DENSITY RESULTS WITH LINEAR COMBINATIONS OF TRANSLATES OF FUNDAMENTAL SOLUTIONS∗

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ABSTRACT. In the present work, we investigate the approximability of solutions of elliptic partial differential equations in a bounded domain Ω by linear combinations of translates of fundamental solutions of the underlying partial differential operator. The singularities of the fundamental solutions lie outside of Ω. The domains under consideration satisfy a rather mild boundary regularity requirement, namely the Segment Condition. We study approximations with respect to the norms of the spaces C^k(Ω), C^k,σ(Ω) and W^{k,p}(Ω), and we establish density and non-density results for elliptic operators with constant coefficients. We also provide applications of our density results related to the method of fundamental solutions and to the theory of universal series.

1. INTRODUCTION

Let \( L = \sum_{|\alpha| \leq m} a_\alpha \partial^\alpha \) be an elliptic operator of order \( m \) with coefficients in \( C^\infty \) possessing a fundamental solution. Felix Browder [Bro62] showed in 1962 that linear combinations of fundamental solutions of \( L \) with singularities in an arbitrary open set outside a bounded open domain \( \overline{\Omega} \) are dense, in the sense of the uniform norm, in the space

\[ \mathcal{X} = \{ u \in C^m(\Omega) : Lu = 0 \text{ in } \Omega \} \cap C(\overline{\Omega}). \]

The domain \( \Omega \) is assumed to satisfy the Cone Condition. Also, the fundamental solution \( e = e(x, y) \) is assumed to be bi–regular, i.e., \( L_x e(\cdot, y) = \delta_y \) and \( L^*_y e(x, \cdot) = \delta_x \), for every \( x, y \), where \( \delta_x \) is the Dirac measure with unit mass at the \( x \) and \( L^* \) is the adjoint of \( L \). (Here \( L_x \) signifies that the differentiation is with respect to \( x \).) Browder’s proof relies on a duality argument, an application of the Hahn–Banach theorem. Browder’s result extends to a partial differential operator \( \mathcal{L} \) possessing a bi–regular fundamental solution, with the property that its adjoint \( \mathcal{L}^* \) satisfies the Condition of uniqueness for the Cauchy problem in the small in \( \Omega \):

\( (U) \). If \( u \in C^m(V) \), where \( V \) is an open connected subset of \( \Omega \) with \( \mathcal{L}^* u = 0 \) and if \( u \) vanishes in a nonempty open subset of \( V \), then \( u \) vanishes everywhere in \( V \).

Weinstock [Wei73] extended Browder’s theorem by showing that the solutions of \( \mathcal{L} u = 0 \) in \( \Omega \), which are also elements of \( C^k(\overline{\Omega}) \), can be approximated by solutions of \( \mathcal{L} u = 0 \) in a neighborhood of \( \overline{\Omega} \), when \( 0 \leq k < m \), where \( m \) is the order of \( \mathcal{L} \). In Weinstock’s work, \( \mathcal{L} \) is assumed to be an elliptic operator with constant coefficients and the domain \( \Omega \) is required to satisfy a weaker condition, the Segment Condition. A survey on extensions of Browder’s theorem and questions on approximability of solutions of elliptic equations by solutions of the same equations in larger domains can be found in [Tar95].

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In this work we extend Browder’s theorem to domains satisfying the segment condition. Our density results are with respect to the spaces $C^{k,\sigma}(\overline{\Omega})$, $k \in \mathbb{N}$, $\sigma \in [0,1)$, as well as the Sobolev spaces $W^{k,p}(\Omega)$, $k \in \mathbb{N}$, $p \in [1,\infty)$. In our results the domain $\Omega$ may possess holes or equivalently its complement may be disconnected. We consider elliptic operators with constant coefficients and we examine the case in which the singularities of the fundamental solutions lie in an open set outside of $\overline{\Omega}$. We observe that analogous density results do not hold with respect to the spaces $W^{k,\sigma}(\Omega)$ and $\Lambda^{k,\sigma}(\overline{\Omega})$. (The spaces $C^{k,\sigma}(\overline{\Omega})$ and $\Lambda^{k,\sigma}(\overline{\Omega})$ are defined in 2.1.) We also provide density results for the case in which the singularities of the fundamental solutions lie on the boundary of a domain $\Omega'$ embracing $\Omega$ - Such results establish the applicability of the method of fundamental solutions. Finally, we present applications of our density results related to the theory of universal series.

The paper is organized as follows. In Section 2 we give definitions of our function spaces, we describe the method of fundamental solutions and we provide definitions and the main result of the abstract theory of universal series. Section 3 contains the main density results. In Section 4 we provide applications of our density results in the theory of universal series. We also provide variations of these results which establish the applicability of the method of fundamental solutions for certain elliptic boundary value problems. In Section 5 we provide a summary of this work and concluding remarks.

In order to avoid overloading the main text, certain proofs are relegated to the Appendix.

2. Preliminaries

2.1. Function spaces.

The spaces $C^k(\overline{\Omega})$. If $\Omega$ is an open bounded domain in $\mathbb{R}^n$, then the space $C^k(\Omega)$, where $k$ is a nonnegative integer, contains all functions $u$ which, together with all their partial derivatives $D^\alpha u$ of orders $|\alpha| \leq k$, are continuous in $\Omega$ and $C^\infty(\Omega) = \bigcap_{k \in \mathbb{N}} C^k(\Omega)$. The space $C^k(\overline{\Omega})$ consists of all functions $u \in C^k(\Omega)$ for which $D^\alpha u$ is uniformly continuous and bounded in $\Omega$ for all $|\alpha| \leq k$, and thus extend continuously to $\overline{\Omega}$. In fact, $C^k(\overline{\Omega})$ is a Banach space with norm $\|u\|_k = \max_{|\alpha| \leq k} \max_{x \in \overline{\Omega}} |D^\alpha u(x)|$.

Hölder spaces. Let now $\sigma \in (0,1)$ and $\Omega$ be an open bounded domain in $\mathbb{R}^n$. The space of Hölder functions $\Lambda^0,\sigma(\overline{\Omega})$ consists of all functions $u$, such that $[u]_{\sigma} = \sup_{\delta>0} \omega_{\sigma}(u,\delta) < \infty$, where

$$\omega_{\sigma}(u,\delta) = \sup_{x,y \in \overline{\Omega}, 0 < |x-y| < \delta} \frac{|u(x)-u(y)|}{|x-y|^\sigma}.$$  

The space $\Lambda^0,\sigma(\overline{\Omega})$ is a Banach space with norm $|u|_{0,\sigma} = |u|_0 + [u]_{\sigma}$, where $|\cdot|_0$ is the norm of $C(\overline{\Omega})$. In general, if $k \in \mathbb{N}$, then the space $\Lambda^k,\sigma(\overline{\Omega})$ consists of all functions $u$ which, together with all their partial derivatives $D^\alpha u$, $|\alpha| \leq k$, belong to $C^{0,\sigma}(\overline{\Omega})$. The space $\Lambda^k,\sigma(\overline{\Omega})$ is a Banach space with respect to the norm given by $|u|_{k,\sigma} = |u|_k + \max_{|\alpha| \leq k} [D^\alpha u]_{\sigma}$. If $u \in \Lambda^k,\sigma(\overline{\Omega})$ and $\lim_{\delta \to 0} \omega_{\delta}(u,\delta) = 0$, then $u$ is called uniformly Hölder continuous of order $(k,\sigma)$ in $\overline{\Omega}$. The set of uniformly Hölder continuous of order $(k,\sigma)$ in $\overline{\Omega}$, which is denoted by $C^{k,\sigma}(\overline{\Omega})$, is a closed subspace of $\Lambda^k,\sigma(\overline{\Omega})$ and thus a Banach space as well. Clearly, if $u \in \Lambda^k,\sigma(\overline{\Omega})$ can be approximated in the $|\cdot|_{k,\sigma}$-norm by functions which are $C^\infty$ in a neighborhood of $\overline{\Omega}$, then $u \in C^{k,\sigma}(\overline{\Omega})$. We extend for $\sigma = 0$ the definition of the space $C^{0,\sigma}(\overline{\Omega})$ by setting $C^{0,0}(\overline{\Omega}) = C(\overline{\Omega})$.

