Darboux polynomials and first integrals of Lotka-Volterra systems

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Abstract

We study Darboux polynomials and First Integrals of Lotka-Volterra systems in three dimensions.

1 Introduction

The problem of integrability of ordinary differential equations is closely related to the problem of finding constants of motion, known as first integrals. For three-dimensional systems the existence of two independent first integrals means that the system is completely integrable (because the phase portrait is then completely characterized). If a three-dimensional system is integrable its solutions have good behaviour and it is possible to obtain global information on its long-term evolution. Since the notion of integrability is based on the existence of first integrals, the following natural question arises. Given a system of ordinary differential equations depending on parameters, how does one recognize the values of the parameters for which the system has first integrals? Many different methods have been used to study the existence of first integrals. They include the use of Lax pairs [16], Painlevé analysis [2], Lie and Noether symmetries [21], [6], the method of Darboux [10], and the Ziglin analysis [23], [24].

The Lotka-Volterra dynamical system, in its general form, is defined by

\[ \dot{x}_i = b_i x_i + \sum_{j=1}^{n} a_{ij} x_j x_i \quad i = 1, \ldots, n, \]  

(1)

where \( a_{ij}, b_i \) are real parameters. This system was introduced by Lotka in order to model a chemical reaction, and by Volterra to model competition among species, in 1925. It has been widely used in applied mathematics and in a large variety of physical topics such as laser physics, plasma physics, neural networks, etc. The integrability of the three-dimensional Lotka-Volterra system has been examined extensively, see for instance Grammaticos et al [13], Almeida et al [1], and Cairó et al [3],[4]. The rational first integrals of degree zero of the three dimensional homogeneous Lotka-Volterra system has been characterized by Moulin-Ollagnier [20]. Moulin-Ollagnier [19] and Labrunie [15] have characterized the polynomial first integrals of a special form of system (1) in
three dimensions, the so-called ABC system defined by
\[
\begin{align*}
\dot{x}_1 &= x_1(Cx_2 + x_3) \\
\dot{x}_2 &= x_2(x_1 + Ax_3) \\
\dot{x}_3 &= x_3(Bx_1 + x_2).
\end{align*}
\]

Another special form of system (1) is the Volterra model, also known as the KM system, defined by
\[
\begin{align*}
\dot{x}_1 &= x_1x_2 \\
\dot{x}_2 &= -x_1x_2 + x_2x_3 \\
\dot{x}_3 &= -x_2x_3.
\end{align*}
\]

This system was originally studied by Volterra in [22]. It was first solved by Kac and van-Moerbeke in [14], using a discrete version of inverse scattering due to Flaschka [11]. In [18] Moser gave a solution of the system using the method of continued fractions and in the process he constructed action-angle coordinates. Equations (3) can be considered as a finite-dimensional approximation of the Korteweg-de Vries (KdV) equation. They also appear in the discretization of conformal field theory. The KM system has a close connection with the Toda lattice
\[
\begin{align*}
\dot{a}_i &= a_i(b_{i+1} - b_i) & i &= 1, 2, 3 \\
\dot{b}_i &= 2(a_i^2 - a_{i-1}^2) & i &= 1, 2, 3.
\end{align*}
\]

In fact, the two systems are connected by a transformation of Hénon. Toda systems are studied in [8]. Damianou and Fernandes examined the relation between Volterra and Toda systems in [9], [7].

In this report we consider a form of the three-dimensional Lotka-Volterra system (1) where the matrix \( A = (a_{ij}) \) is skew-symmetric and study its Darboux polynomials and first integrals. The report is organized as follows: In Section 2 we define Darboux polynomials and first integrals for a system of ordinary differential equations, and give some elementary results. In Section 3 we define a Lotka-Volterra system and present some results about its Darboux polynomials and first integrals. In particular, we find that any Darboux polynomial of degree up to four is the product of two Darboux polynomials of smaller degree, and we determine all independent first integrals of degree up to six. Finally, in Section 4 we make some concluding remarks.

