \), the space \( \text{Lip}(k, \sigma, \Omega) \) contains all the functions \( u : \Omega \to \mathbb{R} \) which are \( k \) times continuously differentiable in \( \Omega \), and all their derivatives up to order \( k \) are bounded and extend continuously to \( \overline{\Omega} \) and \( |D^nu|_{0, \sigma, \Omega} < \infty \). The space \( \text{Lip}(k, \sigma, \Omega) \) is a Banach space when endowed with the norm

\[
|u|_{k, \sigma, \Omega} = |u|_{k, \Omega} + \max_{|\alpha|\leq k} [D^\alpha u]_{0, \sigma, \Omega}.
\]

Unfortunately, the test functions, when restricted to \( \overline{\Omega} \), are not dense in \( \text{Lip}(k, \sigma, \Omega) \), for every \( k \in \mathbb{N} \) and every \( \sigma \in (0, 1) \). In fact, if \( \Omega \) is bounded and \( \sigma \in (0, 1) \), then a function \( u \in \text{Lip}(k, \sigma, \Omega) \) can be approximated by elements of \( \mathcal{D} \) if and only if \( \lim_{\delta \to 0} \omega_\sigma(u, \delta, \Omega) = 0 \) ([Tar95, 1.3.4]). In particular, for \( \sigma = 1 \), a function \( u \in \text{Lip}(k, 1, \Omega) \) can be approximated by elements of \( \mathcal{D} \) if and only if \( u \in C^{k+1}(\overline{\Omega}) \).

Let \( \sigma \in (0, 1) \) and \( k \in \mathbb{N} \). We denote by \( \text{lip}(k, \sigma, \mathbb{R}^n) \) the closure of \( \mathcal{D} \) in \( \text{Lip}(k, \sigma, \Omega) \). If \( \Omega \subset \mathbb{R}^n \) is open and bounded, then we set

\[
\text{lip}(k, \sigma, \Omega) = \left\{ u \in \text{Lip}(k, \sigma, \Omega) : \lim_{\delta \to 0} \omega_\sigma(u, \delta, \Omega) = 0 \right\}.
\]

The elements of \( \text{lip}(k, \sigma, \Omega) \) extend to elements of \( \text{lip}(k, \sigma, \mathbb{R}^n) \), and the norm of the extension operator is bounded ([Ste70, Chapter 6]). If we set \( \mathcal{A} = \text{lip}(k, \sigma, \mathbb{R}^n) \), then \( \mathcal{A} \) satisfies postulates \( [P_1 - P_5] \) and if \( \Omega \subset \mathbb{R}^n \) is open and bounded, then it is readily seen that \( \mathcal{A}(\overline{\Omega}) \) can be identified with \( \text{lip}(k, \sigma, \Omega) \) and \( \mathcal{A}_{\text{loc}}(\Omega) \) coincides with \( \text{lip}(k, \sigma, \Omega) \), which is the Fréchet space, the topology of which is defined by the family of seminorms \( \{ | \cdot |_{k, \sigma, K} : K \text{ compact subset of } \Omega \} \).

4. Density results

In this section, \( \Omega \) shall always be an open bounded domain in \( \mathbb{R}^n \) and \( \mathcal{A} \subset \mathcal{D} \) shall be a Banach space which satisfies postulates \( [P_1 - P_5] \).

4.1. An abstract version of a density result of F. Browder. We first prove the following result:

**Proposition 6.** Let \( \mathcal{A} \) be a Banach space in \( \mathbb{R}^n \) satisfying postulates \( [P_1 - P_5] \) and \( v \in \mathcal{A}(\overline{\Omega})' \), where \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \). Assume further that \( v \) annihilates every \( u \) satisfying \( \mathcal{L} u = \mathcal{L} v = 0 \) in a neighborhood of \( \overline{\Omega} \). If \( \{ U_j \}_{j=0}^J \) is an open covering of \( \overline{\Omega} \), there exist \( \{ \psi_j \}_{j=0}^J \subset \mathcal{A}(\overline{\Omega})' \), such that \( \text{supp } \psi_j \subset \overline{\Omega} \cap U_j \), \( \sum_{j=0}^J \psi_j = v \), and each \( \psi_j \) annihilates every \( u \) which satisfies \( \mathcal{L} u = \mathcal{L} v = 0 \) in a neighborhood of \( \overline{\Omega} \cap U_j \).