Sobolev spaces. If $u,v \in L^1_{\text{loc}}(\Omega)$, such that $\int_{\Omega} u(x) D^\alpha \psi(x) \, dx = (-1)^{|\alpha|} \int_{\Omega} v(x) \psi(x) \, dx$, for every $\psi \in C^\infty_0(\Omega)$, then $v$ is said to be a weak $\alpha$-derivative of $u$. The weak $\alpha$-derivative is uniquely defined almost everywhere (if it exists) and we use for it the same notation as for the classical derivative of $u$. Let $m$ be a positive integer and $p \in [1,\infty)$. The space $W^{m,p}(\Omega)$ consists of all $u$ in $L^p(\Omega)$ for which all the weak derivatives $D^\alpha u$, $|\alpha| \leq m$, belong to $L^p(\Omega)$. The space $W^{m,p}(\Omega)$ is a Banach space with norm $\|u\|_{m,p} = \sum_{|\alpha| \leq m} \|D^\alpha u\|_{p}$, where $\|\cdot\|_p$ is the norm of the space $L^p$. 

2.2. Density results.
2.2. Fundamental solutions. Let $\mathcal{L} = \sum_{|\alpha| \leq m} a_\alpha D^\alpha$ be a partial differential operator in $\mathbb{R}^n$ with constant coefficients of order $m$. A fundamental solution of $\mathcal{L}$ is a function $\psi : \mathbb{R}^n \setminus \{0\}$ satisfying $\mathcal{L} \psi = \delta$, where $\delta$ is the Dirac measure with unit mass at the origin, in the sense of distributions, i.e.,

$$((\mathcal{L} \psi)(\psi) = \int_{\mathbb{R}^n} e(x) \mathcal{L}^* \psi(x) \, dx = \psi(0) = \delta(\psi),$$

for every $\psi \in C_0^\infty(\mathbb{R}^n)$, where $\mathcal{L}^* u = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} a_\alpha D^\alpha u$. The operator $\mathcal{L}^*$ is known as the adjoint of $\mathcal{L}$. It is readily shown that if $\psi$ is a fundamental solution of $\mathcal{L}$, then $\psi(x) = e(x)$, is a fundamental solution of $\mathcal{L}^*$. Also, if $\mathcal{L}$ is elliptic, then $\psi$ is real analytic in $\mathbb{R}^n \setminus \{0\}$ and satisfies, in the classical sense, $\mathcal{L} \psi(x) = 0$ for every $x \in \mathbb{R}^n \setminus \{0\}$. The fundamental solutions produce solutions of the corresponding inhomogeneous equation by convolution. If a distribution $v \in \mathcal{D}'(\mathbb{R}^n)$ has compact support and $\psi$ is a fundamental solution of $\mathcal{L}$, then the convolution of $\psi$ and $v$ is a distribution defined as $(\psi * v)(\psi) = \int_{\mathbb{R}^n} e(x) \mathcal{L}^* \psi(x) \, dx$, for $\psi \in \mathcal{D}(\mathbb{R}^n)$. It is readily proved that

$$\mathcal{L}(e * v) = (\mathcal{L} \psi) * v = \delta * v,$$

in the sense of distributions. Malgrange [Mal56] and Ehrenpreis [Ehr56] independently established in 1955–56 the existence of fundamental solutions for partial differential operators with constant coefficients. In particular, Malgrange established the existence of bi-regular fundamental solutions.

2.3. The Method of Fundamental Solutions: A Trefftz method. Let $\mathcal{L} = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$ be an elliptic partial differential operator in $\Omega \subset \mathbb{R}^n$ of order $m$. In Trefftz methods, the solution of the boundary value problem

$$\begin{align*}
\mathcal{L} u &= 0 & \text{in } \Omega, \\
\mathcal{B} u &= f & \text{on } \partial \Omega,
\end{align*}$$

(2.1a)

(2.1b)

where $\Omega$ is an open domain in $\mathbb{R}^n$ and $\mathcal{B} u = f$ is the boundary condition, is approximated by linear combinations of particular solutions of (2.1a), provided that such linear combinations are dense in the set of all solutions of this equation. Erich Trefftz presented this approach in 1926 [Tre26] as a counterpart of Ritz’s method. A typical Trefftz method is the method of fundamental solutions (MFS). The MFS was introduced by Kupradze and Aleksidze [KA63] in 1963 as the method of generalized Fourier series (метод обобщённых рядов Фурье). In a typical application to the Dirichlet problem for Laplace’s equation (see [FK98, Smy06b]) in a bounded domain $\Omega$, the function

$$e_1(x) = \begin{cases} 
\frac{-\log |x|}{2\pi}, & \text{if } n = 2, \\
\frac{|x|^{2-n}}{(2-n) \omega_{n-1}}, & \text{if } n > 2,
\end{cases}$$

(2.2)

is a fundamental solution of the operator $-\Delta$, where $\omega_{n-1}$ is the area of the surface of the unit sphere $S^{n-1}$ in $\mathbb{R}^n$ and $|\cdot|$ is the Euclidean norm in $\mathbb{R}^n$. In the MFS, the solution of the Dirichlet problem in $\Omega$ is approximated by a finite linear combination of the form $u_N(x; c) = \sum_{j=1}^N c_j e_1(x - y_j)$, where $c = (c_j)_{j=1}^N \subset \mathbb{R}^N$ and $\{y_j\}_{j=1}^N$, the singularities of the fundamental solutions, are located on a pseudo-boundary, i.e., a prescribed boundary $\partial \Omega'$ of a domain $\Omega'$ embracing $\Omega$.

**Definition 1. (The Embracing Pseudo-boundary)** Let $\Omega$, $\Omega'$ be open connected subsets of $\mathbb{R}^n$. We say that $\Omega'$ embraces $\Omega$ if $\overline{\Omega} \subset \Omega'$, and for every connected component $V$ of $\mathbb{R}^n \setminus \overline{\Omega}$, there is an open connected component $\overline{V}'$ of $\mathbb{R}^n \setminus \overline{\Omega}$ such that $\partial \overline{V}' \subset V$. Comprehensive reference lists of applications of the MFS can be found in [Ale91, FKM03, GC99]. The question of the applicability of the MFS, i.e., whether linear combinations of fundamental solutions with singularities lying on a prescribed pseudo-boundary are dense in the set of all solutions of the
corresponding equation has been studied by Kupradze and Aleksidze [KA63], Bogomolny [Bog85] and Smyrlis [Smy06a].

2.4. Universal series. Let $X$ be a Banach space on $\mathbb{K}$ (where $\mathbb{K}$ is $\mathbb{R}$ or $\mathbb{C}$) and $\{u_\ell\}_{\ell \in \mathbb{N}} \subset X$.

Definition 2. A sequence $a = \{a_\ell\}_{\ell \in \mathbb{N}} \in \mathbb{K}^\mathbb{N}$ belongs to the set $\mathcal{U}$ if the sequence of partial sums $\{\sum_{j=0}^{\ell} a_j u_j\}_{\ell \in \mathbb{N}}$ is dense in $X$. The set $\mathcal{U}$ is the class of unrestricted universal series.

Clearly, if $a = \{a_\ell\}_{\ell \in \mathbb{N}}$ is a universal series, then for every $u \in X$, there exists an increasing sequence $\{\lambda_\ell\}_{\ell \in \mathbb{N}}$ such that $\lim_{\ell \to \infty} \sum_{j=0}^{\ell} a_j u_j = u$.

Restricted universal series. Of interest is whether universal series exist in specific subspaces of $\mathbb{K}^\mathbb{N}$. Let $A$ be a linear subspace of $\mathbb{K}^\mathbb{N}$ which is a Fréchet space on $\mathbb{K}$ satisfying the following postulates:

P$_1$ The projections $\pi_\ell : A \to \mathbb{K}$, where $\pi_\ell(\{a_j\}_{j \in \mathbb{N}}) = a_\ell$, are continuous, for all $\ell \in \mathbb{N}$.

P$_2$ Let $G = \{\{a_\ell\}_{\ell \in \mathbb{N}} \in \mathbb{A} : a_\ell \neq 0 \text{ holds only for finitely many } \ell \in \mathbb{N}\}$. Then $G \subset A$.

P$_3$ Let $\{e^j\}_{j \in \mathbb{N}} \subset \mathbb{K}^\mathbb{N}$, where $e^j = (\delta_{ij})_{j \in \mathbb{N}}$. Then $\lim_{\ell \to \infty} \sum_{j=0}^{\ell} a_j e_j = a$, with respect to the distance of $A$, for every $a = \{a_\ell\}_{\ell \in \mathbb{N}} \in A$.