2 Darboux polynomials and first integrals

Let us consider a system of ordinary differential equations
\[
\frac{dx_i}{dt} = v_i(x_1(t), ..., x_n(t)), \quad i = 1, ..., n
\]
where the functions \( v_i \) are smooth on a domain \( U \subset \mathbb{K}^n \). Here \( \mathbb{K} = \mathbb{R} \) or \( \mathbb{K} = \mathbb{C} \), and we denote by \( \mathbb{K}[x] \), \( x = (x_1, ..., x_n) \), the ring of polynomials in \( n \) variables over \( \mathbb{K} \). Let \( \phi : I \to U \) be a solution of (5) defined on an open non-empty interval \( I \) of the real axis. A continuous function \( F : U \to \mathbb{R} \) is called a first integral of system (5) if it is constant along its solution, i.e. if the function \( F \circ \phi \) is constant on its domain of definition for arbitrary solution \( \phi \) of (5). When \( F \) is differentiable, it is a first integral of system (5) if
\[
L_v(F) = \sum_{i=1}^n v_i(x) \frac{\partial F}{\partial x_i}(x) = 0,
\]
where $L_v$ is the Lie derivative along the vector field $v = (v_1, \ldots, v_n)$. If $A$ is any function of $x$, then the Lie derivative of $A$ is the time derivative of $A$, i.e. $\dot{A} = \frac{dA}{dt} = L_v(A)$. The vector field generates a flow $\phi_t$ that maps a subset $U$ of $\mathbb{K}^n$ to $\mathbb{K}^n$ in such a way that a point in $U$ follows the solution of the differential equation. That is, $\phi(x)(t) = v(\phi(x)) \forall x \in U$. The time derivative is also called the derivative along the flow since it describes the variation of a function of $x$ with respect to $t$ as $x$ evolves according to the differential system.

A polynomial $G \in \mathbb{K}[x]$ is called a Darboux polynomial of system (5) if

$$L_v(G) = \Lambda G,$$

for some polynomial $\Lambda \in \mathbb{K}[x]$, which is called the cofactor of $G$. When $\Lambda = 0$, the Darboux polynomial is a first integral. $G$ is said to be a proper Darboux polynomial if $\Lambda \neq 0$. The following propositions ([12]) give some elementary properties of Darboux polynomials.

**Proposition 1.** Let $F \in \mathbb{K}[x], G \in \mathbb{K}[x]$ be non-zero and coprime (i.e. they do not have common divisors different from constants). Then, $F/G$ is a rational first integral if and only if there exists $\Lambda \in \mathbb{K}[x]$ such that $L_v F = \Lambda F$ and $L_v G = \Lambda G$.

Proof. Suppose that $F/G$ is a first integral of system (5), i.e. $L_v(F/G) = 0$. We have

$$L_v(F/G) = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} \left( \frac{F}{G} \right) = \sum_{i=1}^n \frac{v_i}{G} \frac{G}{\partial x_i} - \frac{F}{G} \frac{\partial G}{\partial x_i} = \frac{1}{G} L_v(F) - \frac{F}{G^2} L_v(G),$$

so that $GL_v(F) = FL_v(G)$. Thus $F$ divides $GL_v(F)$, and since $F, G$ are coprime this implies that $L_v(F) = \Lambda F$, for some $\Lambda \in \mathbb{K}[x]$. Then, we also have $L_v(G) = \Lambda G$. Conversely, if $L_v(F) = \Lambda F$ and $L_v(G) = \Lambda G$ for some $\Lambda \in \mathbb{K}[x]$, then $L_v(F/G) = 0$ as can be seen from the above equation and $F/G$ is a first integral. $\square$

**Proposition 2.** Suppose $P_1, P_2$ are arbitrary Darboux polynomials with cofactors $\Lambda_1, \Lambda_2$ respectively. Then,

(i) $P_1P_2$ is a Darboux polynomial with cofactor $(\Lambda_1 + \Lambda_2)$.