Note that, if \( \zeta \in \mathcal{D} \) is a cut-off function, which is equal to one in a suitable neighborhood of \( \overline{\Omega} \), then the zero extension \( \zeta u \) is an element of \( \mathcal{D} \). Next, we observe that the value of \( \langle \zeta u, v \rangle \) does not depend on \( \zeta \), since \( \text{supp } v \subset \overline{\Omega} \). We can thus say that \( v \) annihilates \( u \) if \( \langle \zeta u, v \rangle = 0 \).

**Proof of Proposition 6.** Let \( \{ \psi_j \}_{j=0}^J \) be a partition of unity subordinate to the covering \( \{ U_j \}_{j=0}^J \). If \( e \) is a fundamental solution of \( \mathcal{L} \), then \( \psi_j(x) e \) = 0 for every \( x \in \mathbb{R}^n \setminus \overline{\Omega} \). But \( \theta(x) = \psi_j(x) e = (\delta * v)(x) \). Note that \( \delta \) defines a distribution in \( \mathbb{R}^n \), as a convolution of two distributions, one of which (namely \( v \)) has compact support (i.e., \( \text{supp } v \subset \overline{\Omega} \)). Also, \( \delta \) is a smooth function and satisfies, in the classical
sense, the equation $L^* u = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$. Thus $\text{supp} \vartheta \subset \overline{\Omega}$. Further, $L^* \vartheta = \nu$ in the sense of distributions, since $\vartheta$ is a fundamental solution of $L^*$ (the adjoint of $L$). Set $\nu^j = L^*(\psi^j \vartheta)$, $j = 0, \ldots, J$. Proposition 2 provides that $\{\nu^j\}_{j=0} \subset (A(\overline{\Omega}))'$. It is clear that $\sum_{j=0}^J \nu^j = \nu$ and $\text{supp} \nu^j \subset \overline{\Omega} \cap \overline{\Omega}_j$. Let $u$ be a $C^\infty$–function, defined in a neighborhood $W$ of $\overline{\Omega} \cap \overline{\Omega}_j$, which satisfies the equation $L u = 0$. Let $\zeta \in \mathcal{D}(W)$, which is equal to one in a neighborhood of $\overline{\Omega} \cap \overline{\Omega}_j$. Then the zero extension of $\zeta u$ belongs to $\mathcal{D}$ and we have

$$\langle u, \nu^j \rangle = \langle \zeta u, \nu \rangle = \langle \zeta u, L^*(\psi^j \vartheta) \rangle = \langle L(\zeta u), \psi^j \vartheta \rangle.$$ 

The last term of the above is equal to zero since $L(\zeta u) = 0$ in a neighborhood of the support of $\psi^j \vartheta$. Thus $\nu^j$ annihilates every $u$ satisfying $L u = 0$ in a neighborhood of $\overline{\Omega} \cap \overline{\Omega}_j$. \hfill $\square$

**Remark 3.** Proposition 6 remains valid if in its formulation we replace the phrase "$\nu$ annihilates every $u$ which satisfies $L u = 0$ in a neighborhood of $\overline{\Omega}$ by $\nu$ annihilates $\tau_x e$, for every $x \in \mathbb{R}^n \setminus \overline{\Omega}$ where $e$ is a fundamental solution of $L$.

In our main result the singularities of the fundamental solution lie in an open set outside of $\Omega$:

**Theorem 1.** Let $L$ be an elliptic operator with constant coefficients in $\mathbb{R}^n$ and $e$ be a fundamental solution of $L$. Let also $\Omega$ be an open bounded domain satisfying the segment condition and $U \subset \mathbb{R}^n \setminus \overline{\Omega}$ an open set intersecting all the components of $\mathbb{R}^n \setminus \overline{\Omega}$. Then the set $\mathcal{X}$ of linear combinations of the functions $\tau_x e$, with $y \in U$, is dense in $A^\text{int}_L(\overline{\Omega}) = \{ u \in A(\overline{\Omega}) : L u = 0 \text{ in } \Omega \}$, with respect to the norm of $A(\overline{\Omega})$.