Then $\mathcal{U}_A = \mathcal{U} \cap A$ is the class of restricted universal series.

The main result of the abstract theory of universal series.

Lemma 1. Let $A$ be a Fréchet space satisfying the postulates P$_1$, P$_2$ and P$_3$. Let also $d(\cdot, \cdot)$ be the distance in $A$ and $\|\cdot\|$ the norm of $X$. Then the following are equivalent:

(i) $\mathcal{U}_A \neq \emptyset$.

(ii) For every $u \in X$ and $\epsilon > 0$, there exist $\ell \in \mathbb{N}$ and $c_0, \ldots, c_\ell \in A$ such that

$$\|c_0 u_0 + \cdots + c_\ell u_\ell - u\|_X < \epsilon \quad \text{and} \quad d(c_0 e^0 + \cdots + c_\ell e^\ell, 0) < \epsilon.$$

(iii) $\mathcal{U}_A$ is a dense $G_\delta$ in $A$.

(iv) $\mathcal{U}_A \cup \{0\}$ contains a dense linear subspace of $A$.

Proof. See [NP05, BGENP].

3. Density results

3.1. The main result. The domains in our density results satisfy a rather weak boundary regularity requirement, namely the segment condition:

Definition 3. (The Segment Condition) Let $\Omega$ be an open subset of $\mathbb{R}^n$. We say that $\Omega$ satisfies the segment condition if every $x \in \partial \Omega$ has a neighborhood $U_x$ and a nonzero vector $\xi$ such that, if $y \in U_x \cap \overline{\Omega}$, then $y + t \xi \in \Omega$ for every $t \in (0, 1)$.

Note that the segment condition in weaker than the Cone Condition and allows the boundaries to have corners and cusps. Also, the boundary of the domains which satisfy this condition in $n-1$ dimensional. However, if a domain satisfies the segment condition it cannot lie on both sides of any part of its boundary. In fact, domains satisfying the segment condition coincide with the interior of their closure. Bounded domains satisfying the segment condition can have only finitely many holes.

Definition 4. (The Cone Condition) Let $\Omega$ be an open subset of $\mathbb{R}^n$. We say that its boundary $\partial \Omega$ satisfies the Cone Condition if there is a finite open cover $\{U_\ell\}_{\ell=1}^J$ of $\partial \Omega$ and an $h > 0$, such that, for every $x \in \Omega \cup \bigcup J$, there is a unit vector $\xi_\ell \in \mathbb{R}^n$ such that the cone

$$C_h(\xi_\ell) = \{y = x + r \xi_\ell : r \in (0, h) \text{ and } |\xi - \xi_\ell| < h\},$$

is a subset of $\Omega$. 

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1Certain Sobolev Imbedding Theorems, which we are using in our proofs, require the following boundary requirement:
(i.e., their complement can have finitely many connected components). It is not hard to prove that, if a domain satisfies the segment condition, then every connected component of its complement has a nonempty interior.

In our first result the singularities of the fundamental solution lie in an open set outside of $\Omega$:

**Theorem 1.** Let $\mathcal{L}$ be an elliptic operator in $\mathbb{R}^n$ of order $m$ and $v = e(x)$ be a fundamental solution of $\mathcal{L}$. Let also $\Omega$ be an open bounded domain satisfying the segment condition and $U \subset \mathbb{R}^n \setminus \overline{\Omega}$ an open set intersecting all the components of $\mathbb{R}^n \setminus \overline{\Omega}$. If $k$ a non–negative integer, then the set $\mathcal{X}$ of linear combinations of the functions $q_y(x) = e(x - y)$, with $y \in U$, is dense in

$$\mathcal{Y}_k = \{ u \in C^m(\Omega) : \mathcal{L}u = 0 \} \cap C^k(\overline{\Omega}),$$

with respect to the norm of $C^k(\overline{\Omega})$.

**Proof.** We follow the duality argument of the proof of Theorem 3 in [Bro62]. Both $\mathcal{X}$ and $\mathcal{Y}_k$ are linear subspaces of $C^k(\overline{\Omega})$. From the Hahn–Banach theorem, it suffices to show that $\mathcal{X}^\perp \subset \mathcal{Y}_k^\perp$, i.e.,

$$\text{if } \nu \in (C^k(\overline{\Omega}))' \text{ and } \langle u, \nu \rangle = 0 \text{ for every } u \in \mathcal{X} \text{ then } \langle u, \nu \rangle = 0 \text{ for every } u \in \mathcal{Y}_k.$$  

Let $\nu \in (C^k(\overline{\Omega}))'$ be such that $\langle u, \nu \rangle = 0$, for every $u \in \mathcal{X}$. Clearly, if $x \in U$, then the function $u(y) = e(y - x) = \tau_x e(y)$ belongs to $\mathcal{X}$ and

$$0 = \langle u, \nu \rangle = \langle \tau_x e, \nu \rangle = (\tilde{\nu} * \nu)(x),$$

where $\tilde{\nu}(x) = e(-x)$. The functional $\nu$ defines also a distribution in $\mathbb{R}^n$. Also, $\theta = \tilde{\nu} * \nu$ vanishes in $U$. Note that $\theta$ defines a distribution in $\mathbb{R}^n$, as a convolution of two distributions, one of which (namely $\nu$) is of compact support (i.e., $\text{supp } \nu \subset \overline{\Omega}$). Since $\tilde{\nu}$ is a fundamental solution of $\mathcal{L}^*$ (the adjoint of $\mathcal{L}$), then $\mathcal{L}^* \theta = \nu$, in the sense of distributions. Meanwhile, $\theta$ satisfies the elliptic equation $\mathcal{L}^* u = 0$ in $\mathbb{R}^n \setminus \overline{\Omega}$, and thus it is a real analytic function in $\mathbb{R}^n \setminus \overline{\Omega}$. Let $V$ be a connected component of $\mathbb{R}^n \setminus \overline{\Omega}$. Since $U$ intersects $V$, then $\theta$ vanishes in $V$, and consequently in the whole $\mathbb{R}^n \setminus \overline{\Omega}$, and thus $\text{supp } \theta \subset \overline{\Omega}$.

We now need the following lemma:

**Lemma 2.** Let $\mathcal{L}$ be an elliptic operator with constant coefficients in $\mathbb{R}^n$ and $v = e(x)$ be a fundamental solution of $\mathcal{L}$. Also, let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ satisfying the segment condition and $\nu \in (C^k(\overline{\Omega}))'$. If $\theta = \tilde{\nu} * \nu$ is the convolution of the distributions $\nu$ and $\text{supp } \theta \subset \overline{\Omega}$, then there exists a sequence $\{ \theta^\ell \}_{\ell \in \mathbb{N}} \subset \mathcal{D}'(\mathbb{R}^n)$ with $\text{supp } \theta^\ell \subset \Omega$ and $\{ \mathcal{L}^* \theta^\ell \}_{\ell \in \mathbb{N}} \subset (C^k(\overline{\Omega}))'$, such that $\{ \mathcal{L}^* \theta^\ell \}_{\ell \in \mathbb{N}}$ converges to $\nu$ in the weak* sense of $(C^k(\overline{\Omega}))'$, i.e., for every $u \in C^k(\overline{\Omega})$

$$\lim_{\ell \to \infty} \langle u, \mathcal{L}^* \theta^\ell \rangle = \langle u, \nu \rangle.$$ 

**Proof.** See Appendix.

Let $u \in \mathcal{Y}_k$. Then by virtue of Lemma 2, applied to the operator $\mathcal{L}^*$ which is also elliptic, there exists a sequence of distributions $\{ \theta^\ell \}_{\ell \in \mathbb{N}}$ supported in $\Omega$ with $\{ \mathcal{L}^* \theta^\ell \}_{\ell \in \mathbb{N}} \subset (C^k(\overline{\Omega}))'$, such that

$$\langle u, \nu \rangle = \lim_{\ell \to \infty} \langle u, \mathcal{L}^* \theta^\ell \rangle.$$ 