(ii) If $\Lambda_1 = \Lambda_2$, then $P_1 + P_2$ is a Darboux polynomial with cofactor $\Lambda_1 = \Lambda_2$.

(iii) $P_1^z (z \in \mathbb{C})$ is a Darboux polynomial with cofactor $z\Lambda_1$.

(iv) All irreducible factors of $P_1$ (and $P_2$) are Darboux polynomials.

Proof. (i) We have $L_v(P_1) = \Lambda_1 P_1, L_v(P_2) = \Lambda_2 P_2$. Then,

$$L_v(P_1P_2) = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} (P_1P_2) = P_1 L_v(P_2) + P_2 L_v(P_1) = (\Lambda_1 + \Lambda_2) P_1 P_2.$$

(ii) If $\Lambda = \Lambda_1 = \Lambda_2$, then $L_v(P_1 + P_2) = L_v(P_1) + L_v(P_2) = \Lambda(P_1 + P_2)$.

(iii) We have

$$L_vP_1^z = \sum_{i=1}^n v_i \frac{\partial}{\partial x_i} P_1^z = zP_1^{z-1} L_vP_1 = z\Lambda_1 P_1^z.$$

(iv) Suppose $P_1 = D_1^n D_2$ ($n \in \mathbb{N}$), where $D_1, D_2$ are coprime polynomials and $D_1$ is irreducible. Then,

$$L_v(D_1^n D_2) = D_1^n L_v(D_2) + nD_1^{n-1} D_2 L_v(D_1) = \Lambda_1 D_1^n D_2.$$ 

Since $D_1^n$ divides $nD_1^{n-1} D_2 L_v(D_1)$ and $D_1, D_2$ are coprime, $D_1$ must divide $L_v(D_1)$, i.e. $L_v(D_1) = \mathcal{L}_1 D_1$ for some $\mathcal{L}_1 \in \mathbb{K}[x]$. Similarly, $L_v(D_2) = \mathcal{L}_2 D_2$ for some $\mathcal{L}_2 \in \mathbb{K}[x]$. By induction on $D_2$, all irreducible factors of $P_1$ are Darboux polynomials. $\square$
The following Theorem ([12]) shows that the existence of sufficiently many Darboux polynomials implies the existence of a first integral.

**Theorem 1.** *(Darboux)* Let \( \mathbf{v} = (v_1, \ldots, v_n) \in \mathbb{C}[x]^n \) be a polynomial vector field of degree \( d \) and assume that \( L_\mathbf{v} \) admits \( r \) Darboux polynomials \( P_1, \ldots, P_r \), each of degree \( \leq d-1 \), such that \( L_\mathbf{v}(P_i) = \Lambda_i P_i \), for some \( \Lambda_i \in \mathbb{C}[x] \). The polynomials of degree less than or equal to \( d-1 \) in \( n \) variables form a complex vector space of dimension \( \binom{n+d-1}{n} \). Since \( r > \left( \frac{n+d-1}{n} \right) + n \) if and only if the system admits a rational first integral (i.e. \( \lambda_i \in \mathbb{Z} \)).

**Proof.** Suppose that \( L_\mathbf{v} \) admits \( r \) Darboux polynomials \( P_1, \ldots, P_r \), each of degree \( \leq d-1 \), such that \( L_\mathbf{v}(P_i) = \Lambda_i P_i \), for some \( \Lambda_i \in \mathbb{C}[x] \). The polynomials of degree less than or equal to \( d-1 \) in \( n \) variables form a complex vector space of dimension \( \binom{n+d-1}{n} \). Since \( r > \left( \frac{n+d-1}{n} \right) + n \), there exist \( \lambda_1, \ldots, \lambda_r \in \mathbb{C} \) such that \( \sum_{i=1}^r \lambda_i \Lambda_i = 0 \) and from Proposition 2 it follows that \( L_\mathbf{v} \left( \prod_{i=1}^r P_i^{\lambda_i} \right) = (\sum_{j=1}^r \lambda_j \Lambda_j) \prod_{i=1}^r P_i^{\lambda_i} \). Hence, \( I = \prod_{i=1}^r P_i^{\lambda_i} \) is a first integral. Moreover, it can be shown that if \( r > \left( \frac{n+d-1}{n} \right) + n \), we can choose \( \lambda_i \in \mathbb{Z} \) \( \forall i \) and obtain a rational first integral. \( \square \)