**Proof.** We use the duality argument of the proof of Theorem 3 in [Bro62]. We first observe that $\mathcal{X}$ and $A^\text{int}_L(\overline{\Omega})$ are linear subspaces of $A(\overline{\Omega})$. According to the Hahn–Banach theorem, it suffices to show that $\mathcal{X}^\perp \subset (A^\text{int}_L(\overline{\Omega}))^\perp$, i.e.,

$$\text{if } \nu \in (A(\overline{\Omega}))' \text{ and } \nu(u) = 0 \text{ for every } u \in \mathcal{X}, \text{ then } \nu(u) = 0 \text{ for every } u \in A^\text{int}_L(\overline{\Omega}).$$

Let $\nu \in (A(\overline{\Omega}))'$ annihilating $\mathcal{X}$. If $x \in U$, then $\tau_x e$ belongs to $\mathcal{X}$ and

$$0 = \nu(u) = \langle \tau_x e, \nu \rangle = \langle \varepsilon \vartheta, \nu \rangle(x).$$

Thus the convolution $\vartheta = \varepsilon \vartheta \vartheta$ vanishes in $U$. Meanwhile, $\vartheta$ satisfies the elliptic equation $L^* u = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$, and thus it is a real analytic function in $\mathbb{R}^n \setminus \overline{\Omega}$. Let $V$ be a connected component of $\mathbb{R}^n \setminus \overline{\Omega}$. Since $U$ intersects $V$, then $\vartheta$ vanishes in $V$, and consequently in the whole $\mathbb{R}^n \setminus \overline{\Omega}$. Thus $\nu(\tau_x e) = 0$, for every $x \in \mathbb{R}^n \setminus \overline{\Omega}$.

Since $\Omega$ satisfies the segment condition, for every $x \in \partial \Omega$ there exist a non–zero vector $\xi_x$ and an open neighborhood $U_x$ of $x$, such that, if $y \in U_x \cap \overline{\Omega}$, then $y + t \xi_x \in \Omega$, for every $t \in (0, 1)$. Since $\partial \Omega$ is compact, there is a finite collection of such neighborhoods $\{U_j\}_{j=1}^J$ covering $\partial \Omega$. The collection $\{U_j\}_{j=1}^J$ becomes an open cover of $\overline{\Omega}$ by adding a suitable open set $U_0$, which can be chosen so that $\overline{U}_0 \subset \Omega$. Let $\{\psi^j\}_{j=0}^J$ be an infinitely differentiable partition of unity subordinate to the covering $\{U_j\}_{j=0}^J$ of $\overline{\Omega}$.

Proposition 6, and more specifically its modified version according to Remark 3, provides functionals $\{\psi^j\}_{j=0}^J \subset (A(\overline{\Omega}))'$, such that $K^j = \text{supp} \psi^j \subset \overline{\Omega} \cap \overline{\Omega}_j$, $\sum_{j=0}^J \psi^j = \nu$, and each $\psi^j$ annihilates every $u$ satisfying $L u = 0$ in a neighborhood of $K^j$. Let $\xi_j$ be the translation vector provided by the segment.
condition which corresponds to \( U_i \). We denote by \( \tau_{j,\varepsilon} \), the translation operator by \( \varepsilon \xi_j \), with \( \varepsilon \in \mathbb{R} \), i.e.,

\[
(\tau_{j,\varepsilon} \circ w)(x) = \begin{cases} 
  w(x - \varepsilon \xi_j) & \text{if } j = 1, \ldots, J, \\
  w(x) & \text{if } j = 0,
\end{cases}
\]

where \( w \) is a distribution. We also define \( v^j_{\varepsilon} = \tau_{j,\varepsilon} v^j \) and \( v_\varepsilon = \sum_{j=0}^J v^j_{\varepsilon} \). Clearly, each \( v^j_{\varepsilon} \) annihilates every \( u \) which satisfies \( \mathcal{L}u = 0 \) in a neighborhood of \( \text{supp} \ v^j_{\varepsilon} \). Also, \( \text{supp} \ v_\varepsilon = \mathcal{K}_\varepsilon \subset \Omega \). Note that \( v_\varepsilon \) annihilates every \( u \) which satisfies \( \mathcal{L}u = 0 \) in a neighborhood of \( \mathcal{K}_\varepsilon \).