It suffices to show that $\langle u, \mathcal{L}^* \theta^\ell \rangle = 0$, for every $\ell \in \mathbb{N}$. Let $\varepsilon = \text{dist}(\partial \Omega, \text{supp } \theta^\ell) > 0$ and

$$\Omega^\delta = \{ x \in \Omega : \text{dist}(x, \partial \Omega) \geq \delta \}.$$
Clearly, supp $\theta^f \subset \Omega'$. Let $\zeta \in C^0_0(\Omega)$ such that $\zeta|_{\Omega \setminus \Omega'/3} = 1$ and $\zeta|_{\Omega'/3} = 0$. The functions $u$ and $\zeta u$ agree in a neighborhood of the supp $\theta^f$, and thus $\langle u, L^*\theta^f \rangle = \langle \zeta u, L^*\theta^f \rangle$. On the other hand, $\zeta u \in C^0_0(\Omega)$, since $u$ is real analytic in $\Omega$, and according to the definition of the distribution $L^*\theta^f$:

$$\langle \zeta u, L^*\theta^f \rangle = \langle L(\zeta u), \theta^f \rangle.$$ 

The right–hand side in the above equality is equal to zero since $L(\zeta u) = Lu = 0$ in a neighborhood of the support of $\theta^f$. Therefore

$$\langle u, L^*\theta^f \rangle = \langle L(\zeta u), \theta^f \rangle = 0,$$

and thus $\langle u, \nu \rangle = \lim_{k \to \infty} \langle u, L^*\theta^f \rangle = 0$, which concludes the proof of Theorem 1. \hfill \qed

3.2. Extensions of Theorem 1. Analogous density results are obtainable with respect to Sobolev and Hölder norms.

I. Sobolev norms. For every non–negative integer $k$ and $p \in [1, \infty)$ the set $\mathcal{X}$ (defined in Theorem 1) is dense in

$$\mathcal{Y}_{k,p} = \{ u \in C^m(\Omega) : Lu = 0 \} \cap W^{k,p}(\Omega),$$

with respect to the norm of the Sobolev space $W^{k,p}(\Omega)$. A proof of the corresponding theorem, variation of Theorem 1, is obtainable when following the lines of the proof of Theorem 1. In fact, the steps of the proof of the $W^{k,p}$–density result are either simpler or identical to the steps of the proofs of Theorem 1 and Lemma 2.

However, if $p = \infty$ the corresponding density result does not hold. This is because of the fact that $W^{k,\infty}(\Omega)$ and $C^k(\overline{\Omega})$ share the same norm, and $C^k(\overline{\Omega})$ is a closed subspace of $W^{k,\infty}(\Omega)$. Clearly, $\mathcal{X} \subset C^k(\overline{\Omega})$, and thus the closure of $\mathcal{X}$ with respect to the $W^{k,\infty}$–norm is a subset of $C^k(\overline{\Omega})$.

II. Hölder norms. For every non–negative integer $k$ and $\sigma \in (0, 1)$ the set $\mathcal{X}$ is dense in

$$\mathcal{Y}^{k,\sigma} = \{ u \in C^m(\Omega) : Lu = 0 \} \cap C^{k,\sigma}(\overline{\Omega}),$$

with respect to the norm of the Hölder space $C^{k,\sigma}(\overline{\Omega})$. For the justification of the extension of Theorem 1 to Hölder spaces see Subsection A.2 in the Appendix.

Analogous density result does not hold for the space $\Lambda^{k,\sigma}(\overline{\Omega})$. This is because $\mathcal{X} \subset C^{k,\sigma}(\overline{\Omega})$ and since $C^{k,\sigma}(\overline{\Omega})$ is a closed subspace of $\Lambda^{k,\sigma}(\overline{\Omega})$, then the closure of $\mathcal{X}$, with respect to the norm $| \cdot |_{k,\sigma}$, is a subset of $C^{k,\sigma}(\overline{\Omega})$.

3.3. Uniform approximation by solutions of elliptic equations. Let $\Omega$ be an open bounded subset of $\mathbb{R}^n$ and $\mathcal{L}$ be an elliptic operator with constant coefficients of order $m$. Let also, $k$ be a non-negative integer and $\sigma \in [0, 1)$. We define as $\mathcal{H}^{k,\sigma}_m(\overline{\Omega})$ the space of functions in $\overline{\Omega}$ which can be approximated by functions which are solutions of the equation $\mathcal{L}u = 0$ in a neighborhood of $\overline{\Omega}$, with respect to the $C^{k,\sigma}$–norm in $\overline{\Omega}$. The result that follows, which is an extension of Proposition 5 in [Wei73], shows that membership in $\mathcal{H}^{k,\sigma}_m(\overline{\Omega})$ is a local property.

**Theorem 2.** Let $k$ be a non–negative integer, $\sigma \in [0, 1)$ and $\Omega$ be an open bounded subset of $\mathbb{R}^n$. If $u \in C^{k,\sigma}(\overline{\Omega})$ and if for every $x \in \Omega$ there exists an open neighborhood $U_x$ of $x$ in $\mathbb{R}^n$ such that $u \in \mathcal{H}^{k,\sigma}_m(\Omega \cap \overline{U}_x)$, then $u \in \mathcal{H}^{k,\sigma}_m(\overline{\Omega})$.

**Proof.** It follows immediately from Corollary 1 in the Appendix.

Let also $B^{k,\sigma}_m(\overline{\Omega})$ be the space of functions in $C^{k,\sigma}(\overline{\Omega})$ which satisfy the equation $\mathcal{L}u = 0$ in $\Omega$. The following is an extension of Proposition 7 in [Wei73].
Theorem 3. If \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \) satisfying the segment condition and \( \mathcal{L} \) is an elliptic operator with constant coefficients of order \( m \), then \( B^{k,\sigma}_\mathcal{L}(\Omega) = \mathcal{H}^{k,\sigma}_\mathcal{L}(\overline{\Omega}) \) for every non-negative integer \( k \) and every \( \sigma \in [0,1) \).

Proof. Clearly,
\[
\mathcal{H}^{k,\sigma}_\mathcal{L}(\overline{\Omega}) \subset B^{k,\sigma}_\mathcal{L}(\overline{\Omega}) = \{ u \in C^m(\Omega) : \mathcal{L}u = 0 \} \cap C^{k,\sigma}(\overline{\Omega}).
\]
Let \( \varepsilon \) be a fundamental solution of \( \mathcal{L} \) and \( V \) be an open bounded subset of \( \mathbb{R}^n \setminus \overline{\Omega} \) intersecting all the components of \( \mathbb{R}^n \setminus \overline{\Omega} \). If \( \mathcal{X} \) is the set of linear combination of the form \( \sum_{j=1}^{N} c_j e(x-y_j) \), with \( y_j \in V \), then clearly \( \mathcal{X} \) is a subset of the set of functions satisfying the equation \( \mathcal{L}u = 0 \) in a neighborhood of \( \overline{\Omega} \). Thus, the closure of \( \mathcal{X} \) with respect to the norm of \( C^{k,\sigma}(\overline{\Omega}) \) is a subset of \( \mathcal{H}^{k,\sigma}_\mathcal{L}(\overline{\Omega}) \). Due to Theorem 1 the closure of \( \mathcal{X} \) coincides with \( B^{k,\sigma}_\mathcal{L}(\overline{\Omega}) \), and thus \( B^{k,\sigma}_\mathcal{L}(\overline{\Omega}) = \mathcal{H}^{k,\sigma}_\mathcal{L}(\overline{\Omega}) \). \( \square \)

4. Applications

4.1. Universal series of fundamental solutions. Let \( \Omega \) and \( U \) be an open bounded domains with \( \Omega \) satisfying the segment condition and \( U \) intersecting all the connected components of \( \mathbb{R}^n \setminus \Omega \). Let \( S = \{ y_j \}_{j \in \mathbb{N}} \subset U \) be countable and dense in \( \partial \Omega \). The corresponding function space shall be
\[
\mathcal{X} = \{ u \in C^m(\Omega) : \mathcal{L}u = 0 \text{ in } \Omega \} \cap C^{k,\sigma}(\overline{\Omega}).
\]
Clearly, \( \mathcal{X} \) is a Banach space with respect to the norm of \( C^{k,\sigma}(\overline{\Omega}) \), being a closed subset of \( C^{k,\sigma}(\overline{\Omega}) \).

Let \( \varepsilon \) be a fundamental solution of \( \mathcal{L} \) and \( \chi_j(x) = e(x-y_j) \), where \( y_j \in S \).

Question. Is there a sequence \( a = \{ a_t \}_{t \in \mathbb{N}} \in \mathbb{K}^\mathbb{N} \) such that the sequence of partial sums \( \{ \sum_{j=0}^{t} a_j \chi_j \}_{t \in \mathbb{N}} \) is dense in \( \mathcal{X} \)?