The above Theorem implies the following result for planar vector fields (i.e. vector fields in two variables). The proof of the second assertion of the Theorem is omitted.

**Theorem 2.** Let \( \mathbf{f} = (f_1, f_2) \) with \( f_1, f_2 \in \mathbb{K}[x_1, x_2] \), \( L_\mathbf{f} = f_1 \frac{\partial}{\partial x_1} + f_2 \frac{\partial}{\partial x_2} \), and \( d = \deg(f_1, f_2) \). Let \( P_1, \ldots, P_r \) be Darboux polynomials. If \( r \geq \frac{d(d+1)}{2} + 2 \), then there exist integers \( n_1, \ldots, n_r \) such that

\[
I = \prod_{i=1}^r P_i^{n_i}
\]

is a first integral of \( L_\mathbf{f} \). Moreover, if \( P \) is a Darboux polynomial, then either \( P \) divides the greatest common divisor of \( f_1 \) and \( f_2 \), or there exist \( c_1, c_2 \in \mathbb{K} \) such that \( P \) divides

\[
c_1 \prod_{n_i > 0} P_i^{n_i} - c_2 \prod_{n_i < 0} P_i^{-n_i}.
\]

The degree of irreducible Darboux polynomials for planar vector fields is bounded. This bound may depend on the parameters of the vector field. One problem in computing Darboux polynomials is that this bound is not known a priori. Moreover, there is, to date, no generalization of this result to higher dimensions. It is usually conjectured that such a bound exists in general. Nevertheless, there exist systematic ways to construct elementary first integrals for planar vector fields, and we outline one such method below ([12]), known as the Prelle-Singer algorithm. Here, by an elementary function we mean a function whose integral can be expressed by using finite algebraic combinations of known functions such as log, exp, sin, cos, etc. For example, \( F(x) = \int x e^x dx \) can be expressed as \( F(x) = e^x x^2 \), but \( \int e^{-x^2} dx \) has no such simple form.
Proposition 3. If the planar vector field \( \mathbf{f} = (f_1, f_2) \) has an elementary first integral, then there exists an integer \( n \) and a Darboux polynomial \( J \) such that

\[
L_f(J) = -n \left( \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} \right) J.
\] (8)

The existence of such a Darboux polynomial provides an integrating factor for the first integral and can be used in the following algorithm due to Prelle and Singer:

1. Let \( N = 1 \).
2. Find all the Darboux polynomials \( J_i \) with \( L_f J_i = \Lambda_i J_i \) such that \( \deg(J_i) \leq N \).
3. Decide if there exists a set of constants \( \lambda_i \in \mathbb{K} \), not all zero, such that

\[
\sum_{i=1}^{m} \lambda_i \Lambda_i = 0.
\] (9)

If such \( \lambda_i \)'s exist, then \( I = \prod_{i=1}^{m} J_i^{\lambda_i} \) is a first integral. Otherwise, go to step 4.
4. Decide if there exists a set of constants \( \lambda, \ldots, \lambda_m \in \mathbb{K} \), not all zero, such that

\[
\sum_{i=1}^{m} \lambda_i \Lambda_i = - (\partial x_1 f_1 + \partial x_2 f_2).
\] (10)

Then, \( R = \prod_{i=1}^{m} J_i^{\lambda_i} \) is an integrating factor and an elementary first integral can be obtained by integrating the equations

\[
\frac{\partial I}{\partial x_1} = R f_2, \quad \frac{\partial I}{\partial x_2} = -R f_1.
\] (11)

If there is no such \( R \), and if \( N \) is less than a preset bound then increase \( N \) by 1 and return to step 2.