Next we shall show that \( v_\varepsilon \) converges to \( v \) in the weak* sense of \( (\mathcal{A}(\overline{\Omega}))' \). It suffices to show that

\[
\lim_{\varepsilon \to 0} v^j_{\varepsilon}(u) = v^j(u), \quad \text{for every } u \in \mathcal{A}(\overline{\Omega}) \text{ and } j = 0, \ldots, J.
\]

Let \( u \in \mathcal{A}(\overline{\Omega}) \) and \( \varphi \in \mathcal{D} \). Then

\[
v^j_{\varepsilon}(u) - v^j(u) = (v^j_{\varepsilon}(u) - v^j(\varphi)) + (v^j(\varphi) - v^j(\varphi)) + (v^j(\varphi) - v^j(u)).
\]

The third term of (4.1) can become arbitrarily small if \( \varphi \) is chosen sufficiently close to \( u \), and so can the first term, since \( |v^j_{\varepsilon}|_{\mathcal{A}'} = |v^j|_{\mathcal{A}'} \) due to the fifth postulate. The second term of (4.1) can be written as

\[
v^j(\varphi) - v^j(\varphi) = v^j((\tau_{j,\varepsilon}\xi_j)\varphi) - v^j(\varphi) = v^j((\tau_{j,\varepsilon}\xi_j)\varphi - \varphi).
\]

The right-hand side of the above tends to zero, as \( \varepsilon \to 0 \), since \( \tau_{j,\varepsilon}\xi_j \varphi \to \varphi \), as \( \varepsilon \to 0 \), in the topology of \( \mathcal{D} \), and consequently in the topology of \( \mathcal{A} \) as well, due to the third postulate. The weak* convergence of \( v_\varepsilon \) to \( v \), as \( \varepsilon \to 0 \), is now established.

Finally, let \( u \in \mathcal{A}^{\text{int}}_\mathcal{L}(\overline{\Omega}) \). Clearly, \( u \) can be viewed as an element of \( \mathcal{D}'(\Omega) \), when restricted to \( \Omega \), and since \( \mathcal{L}u = 0 \), in \( \Omega \), then \( u \) can be viewed as a smooth function, when restricted to \( \Omega \). Therefore, \( v_\varepsilon(u) = 0 \), since \( u \) satisfies the equation \( \mathcal{L}u = 0 \) in a neighborhood of \( \mathcal{K}_\varepsilon \). Also,

\[
v(u) = \lim_{\varepsilon \to 0} v_\varepsilon(u) = 0,
\]

which concludes the proof of the Theorem. \( \square \)

Let \( \Omega \), \( \mathcal{L} \), \( \varepsilon \) and \( X \) as in Theorem 1. Clearly, \( \mathcal{A}^{\text{int}}_\mathcal{L}(\overline{\Omega}) \subset \mathcal{A}^{\text{int}}_\mathcal{L}(\overline{\Pi}) \). If \( V \) is an open bounded subset of \( \mathbb{R}^n \setminus \overline{\Pi} \) intersecting all the components of \( \mathbb{R}^n \setminus \overline{\Omega} \), then clearly \( X \) is a subset of the set of functions satisfying the equation \( \mathcal{L}u = 0 \) in a neighborhood of \( \overline{\Pi} \). Thus, the closure of \( X \) with respect to the norm of \( \mathcal{A}(\overline{\Omega}) \) is a subset of \( \mathcal{A}^{\text{ext}}_\mathcal{L}(\overline{\Pi}) \). Due to Theorem 1 the closure of \( X \) coincides with \( \mathcal{A}^{\text{int}}_\mathcal{L}(\overline{\Omega}) \), and thus \( \mathcal{A}^{\text{int}}_\mathcal{L}(\overline{\Pi}) = \mathcal{A}^{\text{ext}}_\mathcal{L}(\overline{\Pi}) \). We have thus shown the following result which is an extension of Proposition 7 in [Wei73].

**Corollary 1.** If \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \) satisfying the segment condition and \( \mathcal{L} \) is an elliptic operator with constant coefficients, then \( \mathcal{A}^{\text{ext}}_\mathcal{L}(\overline{\Omega}) = \mathcal{A}^{\text{int}}_\mathcal{L}(\overline{\Omega}) \).