The set of such sequences \( a \) is denoted by \( \mathcal{U} \). In particular, we are interested in finding sequences \( a = \{ a_t \}_{t \in \mathbb{N}} \) in \( \mathcal{U} \) belonging to specific subspaces of \( \mathbb{K}^\mathbb{N} \). We have the following result:

Theorem 4. Let \( \{ \chi_j \}_{j \in \mathbb{N}} \subset \mathcal{X} \) be as defined above and \( \mathcal{U} \subset \mathbb{K}^\mathbb{N} \) be the class of unrestricted universal series corresponding to \( \chi_j \) at \( j \in \mathbb{N} \). Then \( \mathcal{U} \neq \emptyset \). Furthermore,

(i) \( U_{\mathcal{L}} = U \cap \{ p \} \neq \emptyset \) for every \( p \in (1, \infty) \). In particular, \( U_{\mathcal{L}} \cap \{ p \} = U \cap \{ p \} \neq \emptyset \).

(ii) \( U_{\mathcal{L}} = U \cap \{ p \} \neq \emptyset \).

Also, the sets \( \mathcal{U} \), \( U_{\mathcal{L}}(\mathbb{N}) \) with \( p \in (1, \infty) \) and \( U_{\mathcal{L}}(\mathbb{N}) \) are dense \( C^k \) in \( \mathcal{K}^\mathbb{N} \), \( L^p(\mathbb{N}) \) and \( \mathcal{K}^\mathbb{N} \), respectively, and they contain a dense vector subspace except zero.

Proof. We use the argument of the proof of Theorem 3 in [NS07]. Since \( \cap_{p>1} L^p(\mathbb{N}) \subset L^p(\mathbb{N}) \subset \mathbb{K}^\mathbb{N} \), for every \( q \in (1, \infty) \), and since \( L^p(\mathbb{N}) \), \( \cap_{p>1} L^p(\mathbb{N}) \) and \( \mathbb{K}^\mathbb{N} \) satisfy the postulates for \( A \), it suffices to show that \( \mathcal{U} \cap \{ p \} \neq \emptyset \). The space \( \cap_{p>1} L^p(\mathbb{N}) \) is a Fréchet space with distance
\[
d(a, b) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{\|a-b\|_{1,2^{-j}}}{1 + \|a-b\|_{1,2^{-j}}},
\]
where \( \| \cdot \|_p \) is the norm of \( L^p(\mathbb{N}) \). Let \( \varepsilon > 0 \) and \( u \in \mathcal{X} \). Let \( N \in \mathbb{N} \) be such that \( \sum_{j=N+1}^{\infty} 2^{-j} < \varepsilon/2 \) and \( M \) be a sufficiently large positive integer to be defined later. Due to Theorem 1, there exist \( \ell \in \mathbb{N} \) and \( c_0, \ldots, c_\ell \in \mathbb{K} \) such that \( \sum_{j=0}^{\ell} c_j \mathcal{A}_j - u \|_{k,\sigma} < \varepsilon \), where \( \| \cdot \|_{k,\sigma} \) is the norm of the space \( C^{k,\sigma}(\overline{\Omega}) \).

Since \( S \) is dense in \( U \), for every \( j = 0, \ldots, \ell \) we can find distinct \( \mathcal{A}_j, \ldots, \mathcal{A}_{j+\ell} \), close to \( \mathcal{A}_j \), such that
\[
\left| \frac{1}{M} \sum_{j=0}^{\ell} \sum_{i=1}^{M} c_i \mathcal{A}_i - u \right|_{k,\sigma} < \varepsilon.
\]
Clearly
\[ \sum_{j=0}^{\ell} \sum_{i=1}^{M} \frac{1}{M^p} |c_j|^p = \frac{1}{M^{p-1}} \sum_{j=0}^{\ell} |c_j|^p \rightarrow 0, \]
as \( M \rightarrow \infty \), where \( p = 1 + 2^{-N} \). Thus, we can choose sufficiently large \( M \) so that \( \|c\|_{1+2^{-j}} < \frac{\epsilon}{N} \), for \( j = 1, \ldots, N \), where \( c \) is the finite sequence of \( c_j/M \) corresponding to the coefficients \( \chi_j \)'s. If follows that \( d(c, 0) < \epsilon \) and in combination with (4.1) it implies that \( U_{\gamma_{p+1}}(\Omega) \neq \emptyset \).

Next we show that \( U_{\gamma_1} = \emptyset \). Let \( u \in \mathcal{X} \) and \( W \neq \emptyset \) open subset of \( U \), such that \( \overline{W} \subset U \). Clearly, \( d = \text{dist}(\overline{W}, U) > 0 \). If there exist \( c_0, \ldots, c_\ell \) such that
\[ |c_0| + \cdots + |c_\ell| < \epsilon \quad \text{and} \quad \left| \sum_{j=0}^{\ell} c_j \chi_j - u \right|_{k, \sigma} < \epsilon, \]
and let \( s = \max_{(x,y) \in \overline{W} \times \overline{U}} |e(x-y)|, \) then for \( x \in \overline{W} \) we shall have
\[ |u(x)| - \epsilon \leq \left| \sum_{j=0}^{\ell} c_j \chi_j(x) \right| \leq \sum_{j=0}^{\ell} |c_j| \cdot |e(x-y_j)| = s \epsilon. \]
Thus \( |u(x)| \leq (1 + s) \epsilon \) for every \( x \) in \( \overline{W} \) which leads to contradiction since it requires from \( u \) to be arbitrarily small in any open \( W \) with \( \overline{W} \subset \Omega \).

4.2. Applicability of the method of fundamental solutions. In the MFS, the solution of an elliptic boundary value problem in a bounded domain \( \Omega \) is approximated by linear combinations of translates of a fundamental solution of the underlying operator \( \mathcal{L} \) with singularities lying on a prescribed pseudo–boundary \( \partial \Omega' \). Nevertheless, before applying the MFS, one needs to know that such linear combinations are dense in the solution space of the equation \( \mathcal{L} u = 0 \) in \( \Omega \).

4.2.1. Laplace equation. Unfortunately, such linear combinations are not always dense in the solution space. If, for example, the pseudo–boundary is the unit circle \( \partial D \), then the translates of
\[ e_1(x) = -\frac{1}{2\pi} \log |x|, \]
which is a fundamental solution of \( \mathcal{L} = -\Delta \), with singularities on \( \partial \Omega \), vanish at the origin, and so do their linear combinations. Thus, such linear combinations are not dense in the set of harmonic functions in a disk of radius \( \rho < 1 \) centered at the origin.

On the other hand, in dimensions \( n \geq 3 \), linear combinations of translates of the fundamental solution \( e_1 \) of \( -\Delta \) given by (2.2), with singularities lying on a prescribed pseudo–boundary are dense in the solution space. In particular, we have the following result:

**Theorem 5.** Let \( \Omega, \Omega' \) be open bounded domains in \( \mathbb{R}^n, n \geq 3 \), with \( \Omega \) satisfying the segment condition and \( \Omega' \) embracing \( \Omega \) and let \( k \) be a nonnegative integer. Then the space \( \mathcal{X} \) of finite linear combinations of the form \( \sum_{j=1}^{N} c_j e_1(x-y_j) \), where \( e_1 \) is given by (2.2) and \( \{y_j\}_{j=1}^{N} \subset \partial \Omega' \), is dense in
\[ \mathcal{Y}_k = \{ u \in C^k(\Omega) : \Delta u = 0 \text{ in } \Omega \} \cap C^k(\overline{\Omega}), \]
with respect to the norm of the space \( C^k(\overline{\Omega}) \).