3 Lotka-Volterra systems

Let us consider the Lotka-Volterra system

\[
\begin{align*}
\dot{x}_1 &= x_1(rx_2 + sx_3) \quad \text{(12a)} \\
\dot{x}_2 &= x_2(-rx_1 + tx_3) \quad \text{(12b)} \\
\dot{x}_3 &= x_3(-sx_1 - tx_2) \quad \text{(12c)}
\end{align*}
\]

where \( r, s, t \in \mathbb{R} \). We note that system (12) is of considerable generality. The KM system (3) is a special case of system (12), corresponding to \( r = 1, s = 0, t = 1 \). The periodic KM system is also of the form (12).

It can be seen that the functions \( G_1 = x_1, G_2 = x_2, G_3 = x_3 \) are proper Darboux polynomials of system (12), with corresponding cofactors \( \Lambda_1 = rx_2 + sx_3, \Lambda_2 = -rx_1 + tx_3, \Lambda_3 = -sx_1 - tx_2 \), respectively, and that the function \( G_4 = x_1 + x_2 + x_3 \) is a Darboux polynomial with cofactor 0, that is, \( G_4 \) is a first integral of system (12). These are the only proper Darboux polynomials and first integral of degree one. The progress in computer algebra allows us to find Darboux polynomials of higher degree. We know from Proposition 2(i) that the product of two Darboux polynomials is also a Darboux polynomial, with cofactor the sum of the two respective cofactors. It is not difficult to verify that, up to degree four, all proper Darboux polynomials of system (12) have this structure.
For example, the quadratic Darboux polynomials of system (12) are \( x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, \) and \( x_2x_3. \)

In the case of first integrals, we follow steps 2 and 3 of the Prelle and Singer algorithm outlined above, using the Darboux polynomials \( G_i, \) and their cofactors \( \Lambda_i, \) \( i = 1, 2, 3. \) Rather than deciding whether there exists a set of constants \( \lambda_i \in \mathbb{R} \) such that \( \sum_{i=1}^{3} \lambda_i \Lambda_i = 0 \) (in which case \( I = x_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3} \) is a first integral), we consider various cases depending on the degree of the first integral. We fix the values of the constants \( \lambda_i \) (and hence the degree of the first integral), and evaluate the constants \( r, s, t \) so that equation (9) is satisfied, i.e.

\[
\lambda_1(rx_2 + sx_3) + \lambda_2(-rx_1 + tx_3) + \lambda_3(-sx_1 - tx_2) = 0. \tag{13}
\]

We thus determine the form of system (12) in order to admit specific first integrals.

In the case of quadratic first integrals, there are three possibilities:

(i) \( F = x_1x_2 \) (\( \lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 0, \))

(ii) \( F = x_1x_3 \) (\( \lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 1, \))

(iii) \( F = x_2x_3 \) (\( \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 1. \))

It is easy to verify that in case (i) we have \( r = 0, s = -t, \) in case (ii) \( s = 0, r = t, \) and in (iii) \( t = 0, r = -s. \) Thus, for example, \( F = x_1x_3 \) is a first integral of the system obtained by setting \( s = 0, r = t \) in (12):

\[
\begin{align*}
\dot{x}_1 &= rx_1x_2, \quad \text{(14a)} \\
\dot{x}_2 &= -rx_2(x_1 - x_3) \quad \text{(14b)} \\
\dot{x}_3 &= -rx_2x_3, \quad \text{(14c)}
\end{align*}
\]