The following result is our answer to the main question of the Introduction:

**Theorem 2.** Let \( \Omega, \mathcal{U} \) be open subset of \( \mathbb{R}^n \), with \( \mathcal{U} \) satisfying the segment condition. Assume also that \( \Omega \) is bounded, \( \overline{\Omega} \subset \mathcal{U} \) and \( \overline{\mathcal{U}} \setminus \overline{\Omega} \) does not have any compact components. If \( \mathcal{L} \) is an elliptic operator with constant coefficients, then the set of solutions of \( \mathcal{L}u = 0 \) in \( \mathcal{U} \) is dense in \( \mathcal{A}(\overline{\Omega}) \).

Note that, if \( \overline{\Omega} \subset \mathcal{U} \) and \( \overline{\mathcal{U}} \setminus \overline{\Omega} \) does not have any compact components, then \( \mathbb{R}^n \setminus \mathcal{U} \) intersects every component of \( \mathbb{R}^n \setminus \overline{\Omega} \). Instead of proving Theorem 2 we shall prove the following, slightly stronger, Runge-type theorem:
Theorem 3. Let \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \) satisfying the segment condition and \( L \) be an elliptic operator with constant coefficients. Assume that \( \{V_j\}_{j=1}^N \) are the holes of \( \Omega \) (i.e., the bounded components of \( \mathbb{R}^n \setminus \overline{\Omega} \)), and that \( F = \{\xi_1, \ldots, \xi_J\} \) intersects each of the holes of \( \Omega \). Then the set of solutions of \( Lu = 0 \) in \( \mathbb{R}^n \setminus F \) is dense in \( A^1 \mathcal{L}_{\text{int}}(\overline{\Omega}) \).

Proof. Assuming that \( \xi_j \in V_i \) for every \( j = 1, \ldots, J \), we can find an \( r > 0 \), such that \( \overline{B}(\xi_j, r) \subset V_i \), for every \( j = 1, \ldots, J \). Here, by \( B(\xi, r) \) we denote the closure \( B(\xi, r) \). Since \( \Omega \) is bounded we can find an \( R > 0 \), such that \( \overline{\Omega} \subset B(0, R) \). We set \( W_0 = \Omega \) and

\[
W_\nu = B(0, \nu R) \setminus \bigcup_{j=1}^I \overline{B}(\xi_j, 1), \quad \nu \geq 1,
\]

then clearly, \( W_\nu \subset W_{\nu+1} \) and \( \bigcup_{\nu \in \mathbb{N}} W_\nu = \mathbb{R}^n \setminus F \). Theorem 1 is valid if \( \Omega \) is replaced by \( W_\nu \) and \( U \) by \( W_\nu \), with \( 0 \leq \nu < \nu' \). Let \( \epsilon > 0 \) and \( u \in A(\overline{\Omega}) \) satisfying \( Lu = 0 \) in \( \Omega \). Set \( v_0 = u \). By virtue of Theorem 1 there exists a \( v_1 \in W_1 \) which satisfies \( Lu = 0 \) in \( W_1 \), such that \( |v_0 - R_{W_0} v_1|_{A(W_0)} < \epsilon/2 \).

Recursively, we construct a sequence \( v_\nu, \nu \in \mathbb{N}, \) such that

\[
v_\nu \in A(W_\nu), \quad Lv_\nu = 0 \quad \text{in} \quad W_\nu, \quad \text{and} \quad |v_\nu - R_{W_\nu} v_{\nu+1}|_{A(W_\nu)} < \frac{\epsilon}{2^{\nu+1}}.
\]

Clearly, for every \( \nu \geq 0 \), the sequence \( \{R_{W_\nu} v_\ell\}_{\ell \geq 0} \) is a Cauchy sequence in \( A(W_\nu) \), and denote by \( v_\nu \) its limit. It is also clear that \( R_{W_\nu} w = w_\nu \), for every \( 0 \leq \nu < \nu' \). A unique distribution \( \tilde{w} \in \mathcal{D}'(\mathbb{R}^n \setminus F) \), is then defined from the \( w_\nu \)'s as follows. Let \( \phi \in \mathcal{D}(\mathbb{R}^n \setminus F) \) and \( \text{supp} \phi \subset U_\nu \), then we set \( \langle \phi, w \rangle = \langle \phi, w_\nu \rangle \). Clearly, \( \text{supp} \tilde{w} = 0 \) in \( W_\nu \), for every \( \nu \), and \( \mathcal{L} \tilde{w} = 0 \) in \( \mathbb{R}^n \setminus F \). Also, \( w \in A_{\text{loc}}(\mathbb{R}^n \setminus F) \), since, if \( \phi \in \mathcal{D}(\mathbb{R}^n \setminus F) \), then \( \phi \tilde{w} = \phi w_\nu \), provided that \( \text{supp} \phi \subset W_\nu \), and \( \phi \tilde{w} \) can be approximated, in the norm of \( A \), by test functions. Finally,