**Sketch of proof.** An analytic proof Theorem 1 appears in [Smy06a]. We next describe a proof very similar to the one of Theorem 1. Let \( v \in C^k(\overline{\Omega}) \), such that \( (u, v) = 0 \), for every \( u \in \mathcal{X} \), then the convolution \( \theta = e_1 * v \) is harmonic in \( \mathbb{R}^n \setminus \overline{\Omega} \) and vanishes on \( \partial \Omega' \). Here we have used the fact that
\( \hat{\epsilon}_1 = e_1 \). Also, \( D^\alpha e_1 \) vanishes at infinity for every multi-index \( \alpha \) and consequently, so does \( \hat{\theta} \), since \( \hat{\theta} \), outside of \( \Omega \), is equal to
\[
\hat{\theta}(x) = \langle \tau_\kappa e_1, \nu \rangle = \sum_{|\alpha| \leq k+1} (-1)^{|\alpha|} \int_\Omega D^\alpha_y e_1(x-y) \nu_u(y) \, dy,
\]
where \( \{v_u\}_{|\alpha| \leq k+1} \subset L^q(\Omega) \) and \( q \in n/(n-1) \). (See Fact 1 in the proof of Lemma 2.) If \( V \) is a bounded component of \( \mathbb{R}^n \setminus \overline{\Omega} \), then there is an open set \( V' \), such that \( \overline{V'} \subset V \) and \( \partial V' \subset \partial \Omega' \). Since \( \hat{\theta} \) vanishes on \( \partial V' \subset \partial \Omega' \), it vanishes in \( V' \) as well, due to the maximum principle, and in the whole of \( V \), since \( \hat{\theta} \) is a real analytic function in \( V \). In the case of the unbounded component of \( \mathbb{R}^n \setminus \overline{\Omega} \), using the fact that \( \hat{\theta} \) vanishes at infinity and on a boundary of an unbounded component we obtain similarly that \( \hat{\theta} \) vanishes in the whole component, and therefore in \( \mathbb{R}^n \setminus \overline{\Omega} \). The rest of the proof is identical to the proof of Theorem 1.

4.2.2. Poly–Helmholtz equation. If \( L \) is a higher order elliptic operator, i.e., of order \( 2m \geq 4 \), and \( \kappa \) is a fundamental solution of \( L \), then linear combinations of the translates of \( \kappa \), with singularities lying on a pseudo–boundary, are not, in general, dense in the solution space of \( L \). (See [Smy06a].) In such cases, the MFS approximation contains \( m \) different fundamental solutions corresponding to suitable factors of \( L \). For example, in the case of the poly–Helmholtz operator\(^2\)
\[
L = (\Delta - \kappa_1^2) \cdots (\Delta - \kappa_m^2),
\]
with \( \kappa_i > 0 \) and \( \kappa_i \neq \kappa_j \), when \( i \neq j \), the MFS approximation is of the form
\[
u_0(x) = \sum_{j=1}^m \sum_{\ell=1}^N c_{j\ell} e_i(x-y_{i\ell}, \kappa_j^2),
\]
where \( e_1(\cdot, \kappa^2) \) is a fundamental solution of \( \Delta - \kappa^2 \) given by
\[
e_1(x, \kappa^2) = \begin{cases} \frac{-K_0(\kappa|x|)}{2\pi} & \text{if } n = 2, \\ e^{-\kappa|x|} & \text{if } n = 3, \end{cases}
\]
where \( K_0(r) \) is the modified Bessel function of the second kind. The applicability of the MFS is established by the density result that follows.

**Theorem 6.** Let \( \Omega, \Omega' \) be open bounded domains in \( \mathbb{R}^n \), \( n = 2,3 \), with \( \Omega \) satisfying the segment condition and \( \Omega' \) embracing \( \Omega \), and let \( k \) be a non–negative integer. Further, assume that \( \kappa_i > 0 \) and \( \kappa_i \neq \kappa_j \), when \( i \neq j \). Then the space \( \mathcal{X} \) of finite linear combinations of the form (4.2) where \( \varphi(\cdot, \kappa^2) \) is given by (4.3) and \( y_{i\ell}, \ell = 1, \ldots, N, \) lie on \( \partial \Omega' \), is dense in
\[
\mathcal{K}_k = \{ u \in C^{2m}(\Omega) \setminus \mathcal{L} \} \cup C^{k}(\overline{\Omega}),
\]
with respect to the norm of \( C^k(\overline{\Omega}) \), where \( \mathcal{L} = (\Delta - \kappa_1^2) \cdots (\Delta - \kappa_m^2) \).

**Proof.** We assume that \( m = 2 \). The case \( m > 2 \) can be done inductively. Let \( \kappa, \lambda \in \mathbb{R}^+ \) with \( k \neq \lambda \). A fundamental solution of \( \mathcal{L} = (\Delta - \kappa^2)(\Delta - \lambda^2) \) is
\[
e_2(x) = \frac{e(x, \lambda^2) - e(x, \kappa^2)}{\lambda^2 - \kappa^2},
\]
In fact, it is possible to construct a fundamental solution of \( \mathcal{L} = (\Delta - \kappa_1^2) \cdots (\Delta - \kappa_m^2) \) as a linear combination of \( e(\cdot, \kappa_1^2), \ldots, e(\cdot, \kappa_m^2) \). (See [Tré66].) It is readily shown that \( \mathcal{L} e_2 = e(\cdot, \kappa^2) \), in

\( \text{The poly–Helmholtz operator, which is elliptic, arises from } m \text{–porosity media as well as from } m \text{–layered aquifer systems. See } \text{[CAO94]} \text{ and references therein.} \)
the sense of distributions. As in the proof of Theorem 1, let \( v \in (C^k(\Omega))^\prime \) annihilating \( \mathcal{X} \). Then \( v(\tau_x e(\cdot, \kappa^2)) = 0 \)
for every \( x \in \partial \Omega' \). Thus the convolutions \( \vartheta_1 = e(\cdot, \kappa^2) * v \) and \( \vartheta_2 = e_2 * v \) vanish on \( \partial \Omega' \), and \((\Delta - \lambda^2) \vartheta_2 = \vartheta_1 \), in the sense of distributions in \( \mathbb{R}^n \). The distribution \( \vartheta_1 \) is a real analytic function and satisfies the maximum principle in \( \mathbb{R}^n \setminus \overline{\Omega} \), since it satisfies the equation \((\Delta - \kappa^2) u = 0 \). (See [GT83].) Also, \( e(\cdot, \kappa^2) \) together with its partial derivatives of all orders vanish at infinity. (See [AS92, p. 374–378].) As in the proof of Theorem 5, these imply that \( \vartheta_1 \) vanishes in \( \mathbb{R}^n \setminus \overline{\Omega} \). Consequently, \((\Delta - \lambda^2) \vartheta_2 = 0 \) in \( \mathbb{R}^n \setminus \overline{\Omega} \). Repeating the previous argument we obtain that \( \vartheta_2 \) vanishes in \( \mathbb{R}^n \setminus \overline{\Omega} \). The rest of the proof is identical to the proof of Theorem 1.

\[ \square \]

Remarks 4.1.

(i) The existence of universal series of translates of fundamental solutions of Laplace's equation with singularities on a prescribed pseudo–boundary was established in [NS07].

(ii) If we replace in Theorems 5 and 6 the space \( C^k(\Omega) \) by a Sobolev space or a space of uniformly Hölder continuous functions, then the corresponding density result is still valid.

(iii) If \( \{ p_m \}_{m \in \mathbb{N}} \) is a sequence of positive reals tending to infinity, then every Sobolev space \( W^{k,p}(\Omega) \), \( k \in \mathbb{N}, p \in [1, \infty] \), with \( \Omega \) bounded, contains a space \( W^{m,p_m}(\Omega) \), for sufficiently large \( m \). Baire's Theorem will yield the existence of a series which is universal simultaneously for all Sobolev spaces \( W^{k,p}(\Omega) \), \( k \in \mathbb{N}, p \in [1, \infty] \).

(iv) Similar density results establishing the applicability of the MFS can be obtain for various partial differential operators including the biharmonic (and \( m \)-harmonic) and the Cauchy–Navier system in linear elasticity. (See [Smy06a].)

5. Concluding Remarks

We have investigated the approximability of solutions of elliptic partial differential equations in a bounded domain \( \Omega \) satisfying a rather mild boundary regularity requirement, namely, the segment condition, by linear combinations of translates of fundamental solutions of the underlying partial differential operator. We have provided density results in the case in which the singularities of the fundamental solutions lie in an open set outside of \( \overline{\Omega} \), intersecting all the components of \( \mathbb{R}^n \setminus \overline{\Omega} \), and in the case in which the they lie in a surface embracing \( \overline{\Omega} \). Our density results are with respect to the norms of the spaces \( \overline{\Omega} \) and Sobolev spaces \( W^{k,p}(\Omega) \). We observed that analogous density results with respect to the spaces \( W^{k,\infty}(\Omega) \) and \( L^{k,p}(\Omega) \) do not hold. We have also provided applications of our density results related to the method of fundamental solutions and to the theory of universal series. The study of the approximability of solutions elliptic systems by the translates of their fundamental solutions is the subject of a future paper.