which gives the KM system (3) for \( r = 1. \) The first integrals \( x_1x_2, x_1x_3, x_2x_3 \) are related in the sense that for each pair of the corresponding systems there is a transformation matrix \( M \) which sends one system to the other (in fact, in this case, there are two such matrices). In particular, let

\[
M_{1a} = \begin{pmatrix}
    r/s & 0 & 0 \\
    0 & 0 & r/s \\
    0 & r/s & 0
\end{pmatrix},
M_{1b} = \begin{pmatrix}
    0 & 0 & -r/s \\
    -r/s & 0 & 0 \\
    0 & -r/s & 0
\end{pmatrix},
\]

\[
M_{2a} = \begin{pmatrix}
    0 & 1 & 0 \\
    0 & 0 & 1 \\
    1 & 0 & 0
\end{pmatrix},
M_{2b} = \begin{pmatrix}
    0 & -1 & 0 \\
    -1 & 0 & 0 \\
    0 & 0 & -1
\end{pmatrix}.
\]


The matrices \( M_{1a} \) and \( M_{1b} \) transform the system (14) to the system admitting \( F = x_1x_2 \) as a first integral (obtained by setting \( r = 0, s = -t \) in (12)), and \( M_{2a}, M_{2b} \) transform system (14) to the system admitting \( F = x_2x_3 \) as a first integral (obtained by setting \( t = 0, r = -s \) in (12)).

To see this, let \( \mathbf{X} = (X_1, X_2, X_3) \) and consider as an example \( \mathbf{X} = M_{2a}\mathbf{x}, \) where \( \mathbf{x} = (x_1, x_2, x_3) \) satisfies the equations (14). We have \( X_1 = x_2, \) \( X_2 = x_3, \) \( X_3 = x_1, \) so that \( \dot{X}_1 = -rx_2(x_1 - x_3), \)

\( \dot{X}_2 = -rx_2x_3, \) \( \dot{X}_3 = rx_1x_2. \) Substituting for \( x_1, x_2, x_3, \) and using again the \( \mathbf{x} \)-notation we have

\[
\begin{align*}
\dot{x}_1 &= rx_1(x_2 - x_3) \\
\dot{x}_2 &= -rx_1x_2 \\
\dot{x}_3 &= rx_1x_3
\end{align*}
\]

which is the system obtained by setting \( t = 0, r = -s \) in (12). We therefore regard the three systems to be equivalent, and the system (12) as possessing only one ‘independent’ quadratic first integral.
A similar analysis applies to first integrals of higher degree. If the degree of the first integral is fixed, then one can determine the various groups of equivalent systems, the transformation matrices between systems of the same group, and the form of the first integral that each group admits. We have done this for first integrals of degree up to six. Below we summarize all forms of ‘independent’ first integrals. We have, for each form, \(i, j, k = 1, 2, 3\), and \(i, j, k\) distinct. Note that any two groups of systems admitting first integrals of different degree are independent (i.e. there is no transformation matrix between them).

(a) Degree 3: (1) \(F = x_i^2 x_j\), (2) \(F = x_1 x_2 x_3\),
(b) Degree 4: (1) \(F = x_i^2 x_j\), (2) \(F = x_i^2 x_j x_k\),
(c) Degree 5: (1) \(F = x_i^4 x_j\), (2) \(F = x_i^3 x_j^2\), (3) \(F = x_i^3 x_j x_k\), (4) \(F = x_i^2 x_j^2 x_k\),
(d) Degree 6: (1) \(F = x_i^5 x_j\), (2) \(F = x_i^4 x_j x_k\), (3) \(F = x_i^3 x_j^2 x_k\).

4 Conclusion

In this report we have considered a three-dimensional Lotka-Volterra system, and mainly by using computer algebra we have determined its Darboux polynomials and first integrals of up to degree 4 and 6, respectively. We conjecture that any Darboux polynomial of degree greater than one is reducible, i.e. a product of Darboux polynomials of smaller degree.

References


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