\[
|R_{W_\nu} w - u|_{A(\Omega)} = |R_{W_\nu} w - u|_{A(W_0)} = \lim_{\ell \to \infty} |R_{W_\nu} v_\ell - u|_{A(W_\nu)} \leq |R_{W_\nu} v_1 - u|_{A(W_\nu)} + \sum_{\ell=1}^\infty |R_{W_\nu} v_{\ell+1} - R_{W_\nu} v_\ell|_{A(W_\nu)} \leq \sum_{\ell=1}^\infty \frac{\epsilon}{2^{\ell+1}} = \epsilon,
\]

which concludes the proof. \( \square \)

4.2. Density and non–density results in specific spaces. Theorems 1–3 and Corollary 1 remain valid if \( A(\overline{\Omega}) \) is replaced by any of the spaces:

(i) \( W^{k,p}_n(\Omega), k \in \mathbb{N}, p \in [1, \infty) \).

(ii) \( W^{-k,p}_n(\Omega), k \geq 1, p \in (1, \infty) \).

(iii) \( C^k(\overline{\Omega}), k \in \mathbb{N} \).

(iv) \( \text{lip}(k, \sigma, \overline{\Omega}), k \in \mathbb{N}, \sigma \in (0, 1) \).

However, all the previous results cease to be valid if \( A(\overline{\Omega}) \) is replaced by a semilocal Banach space in which the fourth postulate is not satisfied. Clearly, if a solution of the equation \( Lu = 0 \) can not be approximated by functions which are smooth in a neighborhood of \( \overline{\Omega} \), then they can not be approximated by linear combinations of translates of a fundamental solution of \( L \), since such linear combinations are real analytic in a neighborhood of \( \overline{\Omega} \). In fact, Theorems 1–3 and Corollary 1 are not valid if \( A(\overline{\Omega}) \) is replaced by any of the spaces:

(i) \( W^{k,\infty}_n(\Omega), k \in \mathbb{N} \).

(ii) \( \text{lip}(k, \sigma, \overline{\Omega}), k \in \mathbb{N}, \sigma \in (0, 1) \).

(iii) \( \mathcal{M}(\overline{\Omega}), \) the set of signed Borel measures on \( \overline{\Omega} \), which is a Banach space with norm the total variation of the measure.
4.3. **Mathematical foundation of the method of fundamental solutions.** Let \( L \) be an elliptic partial differential operator in \( \Omega \subset \mathbb{R}^n \) of order \( m \). In Trefftz methods, the solution of the boundary value problem

\[
\begin{align*}
L u &= 0 \quad \text{in } \Omega, \quad \text{(4.2a)} \\
B u &= f \quad \text{on } \partial \Omega, \quad \text{(4.2b)}
\end{align*}
\]

where \( \Omega \) is an open domain in \( \mathbb{R}^n \) and \( Bu = f \) is the boundary condition, is approximated by linear combinations of particular solutions of (4.2a), provided that such linear combinations are dense in the set of all solutions of this equation. A typical Trefftz method is the method of fundamental solutions (MFS), the particular solutions of the partial differential equation under consideration are the fundamental solutions of the corresponding partial differential operator with singularities outside of \( \overline{\Omega} \). The MFS was introduced by Kupradze and Aleksidze [KA63] in 1963 as the method of generalized Fourier series (метод обобщённых рядов Фурье). In the most popular version of the MFS the singularities of the fundamental solutions, are located on a pseudo–boundary, i.e., a prescribed boundary \( \partial \Omega' \) of a domain \( \Omega' \) embracing \( \Omega \).

**Definition 2.** (The Embracing Pseudo-boundary) Let \( \Omega, \Omega' \) be open connected subsets of \( \mathbb{R}^n \). We say that \( \Omega' \) embraces \( \Omega \) if \( \overline{\Omega} \subset \overline{\Omega'} \), and for every connected component \( V \) of \( \mathbb{R}^n \setminus \overline{\Omega} \), there is an open connected component \( V' \) of \( \mathbb{R}^n \setminus \overline{\Omega'} \) such that \( V' \subset V \).