A.1. Proof of Lemma 2.

FAC T 1. If \( \overline{\Omega} \subset \Omega_1 \subset \mathbb{R}^n \), where \( \Omega_1 \) is open and satisfies the cone condition, then every bounded linear functional \( v \) on \( C^k(\Omega) \) defines a distribution in \( \Omega_1 \) which can be expressed as \( v = \sum_{|\beta| \leq k+1} (-1)^{|\beta|} D^\beta v_\beta \), with \( \{ v_\beta \}_{|\beta| \leq k+1} \subset L^q(\Omega_1) \) and \( 1 < q < n/(n-1) \).

If \( v \in \left( C^k(\Omega) \right)^\prime \), then \( v(u) \) defines a bounded linear functional in \( C^k(\Omega_1) \) as well. Sobolev Imbedding Theorem (see [AF03, Theorem 4.12]) implies that \( W^{k+1,p}(\Omega_1) \subset C^k(\Omega_1) \), for every \( p > n \), since \( \Omega_1 \) satisfies the cone condition. Therefore, \( v \) defines a bounded linear functional on \( W^{k+1,p}(\Omega_1) \). Thus \( v \in W^{-k-1,q}(\Omega_1) \), where \( 1/p + 1/q = 1 \). Consequently, there exist \( \{ v_\beta \}_{|\beta| \leq k+1} \subset L^q(\Omega_1) \) with
In our case, the elliptic operator is $\mathcal{L}^*$ and $\vartheta(x) = \epsilon(-x)$ is a fundamental solution of $\mathcal{L}^*$. Also, $v = \mathcal{L}^* \vartheta \in W^{-k-1,q}(\Omega)$, and it is represented as $v = \sum_{|\beta| \leq k+1} (-1)^{|\beta|} D^\beta \vartheta_\beta$, where $\{\vartheta_\beta\}_{|\beta| \leq k+1} \subset L^q(\Omega)$, and, in particular, $\vartheta = \sum_{|\beta| \leq k+1} (-1)^{|\beta|} D^\beta (\vartheta \varphi_\beta)$ in the sense of distributions. Using Weyl's Lemma, we obtain $\vartheta \varphi_\beta \in W^{m,p}(\Omega)$, since $\mathcal{L}^* (\vartheta \varphi_\beta) = \varphi_\beta$. Therefore, $\vartheta = \vartheta \varphi_\beta \in W^{m-k-1,q}(\Omega)$. In order to show that $\vartheta \in W^{m-k-1,p}(\Omega)$ we use the fact that supp $\vartheta \subset \Omega$ and that $\vartheta \in W^{m-k-1,q}(\Omega)$. If $m \leq k+1$, there is nothing to prove, since $W^{k,p}(\Omega) = W^{k,q}(\Omega)$, when $k \leq 0$. On the other hand, if $m-k-1 > 0$, then what needs to be proved is a consequence of the following result (for a proof see [AF03, Theorem 5.29]).

Let $\Omega$ be an open subset of $\mathbb{R}^n$ satisfying the segment condition and $k \geq 1$. Then a function $u$ belongs to $W^{k,p}(\Omega)$ if and only if the zero extension of $u$ belongs to $W^{k,p}(\mathbb{R}^n)$.

**Construction of the sequence** $\{\vartheta_\beta\}_{\beta \in \mathbb{N}}$. Since $\Omega$ satisfies the segment condition, then for every $x \in \Omega$, there exist a vector $\xi_x \in \mathbb{R}^n \setminus \{0\}$ and an open neighborhood $U_x$ of $x$, such that if $y \in U_x \cap \Omega$ then $y + t \xi_x \in \Omega$ for every $t \in (0,1)$. Let $V_x$ be an open set in $\mathbb{R}^n$ satisfying

$$x \in \nabla_x \subset U_x. \quad (A.1)$$

Since $\partial \Omega$ is compact, there is a finite collection of such neighborhoods $\{V_j\}_{j=1}^J$ covering $\partial \Omega$. Let $\{U_j\}_{j=1}^J$ be the corresponding $U_x$'s in (A.1), i.e., $\nabla V_j \subset U_j$. The collection $\{V_j\}_{j=1}^J$ becomes an open cover of $\partial \Omega$ with the addition of another open set $V_0$, such that $V_0 \subset \Omega$. Let $\{\vartheta_j\}_{j=0}^J$ be an infinitely differentiable partition of unity corresponding to the covering $\{V_j\}_{j=0}^J$ of $\partial \Omega$.

It is not hard to see that the distribution $\vartheta_j = \mathcal{L}^* (\vartheta_j \psi) \in \mathcal{D}'(\Omega)$ belongs to the dual of $C^k(\partial \Omega)$. Clearly,

$$\mathcal{L}^* (\psi \vartheta) = \psi \mathcal{L}^* \vartheta + \sum_{|\beta| \leq m-1} \mathcal{L}_\beta \psi \vartheta D^\beta \vartheta = \psi \vartheta + \sum_{|\beta| \leq m-1} \mathcal{L}_\beta \psi \vartheta D^\beta \vartheta,$$

where $\mathcal{L}_\beta$ is a linear partial differential operator with constant coefficients of order not exceeding $m - |\beta|$. Also,

$$D^\beta \vartheta \in W_0^{m-|\beta|,k}(\Omega) \subset W^{m-k,q}(\Omega) \subset (C^k(\partial \Omega))'.$$

Finally, it is clear that if $\mu \in (C^k(\partial \Omega))'$ and $\varphi \in C^\infty(\mathbb{R}^n)$, then $\varphi \mu \in (C^k(\partial \Omega))'$.

We denote by $\tau_{j,\epsilon}$ the translation operator by $\epsilon \xi_j$, where $\epsilon \in [0,1]$, i.e.,

$$\tau_{j,\epsilon} \circ w (x) = \begin{cases} w(x - \epsilon \xi_j) & \text{if } j = 1, \ldots, J, \\ w(x) & \text{if } j = 0, \end{cases}$$

where $w$ is a distribution. We also define $\vartheta_{j,\epsilon} = \tau_{j,\epsilon} \circ (\psi \vartheta)$. It is readily seen that

$$\delta = \delta(\epsilon) = \min_{1 \leq j \leq J} \text{dist} (\partial \Omega, \nabla_j \cap V_j + \epsilon \xi_j) > 0. \quad (A.2)$$
Also, if we let \( \theta_\varepsilon = \sum_{j=0}^I \theta_{j,\varepsilon} \), then we have \( \text{supp} \theta_\varepsilon \subset \Omega^\delta \).

We shall finally show that \( \lim_{\varepsilon \to 0} \mathcal{L}^* \theta_\varepsilon = \mathcal{L}^* \theta = v \), in the weak* sense of \( C^k(\Omega) \). Once this is done, the we may define the \( \ell \)-th term of the sequence \( \{ \theta_\varepsilon \}_{\varepsilon \in \mathbb{N}} \) we need to construct as \( \theta_\varepsilon \) for \( \varepsilon = 1/\ell \). We first observe that \( \mathcal{L}^* \theta_{j,\varepsilon} = \tau_{j,\varepsilon} \circ (\mathcal{L}^*(\psi_j \theta)) \). Thus, if \( v^j = \mathcal{L}^*(\psi_j \theta) \) and \( u \in C^k(\Omega) \), then

\[
\langle u, \mathcal{L}^* \theta_\varepsilon \rangle = \sum_{j=0}^I \langle u, \mathcal{L}^* \theta_{j,\varepsilon} \rangle = \sum_{j=0}^I \langle u, \tau_{j,\varepsilon} \circ v^j \rangle \quad \text{and} \quad \langle u, \mathcal{L}^* \theta \rangle = \sum_{j=0}^I \langle u, v^j \rangle.
\]

It suffices to show that \( \lim_{\varepsilon \to 0} \tau_{j,\varepsilon} \circ v^j = v^j \), in the weak* sense of \( C^k(\Omega) \). We define \( K_j \) to be the closure of \( \bigcup_{\varepsilon \in [0,1]} \text{supp} (\tau_{j,\varepsilon} \circ v^j) \subset \Omega \). Then for every \( \varepsilon \in [0,1] \) and \( u \in C^k(\Omega) \) we have that \( \langle u, \tau_{j,\varepsilon} \circ v^j \rangle = \langle \tau_{j,-\varepsilon} \circ u, v^j \rangle \) and thus

\[
|\langle u, \tau_{j,\varepsilon} \circ v^j \rangle - \langle u, v^j \rangle| = |(\tau_{j,-\varepsilon} \circ u, v^j) - \langle u, v^j \rangle| \leq \|v^j\| \cdot |\tau_{j,-\varepsilon} \circ u - u|_{K_j,k}.
\]