Comprehensive reference lists of applications of the MFS can be found in [FKM03] and references therein. The question of the applicability of the MFS, i.e., whether linear combinations of fundamental solutions with singularities lying on a prescribed pseudo–boundary are dense in the set of all solutions of the corresponding equation has been studied by [Bog85, KA63, Smy06, Smy07].

**Laplace equation.** Unfortunately, such linear combinations are not always dense in the solution space. If, for example, the pseudo–boundary is the unit circle \( \partial D \), then the translates of

\[
e_1(x) = -\frac{1}{2\pi} \log |x|,
\]

which is a fundamental solution of \( L = -\Delta \), with singularities on \( \partial \Omega \), vanish at the origin, and so do their linear combinations. Here, \( | \cdot | \) is the Euclidean norm in \( \mathbb{R}^2 \). Thus, such linear combinations are not dense in the set of harmonic functions in a disk of radius \( \varrho < 1 \) centered at the origin.

On the other hand, in dimensions \( n \geq 3 \), linear combinations of translates of the fundamental solution \( e_1 \) of \( -\Delta \) given by

\[
e_1(x) = -\frac{|x|^{2-n}}{(2-n) \omega_{n-1}}, \quad \text{(4.3)}
\]

where \( \omega_{n-1} \) is the area of the surface of the unit sphere \( S^{n-1} \) in \( \mathbb{R}^n \) and \( | \cdot | \) is the Euclidean norm in \( \mathbb{R}^n \), with singularities lying on a prescribed pseudo–boundary are dense in the solution space. In particular, we have the following result:

**Theorem 4.** Let \( \Omega, \Omega' \) be open bounded domains in \( \mathbb{R}^n \), \( n \geq 3 \), with \( \Omega \) satisfying the segment condition and \( \Omega' \) embracing \( \Omega \). Then the space \( X \) of finite linear combinations of the form \( \sum_{j=1}^{N} c_j \tau_{y_j} e_1 \), where \( e_1 \) is given by (4.3) and \( \{y_j\}_{j=1}^{N} \subset \partial \Omega' \), is dense in

\[
\mathcal{Y} = \{ u \in \mathcal{A}(\overline{\Omega}) : \Delta u = 0 \quad \text{in } \Omega \},
\]

with respect to the norm of \( \mathcal{A}(\overline{\Omega}) \).
Sketch of proof. Let \( v \in \mathcal{A}_0(\Omega) \), such that \( \langle u, v \rangle = 0 \), for every \( u \in \mathcal{X} \). Then the convolution \( \vartheta = \varphi * v \) is harmonic in \( \mathbb{R}^n \setminus \overline{\Omega} \) and vanishes on \( \partial \Omega' \). Here we have used the fact that \( \varphi = \varphi_1 \). Also, \( D^\alpha \varphi \) vanishes at infinity for every multi-index \( \alpha \) and consequently, so does \( \vartheta \), since \( \vartheta \), outside of \( \overline{\Omega} \), is equal to \( \vartheta(x) = \langle \tau_x \varphi_1, v \rangle \). If \( V \) is a bounded component of \( \mathbb{R}^n \setminus \overline{\Omega} \), then there is an open set \( V' \), such that \( \overline{\Omega}' \subset V \) and \( \partial V' \subset \partial \Omega' \). Since \( \vartheta \) vanishes on \( \partial V' \subset \partial \Omega' \), it vanishes in \( V' \) as well, due to the maximum principle, and in the whole of \( V \), since \( \vartheta \) is a real analytic function in \( V \). In the case of the unbounded component of \( \mathbb{R}^n \setminus \overline{\Omega} \), using the fact that \( \vartheta \) vanishes at infinity and on a boundary of an unbounded component we obtain similarly that \( \vartheta \) vanishes in the whole component, and therefore in \( \mathbb{R}^n \setminus \overline{\Omega} \). The rest of the proof is identical to the proof of Theorem 1. \( \square \)

Biharmonic and \( m \)-harmonic equation. In the case of an elliptic operator \( \mathcal{L} \) of order \( 2m > 2 \), the linear combinations of the translates of a fundamental solution of \( \mathcal{L} \), with singularities on a pseudo-boundary, are not, in general dense in the solution space ([Smy06]). In such case, the MFS approximation contains translates of fundamental solutions of \( \mathcal{L} \) in the boundary, are not, in general dense in the solution space ([Smy06]). In such case, the MFS approximation is of the form ([Bog85, FKM03, KA63, Smy06])

\[
u_N = \sum_{i=1}^{M} \sum_{j=1}^{N} c_i^j \tau_y e_i,
\]

where \( N \in \mathbb{N}, \{ y_j \}_{j=1}^{N} \subset \partial \Omega' \) and \( e_i \) is a fundamental solution of \( \mathcal{L}_i = (-\Delta)^i \). In the cases \( n = 3 \)

\[
e_i(x) = \left( -1 \right)^{i-1} \frac{|x|^{2i-3}}{4\pi(2i-3)!},
\]

is a fundamental solution of \( \mathcal{L}_i = (-\Delta)^i \). It is straightforward that \( -\Delta e_{i+1} = e_i \), in the sense of distributions, for every positive integer \( i \). We have the following density result:

**Theorem 5.** Let \( \Omega, \Omega' \) be open bounded domains in \( \mathbb{R}^3 \), with \( \Omega \) satisfying the segment condition and \( \Omega' \) embracing \( \Omega \). Then the space \( \mathcal{X}_m \) of finite linear combinations of the form \( \sum_{\ell=1}^{m} \sum_{j=1}^{N} c_j^\ell \tau_y e_j, \) where \( N \in \mathbb{N}, \{ y_j \}_{j=1}^{N} \subset \partial \Omega' \) and \( e_j \) is given by (4.4), is dense in \( \mathcal{A}^m_{\partial \Omega}(\overline{\Omega}) \), with respect to the norm of \( \mathcal{A}(\overline{\Omega}) \).

**Proof.** We shall use induction on \( m \). If \( m = 1 \), then Theorem 5 reduces to Theorem 4. Assume that Theorem 5 is valid for \( m = \ell \) and \( v \in \mathcal{A}(\overline{\Omega})' \) annihilates \( \mathcal{X}_{\ell+1} \). Since \( \mathcal{X}_{\ell} \subset \mathcal{X}_{\ell+1} \), then \( v \) annihilates all the functions which are \( \ell \)-harmonic in a neighborhood of \( \overline{\Omega} \). Clearly,

\[ -\Delta \vartheta_{\ell+1}(x) = -\Delta (e_{\ell+1} * v)(x) = -((\Delta * e_{\ell+1}) * v)(x) = (e_{\ell} * v)(x) = v(\tau_x e_\ell) = 0, \]

for every \( x \in \mathbb{R}^n \setminus \overline{\Omega} \), since \( \tau_x e_\ell \in \mathcal{X}_\ell \), for every \( x \in \mathbb{R}^n \setminus \overline{\Omega} \). Taylor expansion of \( \tau_x e_{\ell+1}(y) \), provides

\[ \tau_x e_{\ell+1}(y) = e_{\ell+1}(y-x) = e_{\ell+1}(x-y) = \sum_{|\alpha| \leq \ell} \frac{(-1)^{|\alpha|}}{\alpha!} D^\alpha e_{\ell+1}(x) y^\alpha + O \left( \frac{1}{|x|} \right), \]

for \( x \) large. The monomials \( y^\alpha, |\alpha| \leq \ell \), are \( \ell \)-harmonic, and thus \( v(y^\alpha) = 0 \), for \( |\alpha| \leq \ell \). Therefore,

\[ \vartheta_{\ell+1}(x) = (e_{\ell+1} * v)(x) = v(\tau_x e_{\ell+1}) = O \left( \frac{1}{|x|} \right), \]

and consequently, \( \vartheta_{\ell+1} \) vanishes at infinity. Since \( \vartheta_{\ell+1} \) vanishes also on \( \partial \Omega' \) and it is harmonic in \( \mathbb{R}^n \setminus \overline{\Omega} \), then \( \vartheta_{\ell+1} \) vanishes in \( \mathbb{R}^n \setminus \overline{\Omega} \). The rest of the proof is identical to the proof of Theorem 1. \( \square \)
Remark 4. Similar density results establishing the applicability of the MFS can be obtained for various partial differential operators including the modified Helmholtz, poly-Helmholtz and the Cauchy–Navier system in linear elasticity. (See [Smy06].)

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