Here \( \|v^j\| \) is the norm of the functional \( v^j \) in \( (C^k(\Omega))^\prime \) and \( |\cdot|_{K_j,k} \) is the norm of the space \( C^k(K_j) \). Clearly, \( |\tau_{j,-\varepsilon} \circ u - u|_{K_j,k} \), tends to zero as \( \varepsilon \searrow 0 \), due to the uniform continuity of \( D^ku, |x| \leq k \), in \( K_j \).

\[\Box\]

A.2. Justification of the extension of Theorem 1 to Hölder spaces. If \( C^k(\Omega) \) is replaced by \( C^{k,\sigma}(\Omega) \), in the formulation of Theorem 1, a few modifications should be made in the proof. Let \( v \in \left( C^{k,\sigma}(\Omega) \right)^\prime \) which annihilates \( \mathcal{X}_1 \), then \( v \) is still a distribution of compact support in \( \mathbb{R}^n \) and \( \mathcal{X} \neq \emptyset \) still vanishes in \( \mathbb{R}^n \setminus \Omega \). We first need to check whether the corresponding version of Lemma 2 holds by following the steps in its proof.

We first use the fact that if \( \nu \) is an element of the dual of \( C^{k,\sigma}(\Omega) \), then by virtue Sobolev Imbedding Theorem ([AF03, Theorem 4.12]) it may be represented as \( \nu = \sum_{|\beta| \leq k+1} (-1)^{|eta|} \nu_\beta \delta^{\beta} \), with \( \{\nu_\beta\}_{|\beta| \leq k+1} \subset L^\ell(\mathbb{R}^n) \) and \( 1 < \ell < n/(\ell - 1 + \sigma) \). Also, there is an open bounded domain \( \Omega_1 \), such that \( \Omega \subset \Omega_1 \) and \( \text{supp} \nu_\beta \subset \Omega_1 \). Next, assuming that \( \nu \in C^1 \), we obtain (as in FACT II) that \( \theta = \epsilon \ast v \in V^{m-1-k\sigma}_0(\Omega) \). The construction of a sequence \( \{\theta^k\}_{k \in \mathbb{N}} \subset \left( C^{k,\sigma}(\Omega) \right)^\prime \) is also identical to the construction in the proof of Lemma 2. It remains to show that \( \mathcal{L}^* \theta_\varepsilon \to \mathcal{L}^* \theta \) in the weak* sense of \( C^{k,\sigma}(\Omega) \), as \( \varepsilon \searrow 0 \). Clearly,

\[
\langle u, \mathcal{L}^* \theta_\varepsilon \rangle = \sum_{j=0}^I \langle u, \mathcal{L}^* \theta_{j,\varepsilon} \rangle = \sum_{j=0}^I \langle u, \tau_{j,\varepsilon} \circ (\mathcal{L}^*(\psi_j \theta)) \rangle = \sum_{j=0}^I \langle u, \tau_{j,\varepsilon} \circ v^j \rangle \quad \text{while} \quad \langle u, \mathcal{L}^* \theta \rangle = \sum_{j=0}^I \langle u, v^j \rangle,
\]

with \( \theta_{j,\varepsilon}, \psi_j, v^j \) and \( \tau_{j,\varepsilon} \) as in the proof of Lemma 2. It suffices to show that \( \lim_{\varepsilon \to 0} \tau_{j,\varepsilon} \circ v^j = v^j \), in the weak* sense of \( C^{k,\sigma}(\Omega) \). Defining once again \( K_j \) as the closure of \( \bigcup_{\varepsilon \in [0,1]} \text{supp} (\tau_{j,\varepsilon} \circ v^j) \subset \Omega \), then for every \( \varepsilon \in [0,1] \) and \( u \in C^{k,\sigma}(\Omega) \) we have that \( \langle u, \tau_{j,\varepsilon} \circ v^j \rangle = \langle \tau_{j,-\varepsilon} \circ u, v^j \rangle \) and thus

\[
|\langle u, \tau_{j,\varepsilon} \circ v^j \rangle - \langle u, v^j \rangle| = |\langle \tau_{j,-\varepsilon} \circ u, v^j \rangle - \langle u, v^j \rangle| \leq \|v^j\| \cdot |\tau_{j,-\varepsilon} \circ u - u|_{K_j,k,\sigma}.
\]

Here \( \|v^j\| \) is the norm of the functional \( v^j \) in \( \left( C^{k,\sigma}(\Omega) \right)^\prime \) and \( |\cdot|_{K_j,k,\sigma} \) is the norm of the space \( C^{k,\sigma}(K_j) \). If \( |\cdot|_{K_j,k} \) is the norm of \( C^k(K_j) \), then we have that

\[
|\tau_{j,-\varepsilon} \circ u - u|_{K_j,k,\sigma} = |\tau_{j,-\varepsilon} \circ u - u|_{K_j,k} + \sum_{|\alpha| = k} \sup_{x \neq y \in K_j} \frac{|(D^k u(x + \epsilon \xi_j) - D^k u(y))(\cdot)|}{|x-y|^{\sigma}}.
\]
Clearly, the first term of the right–hand side tends to zero as $\varepsilon \searrow 0$, due to the uniform continuity of $D^a u$, $|a| \leq k$, in $K_j$. The second term is dominated by the sum

$$S = \sum_{|\alpha|=k} \sup_{x,y \in K_j \atop |x-y| \geq \varepsilon} \frac{|D^a u(x + \varepsilon \xi_j) - D^a u(x)|}{|x-y|^\sigma} + \sum_{|\alpha|=k} \sup_{x,y \in K_j \atop |x-y| \geq \varepsilon} \frac{|D^a u(y + \varepsilon \xi_j) - D^a u(y)|}{|x-y|^\sigma}$$

$$+ \sum_{|\alpha|=k} \sup_{0 < |x-y| < \varepsilon} \frac{|D^a u(x) - D^a u(y)|}{|x-y|^\sigma} + \sum_{|\alpha|=k} \sup_{0 < |x-y| < \varepsilon} \frac{|D^a u(x + \varepsilon \xi_j) - D^a u(y + \varepsilon \xi_j)|}{|x-y|^\sigma}.$$

It is straightforward that each of the four terms of $S$ tends to zero as $\varepsilon \searrow 0$. Note that the last two terms tend to zero because we have assumed that when $u \in C^{k,\sigma}(\overline{\Omega})$, then

$$\lim_{\varepsilon \searrow 0} \sup_{0 < |x-y| < \varepsilon} \frac{|D^a u(x) - D^a u(y)|}{|x-y|^\sigma} = 0,$$

for every $|\alpha| = k$. The rest of the proof of the Theorem works equally well.

**Remark A.1.** The proof of Lemma 2 in conjunction with the justification of the extension of Theorem 1 to the spaces of uniformly H"older continuous functions, and in particular, the construction of the sequence $\{\theta^{\delta}\}_{\varepsilon \in \mathbb{N}}$, yields the following result:

**Corollary 1.** Let $\mathcal{L}$ be an elliptic operator with constant coefficients in $\mathbb{R}^n$ of order $m$ and $\Omega$ be an open bounded subset of $\mathbb{R}^n$. Let $k$ be a non–negative integer, $\sigma \in [0,1)$ and assume that the functional $\nu \in (C^{k,\sigma}(\overline{\Omega}))'$ annihilates

$$\mathcal{Y}_{k,\sigma} = \{ u \in C^m(\Omega) : \mathcal{L} u = 0 \} \cap C^{k,\sigma}(\overline{\Omega}).$$

If $\{U_j\}_{j=0}^J$ is an open cover of $\overline{\Omega}$, then there exist $\{v^j\}_{j=0}^J \subset (C^{k,\sigma}(\overline{\Omega}))'$ annihilating $\mathcal{Y}_{k,\sigma}$ and satisfying

$$\text{supp } v^j \subset \overline{\Omega} \cap U_j, \quad j = 0, \ldots, J \quad \text{and} \quad \sum_{j=0}^J v^j = v.$$

**References**


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